AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

QUEENSLAND PROGRAMME: PROBLEMS April 2009

1. Let n be a positive integer. Prove that in any collection of $n+1$ distinct positive integers all less than or equal to $2n$ at least two of these are coprime.
   (Integers x and y are *coprime* if their greatest common divisor is 1.)

2. Prove that the expressions 11x + 8y and 7x + 2y are divisible by 17 for the same set of integral values of x and y.

3. Let ABC be a triangle, P and Q points exterior to ABC with triangles BAP and ACQ not overlapping ABC, and R a point in the interior of ABC. Show that if BAP, ACQ and BCR are similar isosceles triangles with bases AB, AC and BC respectively, then the quadrilateral $AQRP$ is a parallelogram.

4. Let $[a]$ denote the integer nearest to a which is less than or equal to it. (Thus $[a] \leq a < [a]+1$.) Prove the identity

   $$[x] + [x + \frac{1}{n}] + [x + \frac{2}{n}] + \ldots + [x + \frac{n-1}{n}] = [nx],$$

   for any positive integer n.

5. Find (with proof) the smallest integer N such that for every integer $m > N$ one can dissect an equilateral triangle into m smaller equilateral triangles (possibly of different sizes).

6. Let $ABCD$ be a cyclic quadrilateral. Let E be the intersection of BA (extended) and CD (extended), and let F be the intersection of CB (extended) and DA (extended). Suppose that A is the incentre of triangle CEF. (The incentre of a triangle is the point of intersection of its angle bisectors.)

   Find the sizes of the angles in triangle ABD.

7. Consider functions f with the following properties:
   (a) f is defined for all integers (only) and takes on real values;
   (b) for all integers x, y, $\quad f(x)f(y) = f(x+y) + f(x-y)$;
   (c) $f(0) \neq 0, \quad f(1) = 5/2$.

   Show that there is a unique function f with these properties, and give the value of $f(x)$, for integer x.

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