AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

QUEENSLAND PROGRAMME: PROBLEMS June 2009

1. Determine the largest positive integer which, for all positive integers $n \geq 4$, is a factor of $n^4(n-1)^3(n-2)^2(n-3)$.

2. Let $f(n)$ be the sum of the first n terms of the sequence:

$$0,\ 1,\ 1,\ 2,\ 2,\ 3,\ 3,\ 4,\ 4,\ 5,\ 5,\ 6,\ 6,\ \ldots.$$

   (a) Give a formula for $f(n)$.
   (b) Prove that $f(s+t) - f(s-t) = st$ where s, t are positive integers and $s > t$.

3. Prove that the sum of all the n-digit postitive integers (for $n > 2$) is

$$494\ \underbrace{99\ldots9}_{n-3\ 9's}\ 55\ \underbrace{00\ldots0}_{n-2\ 0's}.$$

4. Show that for all real numbers x and y,

$$\cos x^2 + \cos y^2 - \cos xy < 3.$$

5. A straight line cuts two concentric circles in the points A, B, C and D in that order; AE and BF are parallel chords, one in each circle; GC is perpendicular to BF at G, and DH is perpendicular to AE at H. Show that $GF = HE$.

6. Within an equilateral triangle of side 15 cm there are 111 points. Prove that it is always possible to cover at least three of these points with a suitably placed round coin of diameter $\sqrt{3}$ cm (part of which may lie outside the triangle).

7. In the triangle ABC the sides are $AB = 33$ cm, $AC = 21$ cm, and $BC = m$ cm, where m is an integer. It is possible to find a point D on AB strictly between A and B and a point E on AC strictly between A and C such that

$$AD = DE = EC = n\,\text{cm},$$

   where n is an integer. What values can m take?

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