AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

QUEENSLAND PROGRAMME: PROBLEMS March 2008

1. What digits should be put instead of zeros in the third and fifth places in the number 3000003 in order to give a number divisible by 13?

2. ABC is a triangle inscribed in a circle. The extensions of the bisectors of the angles A, B, and C intersect the circle at the points D, E and F respectively. Prove that AD is perpendicular to EF.

3. Prove that for any any positive integer n,

$$\left[\frac{n}{3}\right] + \left[\frac{n+2}{6}\right] + \left[\frac{n+4}{6}\right] = \left[\frac{n}{2}\right] + \left[\frac{n+3}{6}\right],$$

   where $[\cdot]$ denotes the greatest integer function (i.e. $[t]$ is the greatest integer less than or equal to t, so $[3] = 3$ and $[\pi] = 3$).

4. Find all natural numbers n for which $2^8 + 2^{11} + 2^n$ is a perfect square.

5. For each function f which is defined for all real numbers and satisfies both
   (i) $f(x.y) = x.f(y) + y.f(x)$ [where '.' means multiplication],
   (ii) $f(x+y) = f(x^7) + f(y^7)$,
   determine the value $f(\sqrt{1999})$.

6. On the successive sides of a square, choose points M, N, P, Q in such a way that all the sides of the quadrilateral $MNPQ$ are equal. Prove that the quadrilateral $MNPQ$ is a square.

7. Exhibit a polynomial $p(n)$ with integer coefficients such that $3^n - p(n)$ is a multiple of 8 for each $n = 1, 2, 3, \ldots$.

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