3) Where Irreducible quadratic factors \( \frac{N(s)}{(s-a)^2 + \alpha^2} \) are involved there are two possible methods of taking the inverse Laplace Transform. Either use the fact that

\[
L^{-1}\left(\frac{s-a}{(s-a)^2 + \alpha^2}\right) = e^{at}\cos(at) \quad \text{and} \quad L^{-1}\left(\frac{\alpha}{(s-a)^2 + \alpha^2}\right) = e^{at}\sin(at)
\]

Or use complex numbers and factorise the denominator.

\[
\frac{N(s)}{(s-a)^2 + \alpha^2} = \frac{N(s)}{(s-a+i\alpha)(s-a-i\alpha)} = \frac{A}{(s-a+i\alpha)} + \frac{B}{(s-a-i\alpha)}
\]

Take the example

\[
F(s) = \frac{3(s-10)}{s^2 - 4s + 20}
\]

A quadratic factor is irreducible if its roots are complex.
Here the roots of \( s^2 - 4s + 20 \) are \( s = 2 \pm 4i \).
If the quadratic factor is irreducible it can always be written in the form \( ((s-a)^2 + \alpha^2) \) by completing the square.

\[
s^2 - 4s + 20 = (s-2)^2 - 4 + 20 = (s-2)^2 + 4^2
\]

So here we can write \( F(s) = \frac{3(s-10)}{s^2 - 4s + 20} = \frac{3(s-2) - 24}{(s-2)^2 + 4^2} \)

Or \( F(s) = \frac{3(s-2)}{(s-2)^2 + 4^2} - \frac{6 \times 4}{(s-2)^2 + 4^2} \)

Now use the fact that \( L(\cos(at)) = \frac{s}{s^2 + \alpha^2} \) and \( L(\sin(at)) = \frac{\alpha}{s^2 + \alpha^2} \)

AND \( L(e^{at}f(t)) = F(s-a) \) to deduce that

\[
L^{-1}\left(\frac{s-2}{(s-2)^2 + 4^2}\right) = e^{2t}\cos(4t) \quad \text{and} \quad L^{-1}\left(\frac{4}{(s-2)^2 + 4^2}\right) = e^{2t}\sin(4t)
\]

So that \( f(t) = 3e^{2t}\cos(4t) - 6e^{2t}\sin(4t) \).

4) Repeated irreducible quadratic factor \( \frac{N(s)}{((s-a)^2 + \alpha^2)^m} \)

Trouble. We don’t even know the inverse transform of \( \frac{1}{(s^2 + \alpha^2)^2} \)!

Once again you can always use complex numbers and that method works.
But a more interesting method involves the differential of the transformed function.

Suppose you know the transform of \( f(t) \), that is \( F(s) = L(f(t)) = \int_0^\infty e^{-st}f(t)dt \).
Then the differential of \( F(s) \) with respect to \( s \) is

\[
\frac{dF(s)}{ds} = \int_0^\infty (-te^{-st})f(t)dt = -\int_0^\infty tf(t)dt
\]

the transform of \(-tf(t)\).

So for instance since \( L(t^3) = \frac{6}{s^4} \) implies that \( -\frac{d}{ds}\left(\frac{6}{s^4}\right) = \frac{4 \times 6}{s^5} = L(t^4) \) which is true!
We can use this to find the inverse transform of repeated irreducible quadratic factors.

\[ L(t \sin(\alpha t)) = -\frac{d}{ds} \left( \frac{\alpha}{(s^2 + \alpha^2)} \right) = \frac{2s\alpha}{(s^2 + \alpha^2)^2} \]

\[ L(t \cos(\alpha t)) = -\frac{d}{ds} \left( \frac{s}{(s^2 + \alpha^2)} \right) = \frac{2s^2}{(s^2 + \alpha^2)^2} - \frac{1}{(s^2 + \alpha^2)} \]

Or

\[ \frac{s^2}{(s^2 + \alpha^2)^2} = \frac{1}{2} \left( \frac{1}{\alpha} L(\sin(\alpha t)) + L(\cos(\alpha t)) \right) = L \left( \frac{1}{2\alpha} \sin(\alpha t) + \frac{1}{2} t \cos(\alpha t) \right) \]

Lastly we can use a bit of algebra

\[ \frac{\alpha^2}{(s^2 + \alpha^2)^2} = \frac{1}{\alpha} L(\sin(\alpha t)) - \frac{1}{2} L(\cos(\alpha t)) - \frac{1}{2\alpha} L(\sin(\alpha t)) = L \left( \frac{1}{2\alpha} \sin(\alpha t) - \frac{1}{2} t \cos(\alpha t) \right) \]

Here are the results

\[ L^{-1} \left( \frac{s}{(s^2 + \alpha^2)^2} \right) = \frac{1}{2\alpha} t \sin(\alpha t) \]

\[ L^{-1} \left( \frac{s^2}{(s^2 + \alpha^2)^2} \right) = \frac{1}{2\alpha} \sin(\alpha t) + \frac{1}{2} t \cos(\alpha t) \]

\[ L^{-1} \left( \frac{\alpha^2}{(s^2 + \alpha^2)^2} \right) = \frac{1}{2\alpha} \sin(\alpha t) - \frac{1}{2} t \cos(\alpha t) \]

I won’t go into the details, but you can continue using this idea to find the inverse transform of \( \frac{N(s)}{((s - a)^2 + \alpha^2)^m} \) for \( m = 3, 4 \).

Lets take an example

\[ F(s) = \frac{s^2 - 7}{(s^2 - 2s + 5)^2} = \frac{s^2 - 7}{((s - 1)^2 + 4)^2} \]

\[ F(s) = \frac{(s - 1)^2 + 2(s - 1) - 6}{((s - 1)^2 + 4)^2} = \frac{(s - 1)^2}{((s - 1)^2 + 4)^2} + \frac{2(s - 1)}{((s - 1)^2 + 4)^2} - \frac{6}{((s - 1)^2 + 4)^2} \]

Now

\[ L^{-1} \left( \frac{(s - 1)^2}{((s - 1)^2 + 4)^2} \right) = e^t \left( \frac{1}{4} \sin(2t) + \frac{1}{2} t \cos(2t) \right) \]

\[ L^{-1} \left( \frac{2(s - 1)}{((s - 1)^2 + 4)^2} \right) = \frac{1}{2} e^t \sin(2t) \]

\[ L^{-1} \left( \frac{1}{((s - 1)^2 + 4)^2} \right) = e^t \left( \frac{1}{16} \sin(2t) - \frac{1}{8} t \cos(2t) \right) \]

Finally

\[ f(t) = e^t \left( \frac{1}{4} \sin(2t) + \frac{1}{2} t \cos(2t) \right) + e^t \sin(2t) - 6e^t \left( \frac{1}{16} \sin(2t) - \frac{1}{8} t \cos(2t) \right) \]

\[ = e^t \left( -\frac{1}{8} \sin(2t) + \frac{5}{4} t \cos(2t) + \frac{1}{2} t \sin(2t) \right) \]
4. The SECOND SHIFTING THEOREM and the Dirac Delta function

The SECOND SHIFTING THEOREM

Consider the function \( f(t) \), taken zero for \( t \) negative and then shifted over to \( k \):

\[
f(t-k)u(t-k) = \begin{cases} 
0 & 0 \leq t < k \\
f(t-k) & k \leq t < \infty
\end{cases}
\]

The Laplace Transform of this function is actually rather simple:

\[
L(f(t-k)u(t-k)) = \int_{0}^{\infty} e^{-st} f(t-k)u(t-k) \, dt = \int_{k}^{\infty} e^{-st} f(t-k) \, dt \quad \text{now change variables } x = t-k
\]

\[
= \int_{0}^{\infty} e^{-s(x+k)} f(x) \, dx = e^{-sk} \int_{0}^{\infty} e^{-sx} f(x) \, dx = e^{-sk} F(s)
\]

So that

\[
L(f(t-k)u(t-k)) = e^{-sk} F(s) \quad \text{SECOND SHIFTING THM.}
\]

So for instance

\[
L^{-1}\left(\frac{e^{-ks}}{s^2}\right) = (t-2)u(t-2) = \begin{cases} 
0 & 0 \leq t < 2 \\
t-2 & 2 \leq t < \infty
\end{cases}
\]

Or consider \( L^{-1}\left(\frac{(1-e^{-s})^2}{s^2}\right) \)

Since

\[
\frac{(1-e^{-s})^2}{s^2} = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}
\]

\[
L^{-1}\left(\frac{(1-e^{-s})^2}{s^2}\right) = t - 2(t-1)u(t-1) + (t-2)u(t-2)
\]

\[
= \begin{cases} 
t + 0 + 0 & 0 \leq t < 1 \\
t - 2(t-1) + 0 & 1 \leq t < 2 \\
t - 2(t-1) + (t-2) & 2 \leq t
\end{cases}
\]

\[
= \begin{cases} 
t & 0 \leq t < 1 \\
2 - t & 1 \leq t < 2 \\
0 & 2 \leq t
\end{cases}
\]
Going the other way.

\[
f(t) = \begin{cases} 
2 & 0 \leq t < \pi \\
0 & \pi \leq t < 3\pi \\
\sin t & 3\pi \leq t 
\end{cases}
\]

Then

\[
f(t) = 2 - 2u(t - \pi) + \sin(t)u(t - 3\pi)
\]

\[
= 2 - 2u(t - \pi) - \sin(t - 3\pi)u(t - 3\pi)
\]

So that

\[
F(s) = \frac{2}{s} - \frac{2e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s^2 + 1}
\]

Consider the LC circuit, where the applied EMF, \( E(t) \), is a function of time.

As before for an RLC circuit

\[
L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} = E(t)
\]

where \( Q(t) \) is the charge at time \( t \).

If \( R = 0 \) the unforced system \( (E(t) = 0) \) has purely oscillatory solutions (centers) and then it is usual to set \( \frac{1}{LC} \) equal to \( \omega^2 \). In which case the unforced system undergoes oscillations with period \( \frac{2\pi}{\omega} \).

So dividing by \( L \) and setting \( \frac{1}{LC} = \omega^2 \) gives

\[
\frac{d^2Q(t)}{dt^2} + \omega^2Q(t) = \frac{E(t)}{L}
\]

Suppose the voltage is switched on for a short time and then switched off. We could assume that it is switched on at \( t = 0 \) and off at say \( t = k \)

\[
\frac{E(t)}{L} = \begin{cases} 
\bar{E}_0 & 0 \leq t < k \\
0 & k \leq t 
\end{cases}
\]

To use Laplace transforms write \( \frac{E(t)}{L} \) in terms of the unit step function. So here that is

\[
\frac{E(t)}{L} = \bar{E}_0(u(t) - u(t - k))
\]

Now let \( \bar{Q}(s) = L(Q(t)) \), the bar is just to distinguish this from \( Q(t) \) and from the the transfer function!