AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

QUEENSLAND PROGRAMME: PROBLEMS May 2009

1. Let N be the positive integer with $2n$ digits ($n > 2$), such that the $n-1$ leftmost digits are ones, the next n digits are twos, and the rightmost digit of N is a four. Show that N is the product of two integers whose digits are all threes except for the rightmost digit, which is 4 in one of them and 6 in the other.

2. Let M be the midpoint of a line segment AB. Let C be any point on AB, and D any point not on AB. Let N be the midpoint of CD, P the midpoint of BD and Q the midpoint of MN. Prove that the line passsing through P and Q bisects AC.

3. A box contains p white balls and q black balls. Beside the box there is a pile of black balls. Two balls are taken out from the box. If they are of the same colour, a black ball from the pile is put in the box. If they are of different colours, the white ball is put back into the box. This procedure is repeated until the last pair of balls is removed from the box and one last ball is put in. What is the probability that this last ball is white ?

4. ABC is a triangle and P is a point inside it. $\angle PAC = \angle PBC$. The perpendiculars from P to BC, CA meet these sides at L, M respectively, and D is the midpoint of AB. Prove that $DL = DM$.

5. Let k be a positive integer. A function f, which is defined only for integers i with $1 \leq i \leq k$, and has $f(i) = 1$ or $f(i) = 2$ for every i is called a *low-valued k-function*. Let $S(k)$ be the number of all low-valued k-functions with the property

$$f(1) + f(2) + ... + f(k) = 1000.$$

Show that $S(500) + S(501) + ... + S(1000) > 10^{150}$.

6. Show that for every integer x, the number

$$\frac{1}{5}x^5 + \frac{1}{3}x^3 + \frac{7}{15}x$$

is an integer.

7. Let x, y, z be real numbers such that

$$x + y + z = 5$$

and

$$xy + yz + zx = 3.$$

Show that $-1 \leq z \leq \frac{13}{3}$.

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