AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

QUEENSLAND PROGRAMME: SOLUTIONS June 2009

1. Determine the largest positive integer which, for all positive integers $n \geq 4$, is a factor of $n^4(n-1)^3(n-2)^2(n-3)$.

Solution. Let D be the number we are looking for. Consider what happens for $n = 4, 5, 6, 7$:

n	$n^4(n-1)^3(n-2)^2(n-3)$
4	$4^4 3^3 2^2 = 2^{10}3^3$
5	$5^4 4^3 3^2 2 = 2^7 3^2 5^4$
6	$6^4 5^3 4^2 3 = 2^8 3^5 5^3$
7	$7^4 6^3 5^2 4 = 2^5 3^3 5^2 7^4$

The largest divisor of just these four numbers is $2^5 3^2$, so D will be a divisor of this. But in fact we can show that $D = 2^5 3^2$. For every $n \geq 4$, one of n, $n-1$, $n-2$ is a multiple of 3, and so $n^4(n-1)^3(n-2)^2$ is a multiple of 3^2. Thus 3^2 is a factor of D. Now we consider the powers of 2. If n is even, $n-2$ is also even, and so $n^4(n-2)^2$ is a multiple of 2^6. If n is odd, both $n-1$ and $n-3$ are even, and being successive even numbers, one of them is a multiple of 4. So $(n-1)^3(n-3)$ is a multiple of $2^3 \times 2 \times 2 = 2^5$. So in either case, $n^4(n-1)^3(n-2)^2(n-3)$ is a multiple of 2^5. Hence 2^5 is a factor of D. It follows that $D = 2^5 3^2 = 288$.

Solved by: Margie Dickson (Indooroopilly SHS), Jonathon Ho (St Josephs Gregory Terrace).

2. Let $f(n)$ be the sum of the first n terms of the sequence:

$$0,\ 1,\ 1,\ 2,\ 2,\ 3,\ 3,\ 4,\ 4,\ 5,\ 5,\ 6,\ 6,\ \ldots.$$

(a) Give a formula for $f(n)$.
(b) Prove that $f(s+t) - f(s-t) = st$ where s, t are positive integers and $s > t$.

Solution. (a) If n is even, we can write

$$\begin{aligned} f(n) &= 0 + 1 + 2 + 3 + \cdots + (\tfrac{n}{2}-2) + (\tfrac{n}{2}-1) + \\ &\quad \tfrac{n}{2} + (\tfrac{n}{2}-1) + (\tfrac{n}{2}-2) + (\tfrac{n}{2}-3) + \cdots + 2 + 1 \\ &= \tfrac{n}{2} \times \tfrac{n}{2} = \tfrac{n^2}{4} \end{aligned}$$

since we have $\frac{n}{2}$ columns, each summing to $\frac{n}{2}$.
And if n is odd, similarly we can write

$$\begin{aligned} f(n) &= 0 + 1 + 2 + 3 + \cdots + \tfrac{n-3}{2} + \tfrac{n-1}{2} + \\ &\quad + \tfrac{n-1}{2} + \tfrac{n-3}{2} + \tfrac{n-5}{2} + \cdots + 2 + 1 \\ &= (\tfrac{n}{2}-1)(\tfrac{n}{2}+1) = \tfrac{n^2-1}{4} \end{aligned}$$

since we have $\frac{n-1}{2}$ columns each summing to $\frac{n+1}{2}$.
(b) Since $(s+t)$ and $(s-t)$ have difference $2t$, which is even, $(s+t)$ and $(s-t)$ are either both even or both odd. In the even case

$$f(s+t) - f(s-t) = \frac{(s+t)^2}{4} - \frac{(s-t)^2}{4} = st.$$

In the odd case

$$f(s+t) - f(s-t) = \frac{(s+t)^2 - 1}{4} - \frac{(s-t)^2 - 1}{4} = st.$$

Solved by: Cynthia Wong (Somerville House), Zack Wei (All Saints Anglican School), Jonathon Ho.

3. Prove that the sum of all the n-digit positive integers (for $n > 2$) is

$$494\ \underbrace{99\cdots9}_{n-3\text{ 9's}}\ 55\ \underbrace{00\cdots0}_{n-2\text{ 0's}}.$$

Solution. The smallest n-digit positive integer is 10^{n-1} and the largest is $10^n - 1$. Hence we want the sum

$$S = 10^{n-1} + (10^{n-1}+1) + (10^{n-1}+2) + \cdots + (10^n - 1).$$

Written backwards,

$$S = (10^n - 1) + (10^n - 2) + (10^n - 3) + \cdots + 10^{n-1}.$$

Note that there are $(10^n - 1) - (10^{n-1} - 1) = 9 \times 10^{n-1}$ terms, so adding the two lines as displayed gives

$$2S = (10^n + 10^{n-1} - 1) \times 9 \times 10^{n-1}.$$

Hence

$$\begin{aligned} S &= 45 \times 10^{n-2} \times (10^n + 10^{n-1} - 1) \\ &= 45 \times 10^{n-2} \times (10^{n-1} \times 11 - 1) \\ &= 495 \times 10^{2n-3} - (100 - 55) \times 10^{n-2} \\ &= (494 + 1) \times 10^{2n-3} - 10^n + 55 \times 10^{n-2} \\ &= 494 \times 10^{2n-3} + 10^{2n-3} - 10^n + 55 \times 10^{n-2} \\ &= 494 \times 10^{2n-3} + (10^{n-3} - 1) \times 10^n + 55 \times 10^{n-2} \\ &= 494 \times 10^{2n-3} + \underbrace{99\cdots9}_{n-3} \times 10^n + 55 \times 10^{n-2} \\ &= 494\ \underbrace{99\cdots9}_{n-3}\ 55\ \underbrace{00\cdots0}_{n-2}. \end{aligned}$$

Solved by: James Hou (The Southport School), Margie Dickson, Cynthia Wong, Zack Wei, Jonathon Ho.

4. Show that for all real numbers x and y,

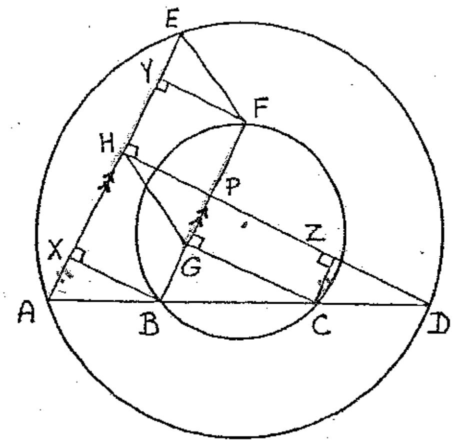
$$\cos x^2 + \cos y^2 - \cos xy < 3.$$

Solution. Since $-1 \le \cos u \le 1$, clearly $\cos x^2 + \cos y^2 - \cos xy \le 3$, and the only way that equality can occur is if $\cos x^2 = \cos y^2 = 1$ and $\cos xy = -1$. For this to happen, $x^2 = 360k^{\circ}$, $y^2 = 360\ell^{\circ}$ and $xy = (180+360m)^{\circ}$, for some integers k, ℓ, m. So $x^2y^2 = 360^2 k\ell = (180+360m)^2$. Hence (dividing by 180^2): $4k\ell = (1+2m)^2 = 1+4m+4m^2$. But this is impossible, since $4k$ is even whereas $1+4m+4m^2$ is odd. Hence equality cannot occur, and so always $\cos x^2 + \cos y^2 - \cos xy < 3$.

Solved by: Zack Wei, Jonathon Ho.

5. A straight line cuts two concentric circles in the points A, B, C and D in that order; AE and BF are parallel chords, one in each circle; GC is perpendicular to BF at G, and DH is perpendicular to AE at H. Show that $GF = HE$.

Solution. Join EF and DG. Let BF and HD meet at P. Let X and Y be the feet of the perpendiculars from B and F to AE. Then $HYFP$ is a rectangle, so XB is parallel to HD, and $HY = PF$. Also $XB = HP = YF$. Let Z be the foot of the perpendicular from C to HD. Then $CZ = GP$.



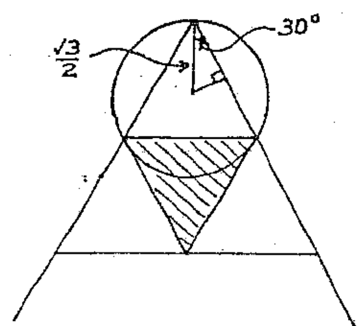
Since the circles are concentric, along chord AC we have $AB = CD$. Since the circles are concentric and the chords BF and AE are parallel, it is easy to see that $AX = YE$ (because the line from the centre of the circles perpendicular to the chords bisects both chords). Also right-angled triangles AXB and CZD are congruent, since $AB = CD$ and $\angle XBA = \angle ZDC$ (because XB is parallel to HD). Thus $AX = CZ$. So we have $YE = AX = CZ = GP$, and thus $YE = GP$. Finally, $HE = HY + YE = PE + GP = GF$.

Solved by: James Hou, Cynthia Wong, Jonathon Ho.

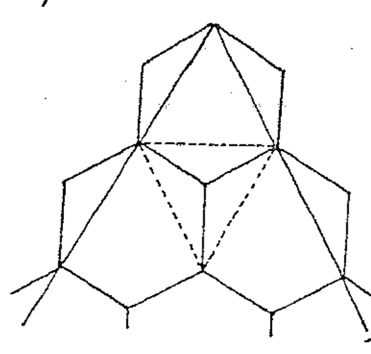
6. Within an equilateral triangle of side 15 cm there are 111 points. Prove that it is always possible to cover at least three of these points with a suitably placed round coin of diameter $\sqrt{3}$ cm (part of which may lie outside the triangle).

Solution. This question should surely be solved by using the pigeon-hole principle, which in its simplest form says that if you have $n+1$ pigeons but only n pigeon-holes to put them in, then one hole must get at least two pigeons.

Here a pigeon-hole will be the region covered by one coin, and the pigeons will be the points. If I can show that the triangle can be completely covered by 55 (or fewer) coins, we shall be finished – if each of these covered at most 2 points, there would be at most 110 points in all. But we have 111 points, so some coin would have to cover at least 3 of them.



Since circles and triangles don't match, I shall cover the triangle with more conveniently shaped figures, each of which just fits inside the coin. The first shape to try is obviously equilateral triangles. The largest equilateral triangle to be covered by a $\sqrt{3}$ coin has its sides of length $2\times\frac{\sqrt{3}}{2}\times\cos 30^0 = 2\times\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2} = \frac{3}{2}$. So to go down the 15 cm side of the big triangle would need 10 small triangles, and the total number of small triangles needed would be $1+3+5+\ldots+19 = \frac{1}{2}\times 10\times(19+1) = 100$. This is larger than 55, so these triangles won't work. But notice in my figure, I don't really need a coin for the shaded triangle – this triangle will be covered by the coins from its neighbouring triangles. Or to put it another way, I can use regular hexagons of diameter $\sqrt{3}$. I shall need 10 of these hexagons to go down the side of the big triangle. The total number of hexagons needed would be $1+2+\ldots+10 = 55$! Thus it follows that indeed the big triangle can be covered by 55 of the coins.



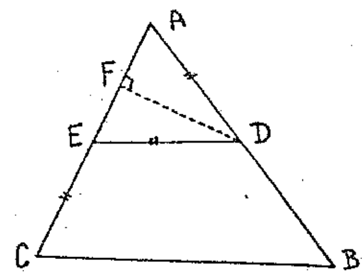
Solved by: Zack Wei, Jonathon Ho.

7. In the triangle ABC the sides are $AB = 33\,\text{cm}$, $AC = 21\,\text{cm}$, and $BC = m\,\text{cm}$, where m is an integer. It is possible to find a point D on AB strictly between A and B and a point E on AC strictly between A and C such that

$$AD = DE = EC = n\,\text{cm},$$

where n is an integer. What values can m take ?

Solution. Since E is between A and C, we must have $\angle BAC < 90^0$. Applying the cosine rule to triangle ABC gives $CB^2 = AB^2 + AC^2 - 2\,AB\,AC\,\cos A$, so



$$m^2 = 33^2 + 21^2 - 2 \times 33 \times 21 \times \cos A. \qquad (1)$$

We can find $\cos A$ from triangle ADF, (where F is the foot of the perpendicular from D to AC). Since $AD = DE$, triangle ADE is isosceles, so F is the midpoint of AE. And $AE = AC - EC = 21 - n$, so $AF = \frac{1}{2}(21 - n)$. Then $\cos A = AF/AD = (21 - n)/2n$. Thus (1) becomes

$$m^2 = 33^2 + 21^2 - 33 \times 21 \times (21 - n)/n. \qquad (2)$$

Since m is an integer, n must divide $33 \times 21 \times (21 - n)$.
Now $\cos A < 1$, so $(21 - n)/2n < 1$ and hence $n > 7$. Also $n < 21$, since E is between A and C. When $7 < n < 21$, a quick check shows that n and $21 - n$ are co-prime. Hence n must divide $33 \times 21 = 3^2 \times 7 \times 11$. So $n = 9, 11$ are the only possibilities.
When $n = 9$, (2) gives $m^2 = 33^2 + 21^2 - 77 \times 12 = 606$, so m is not an integer.
When $n = 11$, (2) gives $m^2 = 33^2 + 21^2 - 63 \times 10 = 900$, so $m = 30$, which is an integer. Thus $m = 30$ is the only value allowed.

Solved by: James Hou, Mitchell Riley (Qld Academy Science Maths and Technology), Margie Dickson, Cynthia Wong, Jonathon Ho.