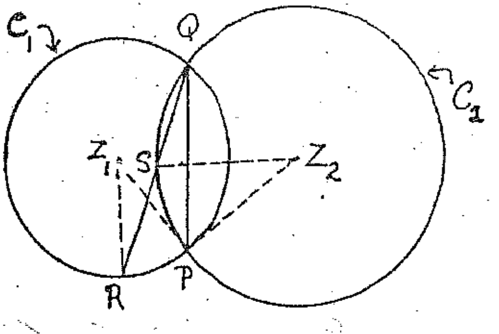
AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

QUEENSLAND PROGRAMME: SOLUTIONS July 2009

1. C_1 and C_2 are circles with centres Z_1 and Z_2 respectively, which intersect in the points P and Q. Let R and S be two distinct points with the properties:
(a) R lies on the circle C_1;
(b) S lies on the circle C_2; and
(c) the angles PZ_1R and PZ_2S are both 15^{O} when measured in the clockwise direction, starting at P.
Show that Q, R and S are on the same line.

Solution. $\angle PQR$ (at the circumference of C_1) is half $\angle PZ_1R$ (at the centre of C_1). Likewise $\angle PQS$ (at the circumference of C_2) is half $\angle PZ_2S$ (at the centre of C_2). Since $\angle PZ_1R = \angle PZ_2S$ ($= 15^{\text{O}}$, which is irrelevant), $\angle PQR = \angle PQS$. Now R and S are on the same side of PQ , since angles PZ_1R and PZ_2S are measured in the same direction. Therefore Q, R and S are on the same line.



If you are careful, you will realize that there is another case, when P and Q are very close together, and then Q is between R and S. Here still $\angle PQS = \frac{1}{2}\angle PZ_2S = \frac{1}{2}\angle PZ_1R$. But $\angle PQR = 180^{\text{O}} - \frac{1}{2}\angle PZ_1R$. (You can see this by considering T on the circumference of C_1: $\angle PTR = \frac{1}{2}\angle PZ_1R$, and $\angle PQR = 180^{\text{O}} - \angle PTR$ since P, Q, R, T are concyclic.) This time $\angle PQR + \angle PQS = 180^{\text{O}}$, so again P, Q and R are on the same line.

Solved by: Jonathon Ho (St Josephs Gregory Terrace), Margie Dickson (Indooroopilly SHS).

2. Determine all real numbers r such that there is precisely one pair (x, y) of real numbers satisfying the conditions:
(i) $y - x = r$;
(ii) $x^2 + y^2 + 2x \leq 1$.

Solution. It follows from (i) that $y = x + r$. Substitute $y = x + r$ in (ii). Then we get the inequality $x^2 + (x + r)^2 + 2x \leq 1$, which can be rewritten as $2x^2 + 2(r + 1)x + r^2 - 1 \leq 0$. Thus one asks for exactly one solution of $2x^2 + 2(r + 1)x + r^2 - 1 \leq 0$. But, for exactly one solution of this inequality, there must be only one solution of the equation $2x^2 + 2(r + 1)x + r^2 - 1 = 0$. Hence we need the discriminant of the expression $2x^2 + 2(r + 1)x + r^2 - 1$ to be equal to 0, that is $4(r + 1)^2 - 8(r^2 - 1) = 0$. Hence $r^2 - 2r - 3 = 0$, so $(r + 1)(r - 3) = 0$. Thus $r = -1$ or $r = 3$. If $r = -1$, then $y = x - 1$ and (ii) turns into $2x^2 \leq 0$. The only solution is $x = 0$, and then $y = -1$. Similarly, if $r = 3$, then $y = x + 3$ and (ii) becomes $(x + 2)^2 \leq 0$. The only solution is $x = -2$, and then $y = 1$.
Thus the two values $r = -1$ and $r = 3$ are the ones for which there is precisely one pair (x, y) satisfying conditions (i) and (ii).

Solved by: Jonathon Ho.

3. Each of the 36 line segments joining 9 distinct points on a circle is coloured either red or blue. Suppose that each triangle determined by 3 of the 9 points contains at least one red side. Prove that there are four points such that the 6 segments connecting them are all red.

Solution. If one point is joined to 4 points A, B, C, D by blue lines, then since there are no blue triangles, all the 6 lines AB, AC, AD, BC, BD, CD must be red, and A, B, C, D give us four suitable points.

So we may assume that every point is connected to at most 3 others by blue lines. Hence every point is connected to at least 5 others by red lines. There cannot be exactly 5 red lines at each point, for otherwise there would be $\frac{1}{2}(9 \times 5) = 22\frac{1}{2}$ red segments all together. Therefore, some point P is joined to at least 6 others, say A, B, C, D, E, F by red lines. Now consider the colours of the segments joining A, B, $\ldots$, F. At least 3 of the 5 lines AB, AC, AD, AE, AF have the same colour. Let these be AB, AC, AD . If this colour is blue, since there are no blue triangles, BC, BD, CD must be red, and P, B, C, D are 4 points with all 6 lines joining them red. On the other hand, if AB, AC, AD are all red, since also at least one side of triangle BCD is red – say BC – we get four points P, A, B, C with all lines joining them red.

Solved by: Jonathon Ho.

4. (a) Prove that if x, y, z are non-negative real numbers, then

$$x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y) \geq 0.$$

(b) Hence or otherwise show that for all real numbers a, b, c

$$a^6 + b^6 + c^6 + 3a^2b^2c^2 \geq 2(b^3c^3 + c^3a^3 + a^3b^3).$$

Solution. (a) By symmetry, we may suppose $x \leq y \leq z$. Put $y = x + u$, $z = y + v$ so $x, y, z, u, v \geq 0$. Then

$$\begin{aligned} x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y) &= x(-u)(-u-v) + y(-v)u + z(u+v)v \\ &= xu^2 + xuv + (z-y)uv + zv^2 \\ &= xu^2 + xuv + uv^2 + zv^2 \\ &\geq 0. \end{aligned}$$

(b) Apply (a) with $x = a^2$, $y = b^2$, $z = c^2$ (so of course $x, y, z \geq 0$):

$$a^2(a^2-b^2)(a^2-c^2) + b^2(b^2-c^2)(b^2-a^2) + c^2(c^2-a^2)(c^2-b^2) \geq 0$$

and hence

$$\begin{aligned} a^6 + b^6 + c^6 + 3a^2b^2c^2 &\geq a^4(b^2+c^2) + b^4(a^2+c^2) + c^4(a^2+b^2) \\ &= a^2b^2(a^2+b^2) + a^2c^2(a^2+c^2) + b^2c^2(b^2+c^2) \\ &\geq a^2b^2(2ab) + a^2c^2(2ac) + b^2c^2(2bc) \\ &= 2(a^3b^3 + a^3c^3 + b^3c^3), \end{aligned}$$

where we used that $a^2 + b^2 \geq 2ab$ [because $(a-b)^2 \geq 0$], etc.

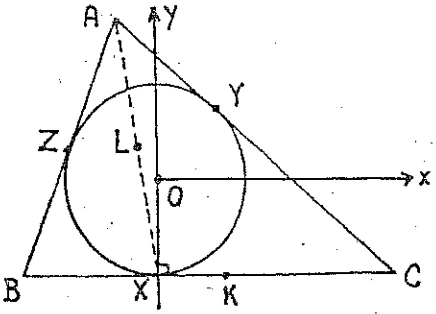
Solved by: Jonathon Ho.

5. If S is a set of positive integers such that the sum of all the numbers in S is n, how large can the product of all the numbers in S be ?

Solution. The question asks for the largest product P whose factors add to n. If $n=1$, then $S=\{1\}$ and $P=1$. So suppose henceforth that $n>1$.
If 1 is one of the factors, delete it and add 1 to another factor, since this increases the product. Thus all factors of P are at least 2 . If a factor $x \geq 4$ appears, then replace x by 2 and $x-2$, since $2(x-2)=2x-4 \geq 2x-x=x$, thus yielding a larger or equal product. Hence all factors of P are 2's or 3's. If the factor 2 occurs more than twice, replace three 2's by two 3's , since $2+2+2=3+3$ yet $2^3<3^2$, so yielding a larger product.
Thus the factors of P must all be 2's or 3's with at most two 2's. Hence

$$P=\begin{cases} 3^k & \text{if } n=3k \\ 3^{k-1}\times 2^2 & \text{if } n=3k+1 \text{ (and } k>0) \\ 3^k \times 2 & \text{if } n=3k+2. \end{cases}$$

6. The incircle of the triangle ABC touches the sides BC, CA, AB at X, Y, Z respectively. Prove that the centre of the circle lies on the straight line through the midpoints of BC and of AX.



Solution. Set up a co-ordinate system with origin O at the centre of the incircle and x-axis parallel to BC (so then X is on the y-axis). Let K, L be the mid-points of BC, AX. Choose the scale so the incircle has equation $x^2+y^2=1$. Then $X=(0,-1)$. Let $B=(b,-1)$ and $C=(c,-1)$, and $A=(a,d)$.
Then $K=(\frac{1}{2}(c+b),-1)$ so the slope of OK is $-1/\frac{1}{2}(c+b)$. We aim to show that the slope of OL is also $-1/\frac{1}{2}(c+b)$, for this will ensure that O, K, L are collinear. Now $L=(\frac{1}{2}a,\frac{1}{2}(d-1))$, so the slope of OL is $(d-1)/a$. Thus we need to express a, d in terms of b, c.
Let the line AB have slope m, so its equation is $y+1=m(x-b)$, or $y=mx-mb-1$. This line meets the circle when $x^2+(mx-mb-1)^2=1$, that is when $(m^2+1)x^2-2m(mb+1)x+(mb+1)^2-1=0$. But it meets the circle in only the one point Z, so this quadratic equation must have only one solution; thus its $B^2=4AC$. So we have $4m^2(mb+1)^2=4(m^2+1)((mb+1)^2-1)$, that is $4(mb+1)^2-4(m^2+1)=0$, and so $m^2(b^2-1)+2mb=0$. Thus $m=0$ or $m=-2b/(b^2-1)$. But $m=0$ gives the line BC, so for AB $m=-2b/(b^2-1)$. Hence the line AB has equation $(b^2-1)(y+1)=-2b(x-b)$. Similarly the line AC has equation $(c^2-1)(y+1)=-2c(x-c)$. We could solve these for the co-ordinates of A. More easily, since $A=(a,d)$ is on the first, $(b^2-1)(d+1)=-2b(a-b)$. We want the value of $(d-1)/a$, so rewrite the equation to involve $d-1$: $(b^2-1)(d-1)=-2b(a-b)-2(b^2-1)=-2ba+2$. Similarly, from the equation of AC, $(c^2-1)(d-1)=-2ca+2$. Subtracting these last two equations gives $(b^2-c^2)(d-1)=-2a(b-c)$, so $(d-1)/a=-2a/(b+c)$. Hence OL and OK have the same slope, as required.

Solved by: Jonathon Ho.

7. A cube is assembled from 27 white cubes. The large cube is then painted black on the outside and disassembled. A blind man reassembles it. What is the probability that the cube is now completely black on the outside ?

Solution. The total number of ways of positioning the cubes (disregarding the orientation of each) is 27! . For any hope of an all black cube, the 6 small cubes with 1 black face must be the centre cubes of the outside faces (6! ways), and the 12 small cubes with 2 black faces must be the centre cubes of the outside edges (12! ways), and the 8 small cubes with 3 black faces must be at the outside corners (8! ways). Hence the probability of having the small cubes in the right positions to perhaps give an all black cube is 6!12!8!/27!.

Now we have to orient the small cubes so that their black faces are all showing. There are none to worry about on the centre cube. For each of the 6 small cubes with 1 black face, there is a probability of 1/6 that the black face is outwards, giving a combined probability of $(1/6)^6$ that they are all outwards. For each of the 12 small cubes with 2 black faces, to be oriented properly they must have their black edge along the edge of the big cube, and there is a probability of 1/12 that this happens, giving a combined probability of $(1/12)^{12}$ that they are all correct. And for each of the 8 small cubes with 3 black faces, they must all have their black corner at the corner of the big cube, with probability 1/8 that this happens, or $(1/8)^8$ that they are all correct. Thus the probability of correctly orienting the 27 small cubes, once they are in their proper positions, is $1/6^6 12^{12} 8^8$. Hence the probability of an all black big cube is

$$\frac{6!12!8!}{27!6^6 12^{12} 8^8} \approx 1.8298 \times 10^{-37}.$$

Solved by: Jonathon Ho, Margie Dickson.