AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

QUEENSLAND PROGRAMME: SOLUTIONS May 2009

1. Let N be the positive integer with $2n$ digits ($n > 2$), such that the $n-1$ leftmost digits are ones, the next n digits are twos, and the rightmost digit of N is a four. Show that N is the product of two integers whose digits are all threes except for the rightmost digit, which is 4 in one of them and 6 in the other.

Solution. Put $x = \underbrace{11\ldots1}_{n-1\ 1's}$. Then $9x = \underbrace{99\ldots9}_{n-1\ 9's}$, so $9x+1 = 10^{n-1}$. And

$$\underbrace{11\ldots1}_{n-1\ 1's}\underbrace{22\ldots2}_{n\ 2's}4 = 10^{n+1}x + 200x + 24.$$

Now

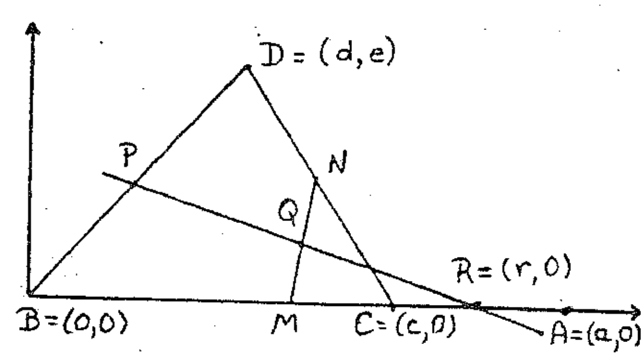
$$\begin{aligned}\underbrace{33\ldots3}_{n-1\ 3's}4 \times \underbrace{33\ldots3}_{n-1\ 3's}6 &= (30x+4)(30x+6)\\ &= 900x^2 + 300x + 24\\ &= 100x(9x+1) + 200x + 24\\ &= 10^{n+1}x + 200x + 24.\end{aligned}$$

This gives the result.

Solved by: James Hou (The Southport School), Margie Dickson (Indooroopilly SHS), Cynthia Wong (Somerville House), Jonathon Ho (St. Joseph's Gregory Terrace).

2. Let M be the midpoint of a line segment AB. Let C be any point on AB, and D any point not on AB. Let N be the midpoint of CD, P the midpoint of BD and Q the midpoint of MN. Prove that the line passsing through P and Q bisects AC.

Solution. Method 1 - co-ordinate geometry. Set up a co-ordinate system as shown, with $B = (0,0)$, $A = (a,0)$, $C = (c,0)$, $D = (d,e)$. Then P is the midpoint of BD so $P = (\frac{1}{2}d, \frac{1}{2}e)$; N is the midpoint of DC so $N = (\frac{1}{2}(c+d), \frac{1}{2}e)$; M is the midpoint of AB so $M = (\frac{1}{2}a, 0)$; Q is the midpoint of MN so $Q = (\frac{1}{4}(a+c+d), \frac{1}{4}e)$. Thus the line PQ has equation



$$\frac{y - \frac{1}{2}e}{-\frac{1}{4}e} = \frac{x - \frac{1}{2}d}{\frac{1}{4}(a+c-d)}$$

and so cuts AB at $R = (r, 0)$, where

$$\frac{-\frac{1}{2}e}{-\frac{1}{4}e} = \frac{r - \frac{1}{2}d}{\frac{1}{4}(a+c-d)}$$

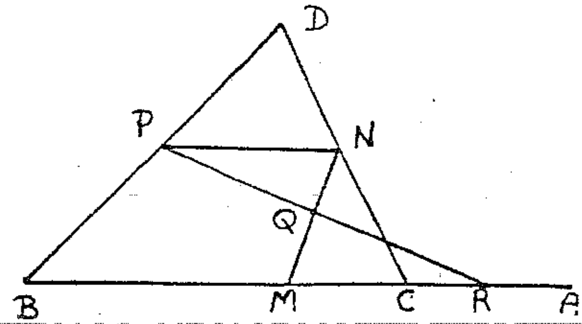
and hence $r = \frac{1}{2}(a+c-d) + \frac{1}{2}d = \frac{1}{2}(a+c)$. Thus R is the midpoint of AC, as required.

Method 2 - plane geometry. Let R be the point where PQ meets AC. In $\triangle BCD$, N and P are midpoints of the sides CD and BD, so PN is parallel to BC and $PN = \frac{1}{2}BC$. In triangles QNP and QMR

(i) $QN = QM$,

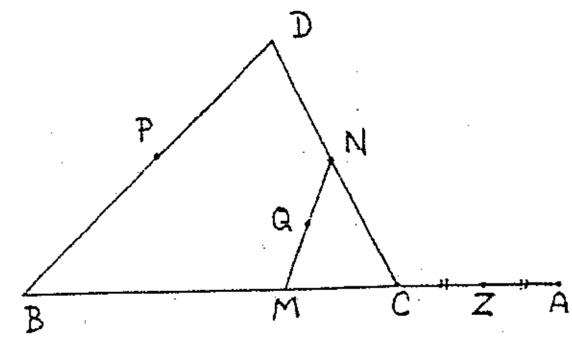
(ii) $\angle NQP = \angle MQR$ (vertically opposite),

(iii) $\angle NPQ = \angle MRQ$ (alternate angles from parallel lines PN and BC).



Hence triangles QNP and QMR are congruent. Thus $MR = PN = \frac{1}{2}BC$. Finally, $AR = AM - MR = \frac{1}{2}AB - \frac{1}{2}BC = \frac{1}{2}(AB - BC) = \frac{1}{2}AC$, and so R is the midpoint of AC.

Method 3 - using vectors. Let Z be the midpoint of AC, and we must show that P, Q, Z are collinear. Take the origin at B, and let $\boldsymbol{a}$ be the vector from B to A and similarly for the other points. Since M, N, P, Q, Z are midpoints, we have $\boldsymbol{m} = \frac{1}{2}\boldsymbol{a}$, $\boldsymbol{n} = \frac{1}{2}(\boldsymbol{c}+\boldsymbol{d})$, $\boldsymbol{p} = \frac{1}{2}\boldsymbol{d}$, $\boldsymbol{z} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{c})$ $\boldsymbol{q} = \frac{1}{2}(\boldsymbol{m}+\boldsymbol{n}) = \frac{1}{4}(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d})$. Then $ZQ = \boldsymbol{q}-\boldsymbol{z} = \frac{1}{4}(\boldsymbol{d}-\boldsymbol{c}-\boldsymbol{a})$ and $ZP = \boldsymbol{p}-\boldsymbol{z} = \frac{1}{2}(\boldsymbol{d}-\boldsymbol{a}-\boldsymbol{c})$. Thus ZQ is a scalar multiple of ZP, and hence Z, P, Q are collinear.



Solved by: James Hou, Cynthia Wong, Jonathon Ho, Zack Wei (All Saints Anglican Middle School).

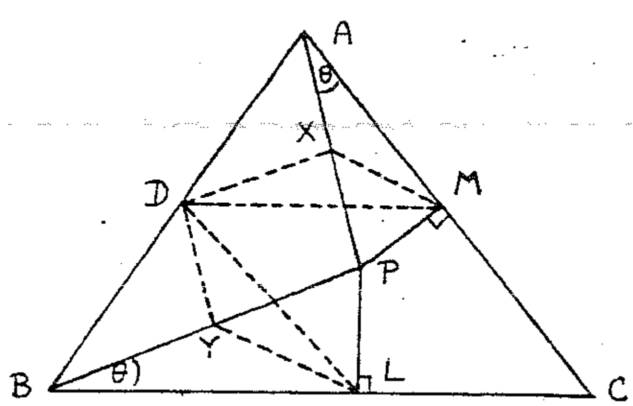
3. A box contains p white balls and q black balls. Beside the box there is a pile of black balls. Two balls are taken out from the box. If they are of the same colour, a black ball from the pile is put in the box. If they are of different colours, the white ball is put back into the box. This procedure is repeated until the last pair of balls is removed from the box and one last ball is put in. What is the probability that this last ball is white?

Solution. The only time the number of white balls in the box changes is when the two balls taken out of the box are both white, and then the number of white balls decreases by two. So if you start with an even number of white balls, it stays even all through, and if you start with an odd number of white balls, it stays odd all through. Therefore if p is even, the last ball cannot be white – the probability is 0. If p is odd, the last ball must be white – the probability is 1.

Solved by: James Hou, Margie Dickson, Cynthia Wong, Zack Wei, Jonathon Ho.

4. ABC is a triangle and P is a point inside it. Angle PAC = angle PBC. The perpendiculars from P to BC, CA meet these sides at L, M respectively, and D is the midpoint of AB. Prove that $DL = DM$.

Solution. Let $\angle PAC = \angle PBC = \theta$. Let X, Y be the midpoints of AP, BP respectively. Since X, Y, D are the midpoints of the sides of triangle APB we have $DX = YP = BY$ and $DY = AX = XP$ also DX and BP are parallel, and DY and AP are parallel.
Since $\angle AMP = 90^0$, X is the centre of a circle through A, M and P. In particular, $XM = AX$.
Similarly, with $\angle BLP = 90^0$, Y is the centre of a circle through B, L and P, so $YL = BY$.
Consider triangles DXM and LYD. Here $DX = BY = YL$, and $XM = AX = DY$.



And $\angle DXM = \angle DXP + \angle PXM = \angle DXP + 2\theta$, also $\angle DYL = \angle DYP + \angle PYL = \angle DYP + 2\theta$. Further, $\angle DXP = \angle DYP$ since $DXPY$ is a parallelogram. Hence $\angle DXM = \angle DYL$. Thus triangles DXM and LYD are congruent (SAS). So $DM = DL$, as required.

Solved by: James Hou, Margie Dickson, Cynthia Wong, Zack Wei, Jonathon Ho.

5. Let k be a positive integer. A function f, which is defined only for integers i with $1 \le i \le k$, and has $f(i) = 1$ or $f(i) = 2$ for every i is called a *low-valued k-function*. Let $S(k)$ be the number of all low-valued k-functions with the property

$$f(1) + f(2) + \ldots + f(k) = 1000.$$

Show that $S(500) + S(501) + \ldots + S(1000) > 10^{150}$.

Solution. We have to show that the number of sequences consisting just of 1's and 2's, whose terms add up to 1000, is greater than 10^{150}. Let $A(n)$ be the number of different sequences consisting just of 1's and 2's, whose terms add up to n, for $n = 1, 2 \ldots$. Then $A(1) = 1$ and $A(2) = 2$. If $n > 2$, then we observe that each such sequence whose terms add up to n is obtained exactly once from a shorter such sequence by adding an extra 2 at the end if the sequence has sum $n - 2$; or by adding an extra 1 at the end if the sequence has sum $n - 1$. Hence we get the formula $A(n) = A(n-1) + A(n-2)$. [You may recognise that the numbers $A(n)$ are the Fibonacci numbers.] We use induction to show:

$$A(2n) > 2^n, \quad \text{for } n \ge 2.$$

It is true for $n = 2$ since $A(4) = A(3) + A(2) > 2A(2) = 2^2$. Make the inductive assumption that $A(2k) > 2^k$. Then for $n = k + 1$,

$$A(2k+2) = A(2k+1) + A(2k) > 2A(2k) > 2 \times 2^k = 2^{k+1}.$$

Hence $A(1000) > 2^{500}$. Since $1024 = 2^{10} > 10^3$, we have $2^{500} > 10^{150}$. Hence $A(1000) > 10^{150}$, as required.

Here is a more direct solution.
For each sequence $s_1, s_2, \ldots, s_{500}$ of 500 terms, each either a 1 or a 2, we define an integer k with $500 \le k \le 1000$ and construct a low-valued k-function f with $f(1) + f(2) + \ldots + f(k) = 1000$, as follows:
Put $S = s_1 + s_2 + \ldots + s_{500}$, so $500 \le S \le 1000$, and define $k = 1500 - S$, so $500 \le k \le 1000$. Then define $f(i)$ for $1 \le i \le k$ by

$$f(i) = \begin{cases} s_1 & \text{if } 1 \le i \le 500 \\ 1 & \text{if } 501 < i \le k. \end{cases}$$

(If $k = 500$, we just have $f(i) = s_i$ for $1 \le i \le 500$.) Then f is indeed a low-valued k-function. Now $f(1) + f(2) + \ldots + f(500) = S$, and $f(501) + f(502) + \ldots + f(k) = k - 500 = 1000 - S$, so that

$$f(1) + f(2) + \ldots + f(k) = S + (1000 - S) = 1000.$$

There are 2^{500} possible sequences $s_1, s_2, \ldots, s_{500}$ and they each give rise to a different function f, so we have constructed 2^{500} different functions, all of which are in the set of functions counted by $S(500) + S(501) + \ldots + S(1000)$. Hence $S(500) + S(501) + \ldots + S(1000) \ge 2^{500} > 10^{150}$.

Solved by: Zack Wei, Jonathon Ho.

6. Show that for every integer x, the number

$$\frac{1}{5}x^5 + \frac{1}{3}x^3 + \frac{7}{15}x$$

is an integer.

Solution. Now

$$\frac{1}{5}x^5 + \frac{1}{3}x^3 + \frac{7}{15}x = \frac{1}{15}(3x^5 + 5x^3 + 7x),$$

so we need to show that both 3 and 5 divide $3x^5 + 5x^3 + 7x$.
Working modulo 3 (i.e. looking at the remainders on division by 3) we have

x	x^2	x^3
0	0	0
1	1	1
2	1	2

So always $x \equiv x^3 \pmod 3$. Hence, modulo 3, $3x^5 + 5x^3 + 7x \equiv 5x^3 + 7x \equiv 5x + 7x \equiv 12x \equiv 0$.
Thus 3 divides $3x^5 + 5x^3 + 7x$.
Similarly, working modulo 5 (i.e. looking at the remainders on division by 5) we have

x	x^2	x^3	x^5
0	0	0	0
1	1	1	1
2	4	3	2
3	4	2	3
4	1	4	4

So always $x \equiv x^5 \pmod 5$. Hence, modulo 5, $3x^5 + 5x^3 + 7x \equiv 3x^5 + 7x \equiv 3x + 7x \equiv 10x \equiv 0$.
Thus 5 divides $3x^5 + 5x^3 + 7x$.
The result follows.

Solved by: James Hou, Margie Dickson, Cynthia Wong, Zack Wei, Jonathon Ho.

7. Let x, y, z be real numbers such that

$$x + y + z = 5$$

and

$$xy + yz + zx = 3.$$

Show that $-1 \le z \le 13/3$.

Solution. From the first equation, $y = 5 - x - z$. From the second equation, $y(x+z) + zx = 3$,

so $$(5 - x - z)(x + z) + zx = 3$$

Hence $$x^2 + (z - 5)x + 3 - 5z + z^2 = 0.$$

For the quadratic $Ax^2 + Bx + C = 0$ to have real solutions, $B^2 - 4AC \ge 0$. Hence

$$(5 - z)^2 - 4(3 - 5z + z^2) \ge 0$$

i.e. $$-3z^2 + 10z + 13 \ge 0$$

i.e. $$(13 - 3z)(1 + z) \ge 0.$$

From the graph of $w = (13 - 3z)(1 + z)$ we see that $w \ge 0$ when $-1 \le z \le 13/3$, so indeed $-1 \le z \le 13/3$ as required.

Solved by: James Hou, Cynthia Wong, Zack Wei, Jonathon Ho.