LINEAR
An ODE is said to be Linear if it is linear in the unknown and it’s derivatives.
But it need not be linear in the independent variable.
For example
\[ \frac{d^2y(t)}{dt^2} = e^t \frac{dy(t)}{dt} - \cos(5t)y(t) + t^5, \]

is linear.

But
\[ \frac{d^4y(t)}{dt^4} = 2y(t) \frac{dy(t)}{dt} - y(t) + 5 \]

is nonlinear, because of the \(2y(t)\frac{dy(t)}{dt}\) term.

And
\[ \frac{d^3u(x)}{dx^3} + \left( \frac{d^5u(x)}{dx^5} \right)^2 - xu(x) + x^2 = 0 \]

is nonlinear, because of the \( \left( \frac{d^5u(x)}{dx^5} \right)^2 \) term.

ORDER
The order of an ODE is the order of the highest derivative.
In the examples above the first is second order, the second fourth order and the third 5th order.

HOMOGENEOUS
A linear ODE is either homogeneous or inhomogeneous.
An ODE is homogeneous if when \( y(t) \) is a solution then so is \( cy(t) \) for any constant \( c \).
This is great when it happens of course because if you can find one solution you immediately have a whole family of others.
Take some examples
\[ \frac{d^2y(t)}{dt^2} + 2 \frac{dy(t)}{dt} - \cos(5t)y(t) = 0 \]

is homogeneous.

Try it. But
\[ \frac{d^2y(t)}{dt^2} = e^t \frac{dy(t)}{dt} - \cos(5t)y(t) + t^5 \]

is inhomogeneous.
And
\[ \frac{d^3u(x)}{dx^3} = \sin(x)u(x) + 5 \]

is inhomogeneous.

Linear, Homogeneous ODE’s obey

THE SUPERPOSITION PRINCIPLE
This means that you can take linear combinations of known solutions to form new solutions. So if \( y_1(t) \) and \( y_2(t) \) are two solutions then \( c_1y_1(t) + c_2y_2(t) \) is also a solution for any constants \( c_1 \) and \( c_2 \).
This also works for systems. If \( \begin{pmatrix} y_{1a} \\ y_{2a} \end{pmatrix} \) and \( \begin{pmatrix} y_{1b} \\ y_{2b} \end{pmatrix} \) are solutions to a 2D linear homogeneous system then for any constants \( c_1 \) and \( c_2 \)
\[ c_1 \begin{pmatrix} y_{1a} \\ y_{2a} \end{pmatrix} + c_2 \begin{pmatrix} y_{1b} \\ y_{2b} \end{pmatrix} \]

is also a solution.

But be careful the superposition principle only applies to Linear Homogeneous systems.
1. Systems of two coupled 1st order ODE’s.

Any second order Linear ODE can be written as a system of two coupled 1st order ODE’s:

\[
\frac{d^2y(t)}{dt^2} + p(t)\frac{dy(t)}{dt} + q(t)y(t) = r(t)
\]

Let \( y_1(t) = y(t), \quad y_2(t) = \frac{dy(t)}{dt} \), then

\[
\frac{dy_1}{dt} = y_2
\]

\[
\frac{dy_2}{dt} = \frac{d^2y(t)}{dt^2} = -p(t)\frac{dy(t)}{dt} - q(t)y(t) + r(t)
\]

Now this system of two first order ODE’s can be written in matrix form:

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-q(t) & -p(t)
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} + 
\begin{bmatrix}
0 \\
r(t)
\end{bmatrix}
\]

If we let

\[
y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\]

and

\[
A(t) = \begin{bmatrix}
0 & 1 \\
-q(t) & -p(t)
\end{bmatrix}
\]

and

\[
r(t) = \begin{bmatrix}
0 \\
r(t)
\end{bmatrix}
\]

Then the system in matrix form is

\[
\dot{y} = A(t)y + r(t)
\]

If \( r(t) = 0 \) the system is homogeneous.
If \( r(t) \neq 0 \) the system is inhomogeneous.
The Homogeneous case with constant coefficients.

Suppose \( \dot{y} = Ay \) with \( A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \) a constant matrix.

We know that the solutions to 2nd order linear ODE’s with constant coefficients are linear combinations of exponentials:

If \( \ddot{y} + p\dot{y} + qy = 0 \) let \( y = e^{\lambda t} \),

then differentiation and substitution gives \( e^{\lambda t}(\lambda^2 + p\lambda + q) = 0 \).

Dividing by \( e^{\lambda t} \) gives the Auxillary equation

\[(\lambda^2 + p\lambda + q) = 0 \quad \text{for} \quad \lambda.\]

Say there are two roots \( \lambda_1 \) and \( \lambda_2 \) then the \textbf{General Solution to the ODE} is

\[y = c_1e^{\lambda_1t} + c_2e^{\lambda_2t}, \quad \text{where} \ c_1 \text{ and } c_2 \text{ are constants.}

(Actually these constants are fixed by the initial conditions \( y(0) \) and \( \dot{y}(0) \) in an initial value problem (IVP).)

\textit{Back to our system}

\[\dot{y} = Ay\]

Try

\[y = xe^{\lambda t} \quad \text{where} \quad x = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{is a constant vector}\]

and \( \lambda \) is a constant scalar.

Sub into the equation

\[LHS = \dot{y} = \lambda xe^{\lambda t} \quad \text{and} \quad RHS = Axe^{\lambda t} \implies \lambda xe^{\lambda t} = Axe^{\lambda t}\]

So dividing by \( e^{\lambda t} \) gives

\[Ax = \lambda x\]

This is called the \textbf{EIGENVALUE EQUATION} for \( A \) and it implies the auxillary equation for \( \lambda!\).

If there are two eigenvalues \( \lambda_1 \) and \( \lambda_2 \) with eigen vectors \( x^{(1)} \) and \( x^{(2)} \) then

\[x^{(1)}e^{\lambda_1t} \quad \text{and} \quad x^{(2)}e^{\lambda_2t}\]

are solutions to

\[\dot{y} = Ay.\]

The \textbf{General Solution to the matrix equation} is a linear combination of these:

\[y = c_1x^{(1)}e^{\lambda_1t} + c_2x^{(2)}e^{\lambda_2t}\]

( from the superposition principle), for constants \( c_1 \) and \( c_2 \).
We now have **TWO** ways to solve **Initial value Problems (IVP’s)**

Take

\[ \dot{y} + 2\dot{y} - 15y = 0, \]

with

\[ y(0) = -1 \quad \text{and} \quad \dot{y}(0) = 13. \]

**METHOD I (OLD Method)**

Let \( y = e^{\lambda t} \implies \dot{y} = \lambda e^{\lambda t} \) and \( \ddot{y} = \lambda^2 e^{\lambda t} \). Substitute into the equation.

\[ \lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} - 15e^{\lambda t} = 0 \]

Divide by \( e^{\lambda t} \)

\[ \lambda^2 + 2\lambda - 15 = 0 \quad \text{Auxiliary Equation.} \]

Solve for \( \lambda; \quad \lambda_1 = 3, \quad \lambda_2 = -5. \)

So \( e^{3t} \) and \( e^{-5t} \) are solutions to the ODE.

**General Solution to the ODE is**

\[ y = c_1 e^{3t} + c_2 e^{-5t}. \]

Then \( \dot{y} \) is found by differentiating \( y \). Here this implies that \( \dot{y} = 3c_1 e^{3t} - 5c_2 e^{-5t}. \)

Now use the initial conditions to find \( c_1 \) and \( c_2. \)

\[ y(0) = -1 \implies c_1 + c_2 = -1 \]
\[ \dot{y}(0) = -1 \implies 3c_1 - 5c_2 = 13. \]

Solve simultaneously to give \( c_1 = 1 \) and \( c_2 = -2. \)

**Solution to IVP is**

\[ y(t) = e^{3t} - 2e^{-5t}. \]