1. (a) \[ \sum_{i=1}^{4} \frac{1}{2^i} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{8}{16} + \frac{4}{16} + \frac{2}{16} + \frac{1}{16} = \frac{15}{16} \]

As \( n \) gets very large, this sum will become very close to 1. (Each extra term which is added takes the sum half-way closer to 1.)

(b) \( (1^2 - 1) + (2^3 - 1) + (3^4 - 1) + (4^5 - 1) = \sum_{i=1}^{4} (i^{i+1} - 1) \).

(Note that there are many different correct answers possible here.)

(c) \( \prod_{j=0}^{5} (-1)^j = (-1)^0 \times (-1)^1 \times (-1)^2 \times (-1)^3 \times (-1)^4 \times (-1)^5 = 1 \times -1 \times 1 \times -1 \times 1 \times -1 = -1 \).

(d) \( (1 - t^2) \cdot (2 - t^2) \cdot (3 - t^2) \cdot (4 - t^2) = \prod_{i=1}^{4} (i - t^2) \).

(e) \[ \frac{7!}{5!2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = \frac{7 \cdot 6}{2 \cdot 1} = 21 \]

2. (a) Assume there are \( n \) people on the field. The first player shakes hands with everyone else, which means that person shakes hands with \( (n - 1) \) other people. Now, the second player has already shaken hands with the first player, so these two players do not shake again. But the second player must shake hands with everyone else, which means \( (n - 2) \) handshakes. Now, the third player has already shaken hands with two people, so must shake hands with \( (n - 3) \) other people. This continues, until the third-last player shakes hands with 2 people, the second-last player shakes with 1 person, and the last person shakes hands with no-one, as this person has already shaken hands with everyone.

Thus the total number of handshakes is \( (n - 1) + (n - 2) + (n - 3) + \ldots + 3 + 2 + 1 \), which equals the given expression.

(b) In summation notation

\[ 1 + 2 + 3 + \ldots + (n - 3) + (n - 2) + (n - 1) = \sum_{i=1}^{n-1} i. \]

(c) **Proof** Let \( P(n) \) be the statement \( \sum_{i=1}^{n-1} i = \frac{n(n - 1)}{2} \).

Then \( P(2) \) is the statement \( \sum_{i=1}^{1} i = \frac{2 \cdot 1}{2} \).

\( P(k) \) is the statement \( \sum_{i=1}^{k-1} i = \frac{k(k - 1)}{2} \).

\( P(k + 1) \) is the statement \( \sum_{i=1}^{k} i = \frac{(k + 1)k}{2} \).

\( P(2) \) is true since \( \sum_{i=1}^{1} i = 1 \) and \( \frac{2 \cdot 1}{2} = 1 \).
\[ \text{L.H.S. of } P(k+1) = \sum_{i=1}^{k} i = \sum_{i=1}^{k} i + k = \frac{k(k-1)}{2} + k \quad \text{(since we assumed } P(k) \text{ is true)} \\
= \frac{k^2 - k}{2} + \frac{2k}{2} \\
= \frac{k^2 + k}{2} \\
= \frac{(k+1)k}{2} \\
= \text{R.H.S. of } P(k+1) \]

Thus, by mathematical induction, for all integers \( n \geq 2 \),
\[ \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}. \]

\[ \text{(d)(i) If each pair of teams played exactly once, then there would be} \]
\[ 1 + 2 + 3 + \ldots + (n-2) + (n-1) = \frac{n(n-1)}{2} \text{ games,} \]

but each pair of teams plays exactly twice so there are \( 2 \cdot \frac{n(n-1)}{2} = n(n-1) \) games.

\[ \text{(d)(ii) If } n \text{ is even and each team plays once per week, then there will be } \frac{n}{2} \text{ games per week (and this is an integer, as } n \text{ is even). Thus the total number of weeks in the whole season is} \]
\[ \frac{n(n-1)}{2} \div \frac{n}{2} = 2(n-1) \text{ weeks all up.} \]

\[ \text{(d)(iii) If } n \text{ is odd and each team plays once per week except one team which has a break, then there will be } \frac{n-1}{2} \text{ games per week (and this is an integer, as } n \text{ is odd). Thus the total number of weeks in the whole season is} \]
\[ \frac{n(n-1)}{2} \div \frac{(n-1)}{2} = 2n \text{ weeks all up.} \]

3. Here are answers to Questions from Section 4.2 and 4.3.

15. Proof Let \( P(n) \) be the statement
\[ \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}, \]
or equivalently
\[ \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}. \]
As we have seen, there are many mathematical statements that can be proved true using the principle of mathematical induction. Let's consider the following example.

Let's prove the statement for all integers \( n \geq 2 \):

\[
1 - \frac{1}{2^2} \cdot \left( 1 - \frac{1}{3^2} \right) \cdot \ldots \cdot \left( 1 - \frac{1}{n^2} \right) = \frac{n+1}{2n}.
\]

**20b. Proof** Let \( P(n) \) be the statement \( n! > n^2 \).

Then \( P(4) \) is the statement \( 4! > 4^2 \).

\( P(k) \) is the statement \( k! > k^2 \).

\( P(k + 1) \) is the statement \( (k + 1)! > (k + 1)^2 \).

\( P(4) \) is true since \( 4! = 24 \) and \( 4^2 = 16 \) and \( 24 > 16 \).

Assume that \( P(k) \) is true and use that to show that \( P(k + 1) \) is true.

\[
\begin{align*}
\text{L.H.S. of } P(k + 1) & = (k + 1)! \\
& = (k + 1)k! \\
& > (k + 1)k^2 \quad \text{(since we assumed } P(k) \text{ is true)} \\
& > (k + 1)(k + 1) \quad \text{(since } k^2 > k + 1 \text{ for } k \geq 4) \\
& = (k + 1)^2 \\
& = \text{R.H.S. of } P(k + 1).
\end{align*}
\]

Thus, by mathematical induction, for all integers \( n \geq 4 \), \( n! > n^2 \).
6. (c) No  (d) Yes  (e) Yes  (g) Yes  (h) No  (i) Yes.
13. (c) Yes, $T \subseteq S$. Every integer that is divisible by 6 is also divisible by 3.
    (d) $R \cap S$ is the set of integers which are divisible by 2 and also divisible by 3; thus
    they are divisible by 6, so $R \cap S = T$.
15. This is too hard to typeset; ask a tutor if you have any trouble with the answers.

5. Let $A(x)$ represent the statement $x \in A$ and $B(x)$ represent the statement $x \in B$. Then
   the statement we are asked to prove is
   \[ \forall x \in \text{a universal set}, (A(x) \rightarrow B(x)) \rightarrow ((A(x) \land B(x)) \rightarrow B(x)). \]
   Thus we use a truth table to investigate the statement form
   \[ (a \rightarrow b) \rightarrow ((a \land b) \rightarrow b). \]

<table>
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<th>$a \land b$</th>
<th>$(a \land b) \rightarrow b$</th>
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Since this statement form is a tautology, the property
   \[ \text{if } A \subseteq B, \text{ then } (A \cap B) \subseteq B \text{ is true.} \]

6. The following three Venn diagrams show $(A - B)$, $(C - B)$ and $(A - B) \cap (C - B)$.

![Venn diagrams](image1)

The following two Venn diagrams show $(A \cap C)$ and $(A \cap C) - B$.

![Venn diagrams](image2)

The Venn diagrams illustrate the fact that
   \[ (A - B) \cap (C - B) = (A \cap C) - B. \]

7.

(a) Here $A = \emptyset$, $B = \{A\} = \{\emptyset\}$ and $C = \{B\} = \{\{\emptyset\}\}$.
   (i) is False  (ii) is True  (iii) is False
   (iv) is True  (v) is True  (vi) is True
   (vii) is True  (viii) is True  (ix) is False
   (x) is True  (xi) is False  (xii) is True

(b) $D = \{\emptyset\}$ and $E = \{\emptyset, \{\emptyset\}\}$.
   (i) is True  (ii) is True  (iii) is True
   (iv) is True  (v) is True

(d) $D = \{\emptyset\}$ and $E = \{\emptyset, \{\emptyset\}\}$. 
   (i) is True  (ii) is True  (iii) is True
   (iv) is True  (v) is True