

AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

QUEENSLAND PROGRAMME: PROBLEMS July 2009

1.  $C_1$  and  $C_2$  are circles with centres  $Z_1$  and  $Z_2$  respectively, which intersect in the points  $P$  and  $Q$ . Let  $R$  and  $S$  be two distinct points with the properties:
  - (i)  $R$  lies on the circle  $C_1$ ;
  - (ii)  $S$  lies on the circle  $C_2$ ; and
  - (iii) the angles  $PZ_1R$  and  $PZ_2S$  are both  $15^\circ$  when measured in the clockwise direction, starting at  $P$ .

Show that  $Q$ ,  $R$  and  $S$  are on the same line.

2. Determine all real numbers  $r$  such that there is precisely one pair  $(x, y)$  of real numbers satisfying the conditions:
  - (i)  $y - x = r$ ;
  - (ii)  $x^2 + y^2 + 2x \leq 1$ .
3. Each of the 36 line segments joining 9 distinct points on a circle is coloured either red or blue. Suppose that each triangle determined by 3 of the 9 points contains at least one red side. Prove that there are four points such that the 6 segments connecting them are all red.
4. (a) Prove that if  $x, y, z$  are non-negative real numbers, then

$$x(x - y)(x - z) + y(y - z)(y - x) + z(z - x)(z - y) \geq 0.$$

- (b) Hence or otherwise show that for all real numbers  $a, b, c$

$$a^6 + b^6 + c^6 + 3a^2b^2c^2 \geq 2(b^3c^3 + c^3a^3 + a^3b^3).$$

5. If  $S$  is a set of positive integers such that the sum of all the numbers in  $S$  is  $n$ , how large can the product of all the numbers in  $S$  be?
6. The incircle of the triangle  $ABC$  touches the sides  $BC$ ,  $CA$ ,  $AB$  at  $X$ ,  $Y$ ,  $Z$  respectively. Prove that the centre of the circle lies on the straight line through the midpoints of  $BC$  and of  $AX$ .
7. A cube is assembled from 27 white cubes. The large cube is then painted black on the outside and disassembled. A blind man reassembles it. What is the probability that the cube is now completely black on the outside?

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