AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

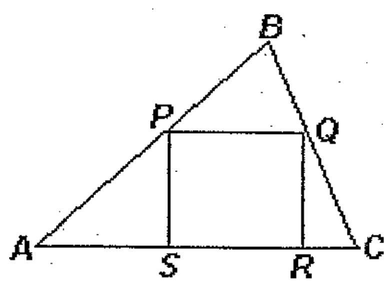
# QUEENSLAND PROGRAMME: PROBLEMS March 2009

1. Find all real numbers $x$ (possibly not integers) which satisfy the equation

$$9^x + 6^x = 2^{2x+1}.$$

2. Three urns are arranged in a row. The left urn contains two red balls and one white ball. The centre urn contains one red ball and one white ball, and the right urn contains one red ball and two white balls. A ball is randomly chosen from the appropriate urn and then discarded. If the chosen ball is white, one moves one urn to the left. If the chosen ball is red, the move is made one urn to the right. The game continues until either an empty urn is reached or until an impossible move is required (i.e. "move left" from the leftmost urn or "move right" from the rightmost urn). Assuming that the game starts at the middle urn, what is the probability that the game ends because an empty urn was reached?

3. A square $PQRS$ is inscribed in the triangle $ABC$ as shown. Prove that the area of the square is less than or equal to half the area of the triangle. When does equality occur?




4. Given a triangle $ABC$, construct the square on $AB$ and the square on $BC$. Let $X$ be the midpoint of $AC$. Show that the lines joining the centres of the squares to $X$ are perpendicular and equal in length.

5. Show that there are no more than nine prime numbers between 10 and $10^{29}$ whose decimal representation is a string of ones (i.e. numbers like 11, or 1111).

6. Let $f$ be a function defined for all real numbers and having non-zero real numbers for its values, such that

$$f(x+2) = f(x-1) \times f(x+5)$$

for all real $x$.
Show that $f$ is a periodic function, i.e. show that there is a number $P$ such that $f(x+P) = f(x)$ for all real numbers $x$.

7. (A real challenge.) What is the smallest value of the positive integer $n$ such that $n$ is the denominator of a fraction whose decimal representation starts with 0.501, i.e. for some integer $a$

$$\frac{a}{n} = 0.501\ldots.$$

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