AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

QUEENSLAND PROGRAMME

MATHEMATICAL NOTES

1. Mathematical Induction

Suppose you were asked to find a general formula for the sum of the first n odd positive integers. You would start by looking at a few special cases:

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \end{aligned}$$

from which you would probably guess the formula

$$1+3+5+\ldots+(2n-1)=n^2.$$

Of course, these four special cases don't *prove* that the formula is correct, they just make it plausible. The easiest way to give a convincing proof that the formula is correct for all n is to use Mathematical Induction.

This is a method for proving that something holds for all positive integers. You have some statement, say $S(n)$, about the integer n which you would like to prove is true for all positive integers n. The Principle of Mathematical Induction says:
Suppose you can show

(i) $S(1)$ is true, and
(ii) for general positive integer k, if $S(k)$ happens to be true, then $S(k+1)$ must also be true.

Then you may conclude that for all n, $S(n)$ is true.
[For by (i) you have shown that $S(1)$ is true. By (ii) with $k=1$, since $S(1)$ is true, you know $S(1+1)$, i.e. $S(2)$, is also true. By (ii) again, with $k=2$, you conclude that $S(2+1)$, i.e. $S(3)$, is true. And so on.]

Let us use induction to prove the formula we guessed above. So here $S(n)$ is the statement $1+3+5+\ldots+(2n+1)=n^2$. Now $S(1)$ is $1=1^2$, which is certainly true. So suppose for some k that $S(k)$ happens to be true, and try to show that then $S(k+1)$ would be true. So we have that $1+3+5+\ldots+(2k-1)=k^2$ (called the *inductive hypothesis*). But then

$$\begin{aligned} 1+3+5+\ldots+(2k+1) &= [1+3+5+\ldots+(2k-1)]+(2k+1) \\ &= k^2+(2k+1) \qquad \text{by the inductive hypothesis} \\ &= (k+1)^2, \end{aligned}$$

just what is needed to make $S(k+1)$ true. Hence, by induction, $S(n)$ is true for all integers $n \geq 1$.

You might like to try the same method on

$$1+2+3+\ldots+n=\frac{1}{2}n(n+1)$$

$$1^2+2^2+3^2+\ldots+n^2=\frac{1}{6}n(n+1)(2n+1).$$

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2. Pigeon-hole Principle

Suppose N objects are to be placed in k pigeon-holes. If $N>k$ then one pigeon-hole contains at least 2 objects. More generally, if $N>km$ then one pigeon-hole contains at least $m+1$ objects (because if every hole had at most m objects, there would be at most km objects in total).

`Example.` In any party of six people, there are either three people all of whom know each other, or three none of whom know each other. (We assume that if X knows Y, then automatically Y knows X.) For let A be one of the people, and divide the other five people into two classes: known by A, and not known by A. By the pigeon-hole principle, since $5>2\times 2$, one of these classes has at least three people in it. Suppose for example at least three people are not known by A, and consider any three people not known by A. If some pair of them do not know each other, together with A we have a set of three people none of whom know each other; whereas if there is no such pair then we have a set of three people all of whom know each other. A similar argument holds for the case that there are at least three people known by A.

3. Divisibility and Congruence

1 *The Fundamental Theorem of Arithmetic:* Every positive integer can be factored into a product of primes, and this factorization is unique up to the order in which the factors are written.

2 *The greatest common divisor:* If a and b are any integers, with $a \neq 0$, one says that *a divides b* or *b is a multiple of a* if there is an integer c such that $ac = b$. Symbolically, "a divides b" may be written $a|b$. An integer d is a *common divisor* of integers m and n if $d|m$ and $d|n$. The *greatest common divisor* of m and n, denoted by (m, n), is the largest integer which divides both m and n. Any common divisor of m and n also divides (m, n). Two positive integers are said to be *coprime* (or *relatively prime*) if their greatest common divisor is 1.

3 *Congruences:* If a and b are two integers whose difference is divisible by a positive integer m then *a is congruent to b modulo m*. This statement is written symbolically as

$$a \equiv b \pmod{m}.$$

If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then

$$a + c \equiv b + d \pmod{m} \qquad \text{and} \qquad ac \equiv bd \pmod{m}.$$

`Example.` $7 \equiv 11 \pmod{4}$ and $9 \equiv 31 \pmod{11}$.

4 *Some modulo results for squares:*

$$\begin{aligned} n^2 &\equiv 0 \pmod{4} && \text{when } n \text{ is even,}\\ n^2 &\equiv 1 \pmod{4} && \text{when } n \text{ is odd.}\\ n^2 &\equiv 0 \pmod{8} && \text{when } n \equiv 0 \pmod{4},\\ n^2 &\equiv 4 \pmod{8} && \text{when } n \equiv 2 \pmod{4},\\ n^2 &\equiv 1 \pmod{8} && \text{when } n \text{ is odd.} \end{aligned}$$

5 *Fermat's Theorem* (Sometimes called Fermat's Little Theorem):
If p is prime and $(a, p) = 1$ then $a^{p-1} \equiv 1 \pmod{p}$.

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4. Polynomials

Let x be a variable which can take any number as value. An expression of the form

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

where $a_0, a_1, \ldots, a_n$ are fixed numbers is called a *polynomial* (in x). The numbers $a_0, a_1, \ldots, a_n$ are the *coefficients*. Those terms $a_i x^i$ where $a_i = 0$ are usually omitted. The term a_0 is the *constant term*. A *constant polynomial* consists only of a constant term a_0. Assuming $a_n \neq 0$, the term $a_n x^n$ is called the *leading term*. The integer n is the *degree* of the polynomial.

You *evaluate* the polynomial $p(x)$ at the given number a by substituting the number a wherever x occurs in $p(x)$ and calculating the value of the resulting numerical expression. This number is called the *value* of $p(x)$ at $x = a$, and is denoted by $p(a)$.
`Example.` For $p(x) = 2x^3 + 4x^2 - x - 7$, its value when $x = -2$ is $p(-2) = 2(-2)^3 + 4(-2)^2 - (-2) - 7 = -16 + 16 + 2 - 7 = -5$.

A *zero* or *root* of the polynomial $p(x)$ is a number r such that the value of $p(x)$ at $x = r$ is zero (thus $p(r) = 0$), and we say that r is a *solution* of the equation $p(x) = 0$.

Let $p(x) = ax^2 + bx + c$ be a quadratic polynomial with real coefficients (and $a \neq 0$). The roots of $p(x)$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The number $\Delta = b^2 - 4ac$ is called the *discriminant* of $p(x)$. There are
- (i) two distinct real roots if $\Delta > 0$;
- (ii) one real root if $\Delta = 0$;
- (iii) no real roots (but two complex roots) if $\Delta < 0$.

The Remainder Theorem:
When the polynomial $p(x)$ is divided by the polynomial $x - a$, the remainder is the number $p(a)$.

The Factor Theorem:
The number a is a root of the polynomial $p(x)$ if and only if $x - a$ is a factor of $p(x)$.

Some useful factorizations:

$$\begin{aligned} &a^2 - b^2 = (a + b)(a - b);\\ &a^2 + 2ab + b^2 = (a + b)^2;\\ &a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})\\ &\hspace{8cm}\text{for all positive integers } n;\\ &a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots - ab^{n-2} + b^{n-1})\\ &\hspace{8cm}\text{for all positive odd integers } n. \end{aligned}$$

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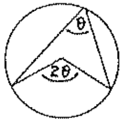
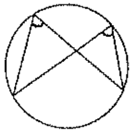
5. Geometry – Circles

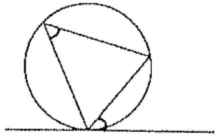
1 The angle at the centre of a circle is double the angle at the circumference subtended by the same arc.



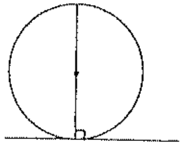
2 Angles subtended at the circumference of a circle by the same arc (or equal arcs) are equal.



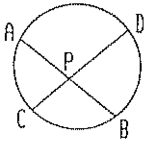
3 The angle between the tangent to a circle and a chord through the point of tangency is equal to the angle subtended by the chord at points on the circumference on the opposite side of the chord (i.e. in the alternate segment).



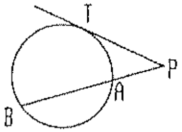
4 The diameter through a point on a circle is perpendicular to the tangent at that point.



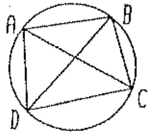
5 If AB, CD are two chords of a circle which cut in a point P (which maybe inside, on, or outside the circle), then $PA.PB = PC.PD$.



6 If P is a point outside a circle and T, A, B are points on the circle such that PT is a tangent and PAB a secant, then $PT^2 = PA.PB$.



7 A quadrilateral is cyclic (i.e. its vertices lie on a circle) if and only if the sum of each pair of opposite angles is 180°. *Ptolemy's Theorem:* If quadrilateral $ABCD$ is cyclic, then $AB.CD + BC.AD = AC.BD$. Conversely, if the sides and diagonals of quadrilateral $ABCD$ satisfy $AB.CD + BC.AD = AC.BD$, then the quadrilateral is cyclic.



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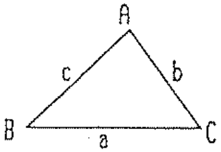
6. Geometry – Triangles

1 Area of triangle ABC

$$\begin{aligned} &= \tfrac{1}{2}\,\text{base} \times \text{height} \\ &= \tfrac{1}{2}ab\sin C = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ac\sin B \\ &= \sqrt{s(s-a)(s-b)(s-c)} \end{aligned}$$

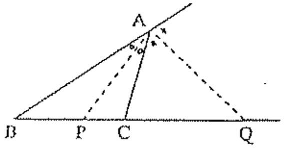
where $s = \frac{1}{2}(a+b+c)$ is the semi-perimeter. The last is *Heron's formula*.



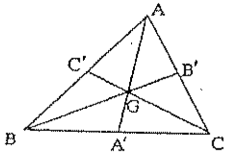
2 The internal (or external) bisector of an angle in a triangle divides the opposite side internally (or externally) in the ratio of the other two sides:

$$\frac{AB}{AC} = \frac{BP}{PC} = \frac{BQ}{QC}.$$

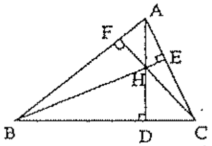
Note that angle PAQ is a right angle.



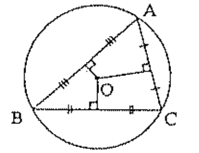
3 The line joining a vertex of a triangle to the midpoint of the opposite side is called a *median*. The three medians meet at the *centroid* of the triangle. The centroid trisects each of the medians; thus $AG = 2GA'$.



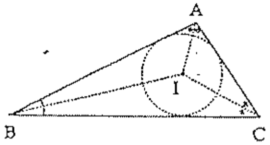
4 The line through a vertex of a triangle perpendicular to the opposite side is called an *altitude* of the triangle. The three altitudes meet at the *orthocentre* of the triangle.



5 The three perpendicular bisectors of the sides of a triangle concur at the *circumcentre* of the triangle. This point is the centre of the *circumcircle* of the triangle.

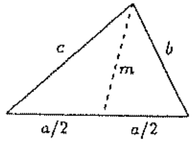


6 The three internal bisectors of the sides of a triangle concur at the *incentre* of the triangle. This point is the centre of the *incircle* of the triangle.



7 If m is the length of a median in the diagram, then

$$b^2 + c^2 = 2m^2 + 2\left(\frac{a}{2}\right)^2.$$



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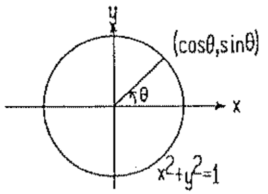
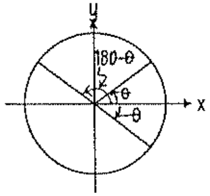
7. Trigonometric Functions and Triangles

1 *Definitions:* Define the sine and cosine functions by: the point on the unit circle $x^2+y^2=1$ making an angle of θ° with the positive x-axis is the point with co-ordinates $(\cos\theta, \sin\theta)$.



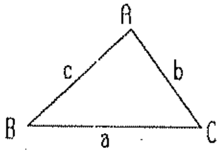


Then, by the definition, $\cos^2\theta+\sin^2\theta=1$. It is also clear that $\sin(\theta+360)=\sin\theta$, $\cos(\theta+360)=\cos\theta$, $\sin(-\theta)=-\sin\theta$, $\cos(-\theta)=\cos\theta$, $\sin(180-\theta)=\sin\theta$, and $\cos(180-\theta)=-\cos\theta$.

2 *Sine and cosine rules:* For triangle ABC, we have

(a) the cosine rule:

$$c^2=a^2+b^2-2ab\cos C,$$

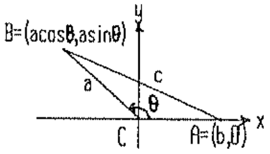


(b) the sine rule:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

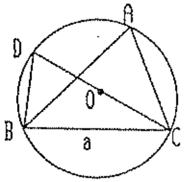
These can be established as follows.

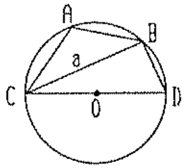
(a) Introduce a co-ordinate system as shown. By the formula for the distance between two points,



$$\begin{aligned} c^2=AB^2&=(a\cos C-b)^2+a^2\sin^2 C\\ &=a^2(\cos^2 C+\sin^2 C)-2ab\cos C+b^2\\ &=a^2-2ab\cos C+b^2. \end{aligned}$$

(b) Circumscribe the triangle ABC by a circle with centre O and radius R.





Draw the diameter OC and the chord BD. Then $\angle CBD$ is an angle in a semi-circle, so $\angle CBD=90^\circ$. Hence, in both figures

$$\sin D=\frac{BC}{CD}=\frac{a}{2R}. \qquad (1)$$

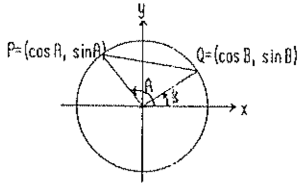
In the first figure, $\angle A=\angle D$ (angles on the same chord BC), and in the second figure, $\angle D=180-\angle A$ (opposite angles of the cyclic quadrilateral $ABCD$). So in both cases, $\sin A=\sin D$. Hence (1) gives that $a/\sin A=2R$. Similarly $b/\sin B=2R$ and $c/\sin C=2R$; thus

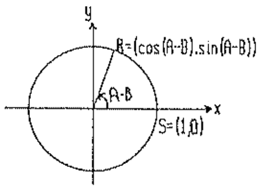
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R.$$

3 *Addition formulae:* There are the addition formulae for sine and cosine:

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B \qquad (2)$$
$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B. \qquad (3)$$

The formula for $\cos(A-B)$ can be obtained as follows:





Here $PQ^2=RS^2$, and by the distance formula

$$\begin{aligned} PQ^2&=(\cos A-\cos B)^2+(\sin A-\sin B)^2\\ &=\cos^2 A-2\cos A\cos B+\cos^2 B+\sin^2 A-2\sin A\sin B+\sin^2 B\\ &=2-2(\cos A\cos B+\sin A\sin B);\\ RS^2&=(\cos(A-B)-1)^2+\sin^2(A-B)\\ &=\cos^2(A-B)-2\cos(A-B)+1+\sin^2(A-B)\\ &=2-2\cos(A-B); \end{aligned}$$

and hence $$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

Putting $A=90^\circ$ in the formula for $\cos(A-B)$ gives

$$\cos(90-B)=\cos 90\cos B+\sin 90\sin B=\sin B,$$

and now replacing B by $90-B$ gives

$$\sin(90-B)=\cos(90-(90-B))=\cos B.$$

Replacing A by $90-A$ in the formula for $\cos(A-B)$ and using that $\cos(90-A-B)=\sin(A+B)$ etc gives the formula for $\sin(A+B)$, and replacing B by $-B$ gives $\cos(A+B)$ and $\sin(A-B)$.

Also, adding the formulae for $\sin(A+B)$ and $\sin(A-B)$ gives: $\sin(A+B)+\sin(A-B)=2\sin A\cos B$, and putting $\alpha=A+B$ and $\beta=A-B$ gives

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2};$$

similarly

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

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8. Inequalities

1 *Rules:* There are the following simple rules for manipulating inequalities: for real numbers a, b, c,

(1) if $a < b$ then $a \pm c < b \pm c$,
(2) if $a < b$ and $c > 0$ then $ac < bc$,
(3) if $a < b$ and $c < 0$ then $ac > bc$,
(4) always $a^2 \geq 0$.

2 *AM–GM inequality:* Since $a^2 + b^2 \geq 2ab$ (from $(a-b)^2 \geq 0$) then for positive x and y,

$$\frac{1}{2}(x+y) \geq \sqrt{xy}.$$

More generally, if $x_1, x_2, \ldots, x_n > 0$ then

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}\ ,$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.
The expression $\dfrac{x_1 + x_2 + \ldots + x_n}{n}$ is called the *arithmetic mean* (AM) of the n numbers $x_1, \ldots, x_n$, and $\sqrt[n]{x_1 x_2 \ldots x_n}$ the *geometric mean* (GM). So this inequality says that AM≥GM.

3 *Harmonic mean:* If $x_1, x_2, \ldots, x_n > 0$ then

$$\sqrt[n]{x_1 x_2 \ldots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}\ ,$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.
The expression

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

is called the *harmonic mean* (HM) of the positive numbers $x_1, x_2, \ldots, x_n$.
Thus AM≥GM≥HM.

5 *The Cauchy–Schwarz inequality:* For all real numbers $a_1, \ldots, a_n$, $b_1, \ldots, b_n$

$$(a_1 b_1 + \ldots + a_n b_n)^2 \leq (a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2),$$

with equality if and only if $a_1 = rb_1$, $a_2 = rb_2$, …, $a_n = rb_n$ for some constant r.

6 For the real quadratic function $f(x) = ax^2 + bx + c$, if the discriminant $b^2 - 4ac$ is negative, then

(i) when $a > 0$, $f(x) > 0$ for all x,
(ii) when $a < 0$, $f(x) < 0$ for all x.

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9. Complex Numbers

For the complex numbers, you have a new "number" i with the property that $i^2 = -1$. A complex number z is built from two real numbers x and y, and written $z = x + iy$. Here x is called the *real part* of z and y the *imaginary part*, written $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

Two complex numbers are *equal* if and only if they have the same real parts and the same imaginary parts.

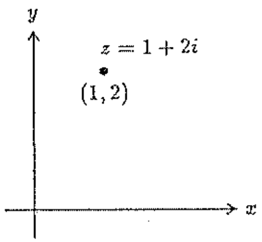
You *add* two complex numbers by adding their real parts and adding their imaginary parts, for example

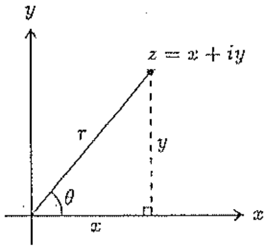
$$(3+2i) + (-2+\tfrac{1}{2}i) = (3-2) + (2+\tfrac{1}{2})i = 1 + \tfrac{5}{2}i.$$

You *multiply* two complex numbers by multiplying out as you would expect, and simplifying using $i^2 = -1$, for example

$$\begin{aligned}(3+2i)(1-4i) &= 3\times 1 + 3\times(-4)i + 2\times 1i + 2\times(-4)i^2\\ &= 3 - 12i + 2i - 8(-1)\\ &= 11 - 10i.\end{aligned}$$

You can picture $z = x + iy$ as the point (x, y) in the plane. This picture is called an *Argand diagram*. The vertical axis is called the Imaginary axis and the horizontal axis is called the Real axis.





The point $z = x + iy$ is also conveniently described by giving its distance r from the origin and the angle θ as shown (measured anticlockwise from the positive x–axis). Here r is called the *modulus* of z and θ an *argument* of z, written $r = |z|$ and $\theta = \arg z$. By simple trigonometry, $x = r\cos\theta$ and $y = r\sin\theta$. Hence

$$z = x + iy = r(\cos\theta + i\sin\theta).$$

This is often abbreviated to

$$z = r\operatorname{cis}\theta,$$

and is called the *modulus–argument form* or the *polar form* of the complex number z.

Multiplication is easy in the modulus–argument form: if $z_1 = r_1\operatorname{cis}\theta_1$ and $z_2 = r_2\operatorname{cis}\theta_2$, then $z_1z_2 = r_1r_2\operatorname{cis}(\theta_1+\theta_2)$. For we have

$$\begin{aligned}z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2)\\ &= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1))\\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))\\ &= r_1r_2\operatorname{cis}(\theta_1+\theta_2).\end{aligned}$$

Repeated use leads to *De Moivre's Theorem* for finding powers of complex numbers: for $z = r\operatorname{cis}\theta$ and positive integer n,

$$z^n = r^n\operatorname{cis}(n\theta).$$

Using de Moivre's Theorem gives an easy way of obtaining some of the identities involving the trig functions, for example that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. Consider $\operatorname{cis} 3\theta$:

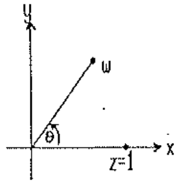
$$\begin{aligned}\cos 3\theta + i\sin 3\theta = \operatorname{cis} 3\theta &= (\operatorname{cis}\theta)^3 \qquad \text{(by de Moivre)}\\ &= (\cos\theta + i\sin\theta)^3\\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta\\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta).\end{aligned}$$

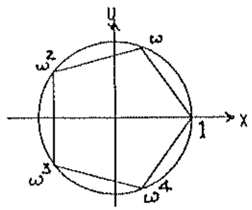
So equating the imaginary parts:

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3(1-\sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta,$$

as claimed.

Also from de Moivre's Theorem we can find the *roots of unity*: these are the complex numbers w for which $w^n = 1$. Let us consider just the 5th roots: numbers $w = r\operatorname{cis}\theta$ with $w^5 = 1$. Since $w^5 = r^5\operatorname{cis}5\theta$, we have $r^5\operatorname{cis}5\theta = 1$. So $r = 1$ and from the diagram, θ has to be such that 5θ is along the x-axis. So $\theta = 0$, $\frac{1}{5}\times 360°$, $\frac{2}{5}\times 360°$, $\frac{3}{5}\times 360°$, $\frac{4}{5}\times 360°$, $\frac{5}{5}\times 360°, \ldots$. Thus $w = 1$, $\operatorname{cis}72°$, $\operatorname{cis}144°$, $\operatorname{cis}216°$, $\operatorname{cis}288°$ are the five distinct 5th roots of unity. (Note $\operatorname{cis}(\frac{5}{5}\times 360° = \operatorname{cis}360° = 1$, etc.) If we put $\omega = \operatorname{cis}72°$, note $\operatorname{cis}144° = \omega^2$, $\operatorname{cis}216° = \omega^3$, $\operatorname{cis}288° = \omega^4$. So the solutions to $w^5 = 1$ are $w = 1, \omega, \omega^2, \omega^3, \omega^4$. Note that these 5th roots of unity are equally spaced around the unit circle in the Argand diagram. Observe also that the polynomial $z^5 - 1$ factors as $(z-1)(z^4+z^3+z^2+z+1)$. Since $\omega^5 - 1 = 0$, we have $(\omega-1)(\omega^4+\omega^3+\omega^2+\omega+1) = 0$. Now $\omega \neq 1$, so $\omega^4+\omega^3+\omega^2+\omega+1 = 0$.





AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

QUEENSLAND PROGRAMME: TEACHER SOLUTIONS April 2003

1. Let n be a positive integer. Prove that in any collection of $n+1$ distinct positive integers all less than or equal to $2n$ at least two of these are coprime.
[x and y are *coprime* if their greatest common divisor is 1.]

Solution. The easiest way is to observe that with $n+1$ numbers chosen from $\{1, 2, 3, \ldots, 2n\}$ there must be two consecutive numbers. (For otherwise, the numbers are at least two apart, so that the largest must must be at least $2n$ bigger than the smallest, and thus the largest chosen would have to be at least $2n+1$.) And two consecutive integers are coprime, for if y is a factor of both x and $x+1$ then $x = ky$ and $x+1 = \ell y$, thus $1 = y(\ell - k)$ so that $y = 1$.

2. Prove that the expressions $11x+8y$ and $7x+2y$ are divisible by 17 for the same set of integral values of x and y.

Solution. Suppose $11x+8y$ is divisible by 17, say $11x+8y = 17k$. Then $y = (17k - 11x)/8$, so $7x + 2y = 7x + (17k - 11x)/4 = (17k + 17x)/4$. Hence $4(7x + 2y) = 17(k + x)$. Thus 17 is a prime divisor of $4(7x + 2y)$, and hence $7x + 2y$ is divisible by 17 (since 17 is not a divisor of 4). Similarly, if we suppose $7x + 2y$ is divisible by 17, say $7x + 2y = 17\ell$, then $y = (17\ell - 7x)/2$, so $11x + 8y = 11x + 4(17\ell - 7x) = 17(4\ell - x)$. Hence 17 is a divisor of $11x + 8y$.

3. Let ABC be a triangle, P and Q points exterior to ABC with triangles BAP and ACQ not overlapping ABC, and R a point in the interior of $\triangle ABC$. Show that if BAP, ACQ and BCR are similar isosceles triangles with bases AB, AC and BC respectively, then the quadrilateral $AQRP$ is a parallelogram.

Solution. Since $\triangle ACQ$ and $\triangle BCR$ are similar

$$CQ/CR = CA/CB. \qquad (1)$$

Also, since $\triangle BAP$ and $\triangle BCR$ are similar

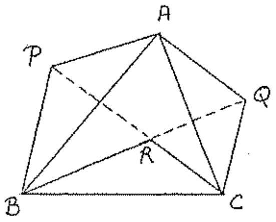
$$BP/BR = BA/BC. \qquad (2)$$

Since $\angle QCA = \angle RCB$,

$$\angle QCR = \angle ACB. \qquad (3)$$

Similarly,

$$\angle PBR = \angle ABC. \qquad (4)$$



By (1) and (3) triangles QRC and ABC have an equal angle with the corresponding sides in the same ratio. Hence they are similar. Likewise by (2) and (4) triangles ABC and PBR are similar. Hence triangles QRC and PBR are similar. But $\triangle BCR$ is isosceles. Therefore $BR = CR$ and triangles QRC and PBR in fact are congruent. So we have

$$PR = QC = AQ \qquad \text{and} \qquad QR = PB = PA.$$

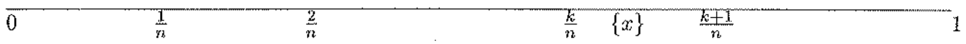
Therefore $AQRP$ is a parallelogram.

4. Let $[a]$ denote the integer nearest to a which is less than or equal to it. (Thus $[a] \le a < [a]+1$.) Prove the identity

$$[x] + [x + \frac{1}{n}] + [x + \frac{2}{n}] + \ldots + [x + \frac{n-1}{n}] = [nx],$$

for any positive integer n.

Solution. Write $\{x\} = x - [x]$, so $0 \le x < 1$. ($\{x\}$ is called the *fractional part* of x.) Divide the interval from 0 to 1 into n equal parts. Then $\{x\}$ will be in one of these parts, say $\frac{k}{n} \le \{x\} < \frac{k+1}{n}$, where k is an integer, $0 \le k \le n-1$.



For integers i with $0 \le i \le n-(k+1)$, (and there are $n-k$ such integers i),

$$0 \le \frac{k}{n} + \frac{i}{n} \le \{x\} + \frac{i}{n} < \frac{k+1}{n} + \frac{i}{n} = \frac{k+1+i}{n} \le \frac{n}{n} = 1,$$

so $[\{x\} + \frac{i}{n}] = 0$ and hence $[x + \frac{i}{n}] = [[x] + \{x\} + \frac{i}{n}] = [x] + [\{x\} + \frac{i}{n}] = [x]$.
Also, for integers i with $n-k \le i \le n-1$, (and there are k such integers i),

$$1 = \frac{k}{n} + \frac{n-k}{n} \le \frac{k}{n} + \frac{i}{n} \le \{x\} + \frac{i}{n} < 1 + 1 = 2,$$

so $[\{x\} + \frac{i}{n}] = 1$ and hence $[x + \frac{i}{n}] = [x] + [\{x\} + \frac{i}{n}] = [x] + 1$. Thus

$$[x] + [x + \frac{1}{n}] + [x + \frac{2}{n}] + \ldots + [x + \frac{n-1}{n}] = (n-k)[x] + k([x]+1) = n[x] + k.$$

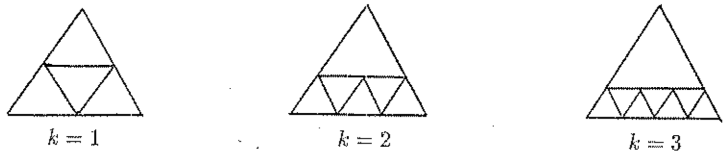
On the other hand, $[nx] = [n[x] + n\{x\}] = n[x] + [n\{x\}]$, (since $n[x]$ *is* an integer). But from $\frac{k}{n} \le \{x\} < \frac{k+1}{n}$, $k \le n\{x\} < k+1$ so $[n\{x\}] = k$. Thus

$$[nx] = n[x] + k = [x] + [x + \frac{1}{n}] + [x + \frac{2}{n}] + \ldots + [x + \frac{n-1}{n}],$$

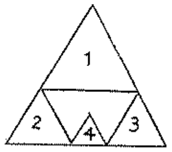
as required.

5. Find (with proof) the smallest integer N such that for every integer $m > N$ one can dissect an equilateral triangle into m smaller equilateral triangles (possibly of different sizes).

Solution. If you have an equilateral triangle, you can put a row of $2k+1$ equilateral triangles along one edge ($k \ge 1$) , by dividing that edge into $k+1$ equal parts, as shown.


$k = 1$ $\qquad$ $k = 2$ $\qquad$ $k = 3$

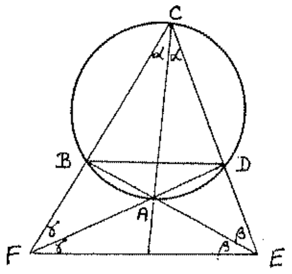
This gives a subdivision into an even number $2k+2$ of triangles; thus $4, 6, 8, \ldots$ small triangles are possible. Take one of these subdivisions, and divide one of the triangles into 4; this increases the number of equilateral triangles in the subdivision by 3. Hence from the $2k+2$ subdivision you get one with $2k+5$ triangles; thus $7, 9, 11, \ldots$ small triangles are possible. So the numbers of small triangles in the dissections found so far are $4, 6, 7, 8, 9, 10, \ldots$.
However, there is no subdivision into 5 equilateral triangles. For if each side is divided into only 2 parts, you must have 4 equilateral triangles, and if at least one side is divided into 3 parts you get at least 4 triangles and a non-triangular region, which needs at least 2 triangles to cover it – giving at least 6 triangles. So 5 triangles cannot happen.
Thus $N = 5$ is the answer to the question.



6. Let $ABCD$ be a cyclic quadrilateral. Let E be the intersection of BA (extended) and CD (extended), and let F be the intersection of CB (extended) and DA (extended). Suppose that A is the incentre of triangle CEF. (The incentre of a triangle is the point of intersection of its angle bisectors.)

Find the sizes of the angles in triangle ABD.

Solution. Let $\angle BCD = 2\alpha$, $\angle CEF = 2\beta$, $\angle CFE = 2\gamma$, so the angles are as shown on the diagram.
Now $\angle ADB = \angle ACB = \alpha$ (angles subtended by chord AB) and $\angle ABD = \angle ACD = \alpha$ (angles subtended by chord AD). Thus in triangle ABD, we have $\angle ABD = \angle ADB = \alpha$, and $\angle BAD = 180^{\circ} - 2\alpha$. So all we need to do is to determine α.
In triangle AEF, $\angle FAE = 180 - \beta - \gamma$. And $\angle BAD = \angle FAE$ (vertically opposite angles), so $180-2\alpha = 180-\beta-\gamma$, and hence $2\alpha = \beta+\gamma$. From triangle BCE, $\angle EBC = 180 - 2\alpha - \beta$. From triangle FDC, $\angle FDC = 180 - 2\alpha - \gamma$. Since $ABCD$ is a cyclic quadrilateral, $\angle ABC + \angle ADC = 180$, i.e. $\angle EBC + \angle FDC = 180$. Thus $(180 - 2\alpha - \beta) + (180 - 2\alpha - \gamma) = 180$. Hence $4\alpha + \beta + \gamma = 180$, so $6\alpha = 180$ (since $2\alpha = \beta + \gamma$). Thus $\alpha = 30$. So the angles in the triangle ABD are 30°, 30° and 120°.



7. Consider functions f with the following properties:
(a) f is defined for all integers (only) and takes on real values;
(b) for all integers x, y, $f(x)f(y) = f(x+y) + f(x-y)$;
(c). $f(0) \neq 0$, $f(1) = 5/2$.
Show that there is a unique function f with these properties, and give the value of $f(x)$, for integer x.

Solution. Put $x = 1$ and $y = 0$ in (b) to find $f(1)f(0) = f(1) + f(1)$, and since $f(1) = 5/2 \neq 0$ we have $f(0) = 2$. (So being told that $f(0) \neq 0$ is unnecessary.) Now put $x = 0$ in (b) to find $f(0)f(y) = f(y) + f(-y)$ and so since $f(0) = 2$ we have $f(-y) = f(y)$. Hence we need only find $f(n)$ for positive integers n. Use (b) again with $x = n$ and $y = 1$ to find $f(n)f(1) = f(n+1) + f(n-1)$, so

$$f(n+1) = \tfrac{5}{2}f(n) - f(n-1). \qquad (*)$$

From (*) we can find the remaining values. Put $n = 1$ in (*) and we get $f(2) = \frac{5}{2} \times \frac{5}{2} - 2 = \frac{17}{4}$. Put $n = 2$ in (*) and we get $f(3) = \frac{5}{2} \times \frac{17}{4} - \frac{5}{2} = \frac{65}{8}$. Put $n = 3$ in (*) and we have $f(4) = \frac{5}{2} \times \frac{65}{8} - \frac{17}{4} = \frac{257}{16}$. Hence we guess that $f(n) = 2^n + 2^{-n}$ (for positive integers n).
Having made this educated guess, we now need to prove that it is correct. Here we can do this by induction. Looking at the form of (*), I can see that the statement that I should actually try to prove by induction is

$$S(n) \; : \quad f(n) = 2^n + 2^{-n} \text{ and } f(n-1) = 2^{n-1} + 2^{-(n-1)}.$$

Since $f(1) = \frac{5}{2} = 2^1 + 2^{-1}$ and $f(0) = 2 = 2^0 + 2^{-0}$, certainly $S(0)$ is true.
Now suppose that $S(k)$ is true, and try to deduce that $S(k+1)$ must also be true. So suppose $f(k) = 2^k + 2^{-k}$ and $f(k-1) = 2^{k-1} + 2^{-(k-1)}$. Then by (*),

$$f(k+1) = \frac{5}{2}f(k) - f(k-1) = (2 + 2^{-1})(2^k + 2^{-k}) - (2^{k-1} + 2^{-(k-1)}) = 2^{k+1} + 2^{-(k+1)};$$

thus $f(k+1) = 2^{k+1} + 2^{-(k+1)}$ and $f(k) = 2^k + 2^{-k}$, so $S(k+1)$ would be true. Hence, by induction, $S(n)$ is true for all positive integers n. So in particular, $f(n) = 2^n + 2^{-n}$. Thus if (a), (b), (c) are to hold, the only possibility is $f(x) = 2^x + 2^{-x}$ for all integers x. And it is easy to check that $f(x) = 2^x + 2^{-x}$ does have these three properties, and hence it is the unique function satisfying (a), (b) and (c).