What if the eigenvalues are complex?

Usually what is wanted is a real solution, which can be expressed in complex form or in real form.

For example, $\ddot{y} + 4y = 0$, when written in matrix form becomes

$$
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} = 
\begin{pmatrix}
0 & 1 \\
-4 & 0
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
$$

Now the eigenvalues of $A = 
\begin{pmatrix}
0 & 1 \\
-4 & 0
\end{pmatrix}$

are given by

$$
\det(A - \lambda I) = 0 \implies \lambda^2 + 4 = 0 \implies \lambda = \pm 2i.
$$

The eigenvectors are also complex, however because the eigenvalues are complex conjugates of each other the eigenvectors are also complex conjugates of each other. (For any real matrix with complex eigenvalues the eigenvalues and vectors are complex conjugates of each other.)

$$(A - \lambda I)x = \begin{pmatrix} \mp 2i & 1 \\ -4 & \mp 2i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

So that

$$
\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ \pm 2i \end{pmatrix}
$$

and the general solution in complex form is

$$
y = c_1 \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} + c_2 \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{-2it}
$$

But note that here $c_1$ and $c_2$ are assumed to be complex constants.

Real Solutions

For a real $y$ the constant $c_2$ must be the complex conjugate of $c_1$.

Then $c_1 \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it}$ and $c_2 \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{-2it}$ are complex conjugates of each other so their sum is real. (If $z = z_r + iz_i$, then it’s complex conjugate $z^* = z_r - iz_i$ and $z + z^* = 2z_r$, which is real.) Alternatively use the fact that, by linearity, the real and imaginary parts of either solution must also be a solution.

Taking the first solution $\begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it}$, since $e^{2it} = \cos(2t) + i \sin(2t)$

$$
\begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} = \begin{pmatrix} \cos(2t) + i \sin(2t) \\ -2\sin(2t) + 2i\cos(2t) \end{pmatrix} = \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix} + i \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}
$$

Then the general solution in real form is a linear combination of the real and imaginary parts;

$$
y = d_1 \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix} + d_2 \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}
$$

for real constants $d_1$ and $d_2$.

In fact you can show that $d_1 = c_1 + c_2$ and $d_2 = i(c_1 - c_2)$, which are real if $c_2$ is the complex conjugate of $c_1$. (See if you can prove it!)
2 Theory and Theorems for first order systems

Take a general n-dimensional system

\[
\frac{dy_1}{dt} = f_1(t, y_1, y_2, \ldots, y_n)
\]

\[
\frac{dy_2}{dt} = f_2(t, y_1, y_2, \ldots, y_n)
\]

etc

\[
\frac{dy_n}{dt} = f_n(t, y_1, y_2, \ldots, y_n).
\]

We can write this as

\[\dot{y} = f(t, y),\]

where \(y\) and \(f\) are n-dimensional vectors and the \(f_i(t, y_1, y_2, \ldots, y_n)\) are not necessarily linear functions if \(y_i\) or \(t\).

For an Initial Value Problem (IVP) there is an initial condition for each \(y_i;\)
\(y_i(t_0) = K_i\) or \(y(t_0) = K\). So IVP is written as

\[\dot{y} = f(t, y) \quad \text{with} \quad y(t_0) = K\]

**Existence Uniqueness**

Basically if \(f\) is smooth then there is one and only one solution for each initial condition. It may not exist for all time, but it must exist in some open time neighbourhood of \(t_0\).

**Existence Uniqueness Theorem**

Let \(f_i\) be continuous functions with continuous partial derivatives with respect to \(y_i\) in some domain of \((t, y_1, y_2, \ldots, y_n)\) space containing \((t_0, K_1, K_2, \ldots, K_n)\).

Then The IVP

\[\dot{y} = f(t, y) \quad \text{with} \quad y(t_0) = K\]

has a unique solution on some interval \(t_0 - \alpha < t < t_0 + \alpha\).

**Note** Solutions may not exist for all time.

\[
\frac{dy}{dt} = y^2 \implies y = \frac{-1}{t + c}
\]

So for an IVP with \(y(0) = 1\) then \(c = -1\) and \(y = \frac{1}{1 - t}\)

which only exists for \(t < 1\).

**Note** If \(f_i\) are NOT continuous at \((t_0, y_i(0))\) the Existence Uniqueness Theorem is not satisfied.

For example if \(\frac{dy}{dt} = \frac{2y}{t} \implies y = ct^2\)

the IVP with \(y(0) = 1\) has no solution and

the IVP with \(y(0) = 0\) has an infinite number of solutions; \(y = ct^2\) for any constant \(c\).

**Note** If \(\frac{\partial f_i}{\partial y_n}\) are not continuous at \((t_0, y_i(0))\) the Existence Uniqueness Theorem is not satisfied.

For example if \(\frac{dy}{dt} = 2\sqrt{y} \implies y = (t + c)^2\)

The IVP with \(y(0) = 0\) is not unique. Both \(y = t^2\) and \(y = 0\) are solutions.
3 Homogeneous Constant Coefficient Linear 2-dimensional Systems and The Phase Plane

We have a vague idea of the types of behaviour we can expect from linear constant coefficient ODE’s because they have exponential solutions. So we expect exponential decay (from terms like $e^{-3t}$) or exponential growth (from terms like $e^{2t}$),

oscillatory behaviour (from terms like $\sin(5t)$) and decaying or growing oscillatory behaviour (from terms like $e^{-3t}\cos(t)$).

But in a 2-dimensional system there are always two fundamental solutions and one may grow while the other decays meaning that different initial conditions may give different behaviour. We really need to consider the 3-dimensional space $(t, y_1, y_2)$. But that is too complicated so we consider $(y_1(t), y_2(t))$ as coordinates in $(y_1, y_2)$ space, which is called Phase Space.

The Linear Pendulum is a good visual model.

\[ \ddot{\theta} = -\frac{g}{l}\theta \]

where $g$ is gravity, $l$ is length and $\theta$ is the angle the pendulum makes with the vertical. Take $\frac{g}{l} = 9$ say then letting $y_1 = \theta$ and $y_2 = \dot{\theta}$ in matrix notation we have

\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-9 & 0
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
\]

So that $\det(A - \lambda I) = 0 \rightarrow \lambda^2 + 9 = 0$.

So $\lambda_{\pm} = \pm 3i$ and $x^\pm = \begin{pmatrix} 1 \\ \pm 3i \end{pmatrix}$ and the complex solutions are $\begin{pmatrix} 1 \\ \pm 3i \end{pmatrix} e^{\pm 3it}$.

Finally the real solutions are

\[
\begin{pmatrix}
\cos(3t) \\
-3\sin(3t)
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
\sin(3t) \\
3\cos(3t)
\end{pmatrix}
\]

In both cases $y_1^2 + y_2^2/9 = 1$. Now the General solution is a linear combination of these and in fact you can show that in general the solutions satisfy $y_1^2 + y_2^2/9 = c^2$, for some constant $c$. In the $(y_1, y_2)$ plane these are ellipses.

As time progresses the coordinate $(y_1, y_2)$ move around the ellipse. When it is positioned on the $y_1$ axis its position ($\theta$) is a maximum, but its speed is zero. When it is positioned on the $y_2$ axis it’s position is zero and its speed is a maximum.
The Phase Portrait
Each initial condition gives a curve in phase space, which is called a trajectory. These 
trajectories, ellipses here, represent solutions to the ODE in phase space. You can build 
up a complete picture by taking lots of different initial conditions, each of which will give 
you a trajectory in phase space. This is called the Phase Portrait of the system. Here 
the phase portrait is simply lots of ellipses of the form \( y_1^2 + \frac{y_2^2}{9} = c^2 \), plus the origin.

The Phase Plane representation of the solutions doesn’t tell you everything. It cannot say how fast you move along a phase curve. However we do usually indicate the direction of increasing time by an arrow.

Existence Uniqueness For a constant coefficient system that is smooth Two Trajectories cannot cross otherwise they would violate existence uniqueness because at the point where they cross there are two different solutions coming out of one point.

The Trivial Solution i.e. \( y=0 \)
A system of the form

\[
\dot{y} = Ay, \quad \text{where} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\]

and \( A \) is a \( 2 \times 2 \) constant matrix has the trivial solution.
That is if \( y_1(t_0) = 0 \) and \( y_2(t_0) = 0 \) then \( y_1(t) = 0 \) and \( y_2(t) = 0 \) for all time.

This is one trajectory that is easy to plot!

Other that this there are 6 qualitatively different phase portraits, apart from special cases.

Four are concerned with real eigenvalues and two with complex eigenvalues.