AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE
QUEENSLAND PROGRAMME: SOLUTIONS April 2009

1. Let n be a positive integer. Prove that in any collection of $n+1$ distinct positive integers all less than or equal to $2n$ at least two of these are coprime.
[x and y are *coprime* if their greatest common divisor is 1.]

Solution. The easiest way is to observe that with $n+1$ numbers chosen from $\{1, 2, 3, \ldots, 2n\}$ there must be two consecutive numbers. (For otherwise, the numbers are at least two apart, so that the largest must must be at least $2n$ bigger than the smallest, and thus the largest chosen would have to be at least $2n+1$.) And two consecutive integers are coprime, for if y is a factor of both x and $x+1$ then $x = ky$ and $x+1 = \ell y$, thus $1 = y(\ell - k)$ so that $y = 1$.

Solved by: Mitchell Riley(Qld Academy for Science, Mathematics and Technology), Jonathon Ho(St Joseph's College, Gregory Terrace).

2. Prove that the expressions $11x+8y$ and $7x+2y$ are divisible by 17 for the same set of integral values of x and y.

Solution. Suppose $11x+8y$ is divisible by 17, say $11x+8y = 17k$. Then $y = (17k-11x)/8$, so $7x+2y = 7x+(17k-11x)/4 = (17k+17x)/4$. Hence $4(7x+2y) = 17(k+x)$. Thus 17 is a prime divisor of $4(7x+2y)$, and hence $7x+2y$ is divisible by 17 (since 17 is not a divisor of 4). Similarly, if we suppose $7x+2y$ is divisible by 17, say $7x+2y = 17\ell$, then $y = (17\ell - 7x)/2$, so $11x+8y = 11x+4(17\ell-7x) = 17(4\ell - x)$. Hence 17 is a divisor of $11x+8y$.

`Sneaky method.` Observe that

$$4(7x+2y) = 17x + (11x+8y).$$

Since 17 is not a divisor of 4, it follows immediately from this equation that 17 is a divisor of $7x+2y$ if and only if 17 is a divisor of $11x+8y$.

Solved by: James Hou(The Southport School).

3. Let ABC be a triangle, P and Q points exterior to ABC with triangles BAP and ACQ not overlapping ABC, and R a point in the interior of $\triangle ABC$. Show that if BAP, ACQ and BCR are similar isosceles triangles with bases AB, AC and BC respectively, then the quadrilateral $AQRP$ is a parallelogram.

Solution. Since $\triangle ACQ$ and $\triangle BCR$ are similar

$$CQ/CR = CA/CB. \qquad (1)$$

Also, since $\triangle BAP$ and $\triangle BCR$ are similar

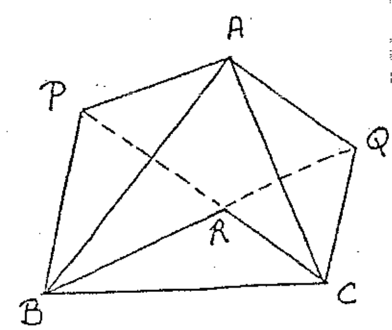
$$BP/BR = BA/BC. \qquad (2)$$

Since $\angle QCA = \angle RCB$,

$$\angle QCR = \angle ACB. \qquad (3)$$

Similarly,

$$\angle PBR = \angle ABC. \qquad (4)$$



By (1) and (3) triangles QRC and ABC have an equal angle with the corresponding sides in the same ratio. Hence they are similar. Likewise by (2) and (4) triangles ABC and PBR are similar. Hence triangles QRC and PBR are similar. But $\triangle BCR$ is isosceles. Therefore $BR = CR$ and triangles QRC and PBR in fact are congruent. So we have

$$PR = QC = AQ \qquad \text{and} \qquad QR = PB = PA.$$

Therefore $AQRP$ is a parallelogram.

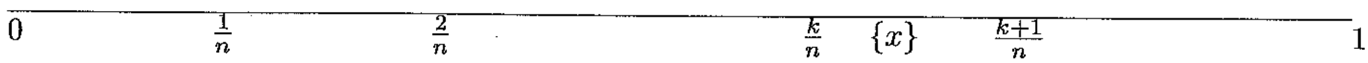
Solved by: James Hou, Jonathon Ho, Cynthia Wong(Somerville House).

4. Let $[a]$ denote the integer nearest to a which is less than or equal to it. (Thus $[a] \leq a < [a]+1$.) Prove the identity

$$[x]+[x+\frac{1}{n}]+[x+\frac{2}{n}]+\ldots+[x+\frac{n-1}{n}]=[nx],$$

for any positive integer n.

Solution. Write $\{x\} = x - [x]$, so $0 \leq x < 1$. ($\{x\}$ is called the *fractional part* of x.) Divide the interval from 0 to 1 into n equal parts. Then $\{x\}$ will be in one of these parts, say $\frac{k}{n} \leq \{x\} < \frac{k+1}{n}$, where k is an integer, $0 \leq k \leq n-1$.



For integers i with $0 \leq i \leq n-(k+1)$, (and there are $n-k$ such integers i),

$$0 \leq \frac{k}{n}+\frac{i}{n} \leq \{x\}+\frac{i}{n} < \frac{k+1}{n}+\frac{i}{n} = \frac{k+1+i}{n} \leq \frac{n}{n} = 1,$$

so $[\{x\}+\frac{i}{n}] = 0$ and hence $[x+\frac{i}{n}] = [[x]+\{x\}+\frac{i}{n}] = [x]+[\{x\}+\frac{i}{n}] = [x]$.
Also, for integers i with $n-k \leq i \leq n-1$, (and there are k such integers i),

$$1 = \frac{k}{n}+\frac{n-k}{n} \leq \frac{k}{n}+\frac{i}{n} \leq \{x\}+\frac{i}{n} < 1+1 = 2,$$

so $[\{x\}+\frac{i}{n}] = 1$ and hence $[x+\frac{i}{n}] = [x]+[\{x\}+\frac{i}{n}] = [x]+1$. Thus

$$[x]+[x+\frac{1}{n}]+[x+\frac{2}{n}]+\ldots+[x+\frac{n-1}{n}] = (n-k)[x]+k([x]+1) = n[x]+k.$$

On the other hand, $[nx] = [n[x]+n\{x\}] = n[x]+[n\{x\}]$, (since $n[x]$ *is* an integer). But from $\frac{k}{n} \leq \{x\} < \frac{k+1}{n}$, $k \leq n\{x\} < k+1$ so $[n\{x\}] = k$. Thus

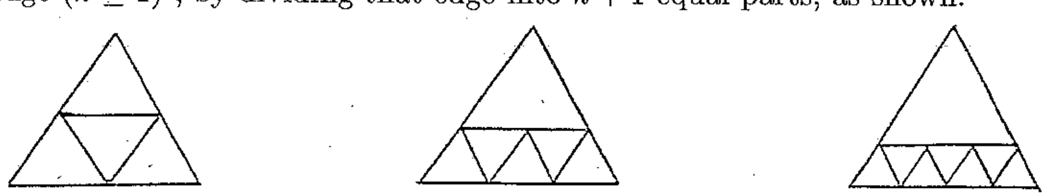
$$[nx] = n[x]+k = [x]+[x+\frac{1}{n}]+[x+\frac{2}{n}]+\ldots+[x+\frac{n-1}{n}],$$

as required.

Solved by: Jonathon Ho.

5. Find (with proof) the smallest integer N such that for every integer $m > N$ one can dissect an equilateral triangle into m smaller equilateral triangles (possibly of different sizes).

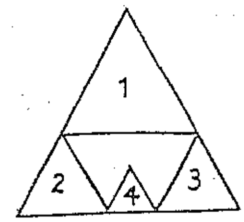
Solution. If you have an equilateral triangle, you can put a row of $2k+1$ equilateral triangles along one edge ($k \geq 1$) , by dividing that edge into $k+1$ equal parts, as shown.



$k=1$ $\quad$ $k=2$ $\quad$ $k=3$

This gives a subdivision into an even number $2k+2$ of triangles; thus $4, 6, 8, \ldots$ small triangles are possible. Take one of these subdivisions, and divide one of the triangles into 4; this increases the number of equilateral triangles in the subdivision by 3. Hence from the $2k+2$ subdivision you get one with $2k+5$ triangles; thus $7, 9, 11, \ldots$ small triangles are possible. So the numbers of small triangles in the dissections found so far are $4, 6, 7, 8, 9, 10, \ldots$.

However, there is no subdivision into 5 equilateral triangles. For if each side is divided into only 2 parts, you must have 4 equilateral triangles, and if at least one side is divided into 3 parts you get at least 4 triangles and a non-triangular region, which needs at least 2 triangles to cover it – giving at least 6 triangles. So 5 triangles cannot happen.



Thus $N = 5$ is the answer to the question.

Solved by: James Hou, Jonathon Ho, Cynthia Wong, Margie Dickson(Indooroopilly SHS).

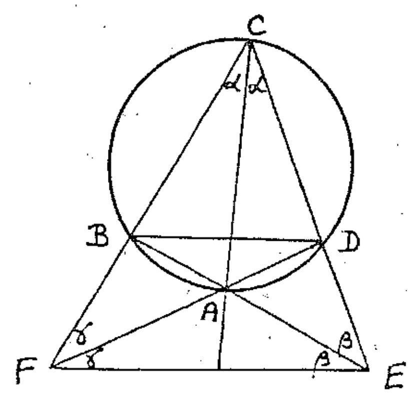
6. Let $ABCD$ be a cyclic quadrilateral. Let E be the intersection of BA (extended) and CD (extended), and let F be the intersection of CB (extended) and DA (extended). Suppose that A is the incentre of triangle CEF. (The incentre of a triangle is the point of intersection of its angle bisectors.)

Find the sizes of the angles in triangle ABD.

Solution. Let $\angle BCD = 2\alpha$, $\angle CEF = 2\beta$, $\angle CFE = 2\gamma$, so the angles are as shown on the diagram.

Now $\angle ADB = \angle ACB = \alpha$ (angles subtended by chord AB) and $\angle ABD = \angle ACD = \alpha$ (angles subtended by chord AD). Thus in triangle ABD, we have $\angle ABD = \angle ADB = \alpha$, and $\angle BAD = 180^{\text{o}} - 2\alpha$. So all we need to do is to determine α.

In triangle AEF, $\angle FAE = 180 - \beta - \gamma$. And $\angle BAD = \angle FAE$ (vertically opposite angles), so $180 - 2\alpha = 180 - \beta - \gamma$, and hence $2\alpha = \beta + \gamma$. From triangle BCE, $\angle EBC = 180 - 2\alpha - \beta$. From triangle FDC, $\angle FDC = 180 - 2\alpha - \gamma$. Since $ABCD$ is a cyclic quadrilateral, $\angle ABC + \angle ADC = 180$, i.e. $\angle EBC + \angle FDC = 180$. Thus $(180 - 2\alpha - \beta) + (180 - 2\alpha - \gamma) = 180$. Hence $4\alpha + \beta + \gamma = 180$, so $6\alpha = 180$ (since $2\alpha = \beta + \gamma$). Thus $\alpha = 30$. So the angles in the triangle ABD are 30^{o}, 30^{o} and 120^{o}.



Solved by: James Hou, Jonathon Ho, Cynthia Wong, Margie Dickson.

7. Consider functions f with the following properties:
(a) f is defined for all integers (only) and takes on real values;
(b) for all integers x, y, $f(x)f(y) = f(x+y) + f(x-y)$;
(c) $f(0) \neq 0$, $f(1) = 5/2$.
Show that there is a unique function f with these properties, and give the value of $f(x)$, for integer x.

Solution. Put $x = 1$ and $y = 0$ in (b) to find $f(1)f(0) = f(1) + f(1)$, and since $f(1) = 5/2 \neq 0$ we have $f(0) = 2$. (So being told that $f(0) \neq 0$ is unnecessary.) Now put $x = 0$ in (b) to find $f(0)f(y) = f(y) + f(-y)$ and so since $f(0) = 2$ we have $f(-y) = f(y)$. Hence we need only find $f(n)$ for positive integers n. Use (b) again with $x = n$ and $y = 1$ to find $f(n)f(1) = f(n+1) + f(n-1)$, so

$$f(n+1) = \tfrac{5}{2}f(n) - f(n-1). \qquad (*)$$

From (*) we can find the remaining values. Put $n = 1$ in (*) and we get $f(2) = \frac{5}{2} \times \frac{5}{2} - 2 = \frac{17}{4}$. Put $n = 2$ in (*) and we get $f(3) = \frac{5}{2} \times \frac{17}{4} - \frac{5}{2} = \frac{65}{8}$. Put $n = 3$ in (*) and we have $f(4) = \frac{5}{2} \times \frac{65}{8} - \frac{17}{4} = \frac{257}{16}$. Hence we guess that $f(n) = 2^n + 2^{-n}$ (for positive integers n).
Having made this educated guess, we now need to prove that it is correct. Here we can do this by induction. (See the notes I sent you on Induction.) Looking at the form of (*), I can see that the statement that I should actually try to prove by induction is

$$S(n) \ : \quad f(n) = 2^n + 2^{-n} \text{ and } f(n-1) = 2^{n-1} + 2^{-(n-1)}.$$

Since $f(1) = \frac{5}{2} = 2^1 + 2^{-1}$ and $f(0) = 2 = 2^0 + 2^{-0}$, certainly $S(0)$ is true.
Now suppose that $S(k)$ is true, and try to deduce that $S(k+1)$ must also be true. So suppose $f(k) = 2^k + 2^{-k}$ and $f(k-1) = 2^{k-1} + 2^{-(k-1)}$. Then by (*),

$$f(k+1) = \frac{5}{2}f(k) - f(k-1) = (2 + 2^{-1})(2^k + 2^{-k}) - (2^{k-1} + 2^{-(k-1)}) = 2^{k+1} + 2^{-(k+1)};$$

thus $f(k+1) = 2^{k+1} + 2^{-(k+1)}$ and $f(k) = 2^k + 2^{-k}$, so $S(k+1)$ would be true. Hence, by induction, $S(n)$ is true for all positive integers n. So in particular, $f(n) = 2^n + 2^{-n}$. Thus if (a), (b), (c) are to hold, the only possibility is $f(x) = 2^x + 2^{-x}$ for all integers x. And it is easy to check that $f(x) = 2^x + 2^{-x}$ does have these three properties, and hence it is the unique function satisfying (a), (b) and (c).

Solved by: James Hou, Jonathon Ho, Cynthia Wong, Mitchell Riley.