q-HYPERGEOMETRIC PROOFS OF POLYNOMIAL ANALOGUES OF THE TRIPLE PRODUCT IDENTITY, LEBESGUE'S IDENTITY AND EULER'S PENTAGONAL NUMBER THEOREM

S. OLE WARNAAR

Abstract. We present alternative, q-hypergeometric proofs of some polynomial analogues of classical q-series identities recently discovered by Alladi and Berkovich, and Berkovich and Garvan.

1. Introduction

In two recent papers, Alladi and Berkovich [2] and Berkovich and Garvan [4] proved the following three polynomial identities:

$$\sum_{n=0}^{L} \frac{z^{-n}+z^{1+n}}{1+z}\, q^{T_n} = \sum_{i,j,k\geq 0} (-1)^k z^{i-j} q^{T_i+T_j+T_k} \begin{bmatrix} L-i \\ j \end{bmatrix} \begin{bmatrix} L-j \\ k \end{bmatrix} \begin{bmatrix} L-k \\ i \end{bmatrix}, \tag{1}$$

$$\sum_{i,j\geq 0} (-1)^j z^{2j} q^{T_i+T_j} \begin{bmatrix} L-j \\ i \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} = \sum_{i,j,k\geq 0} (-1)^j z^{i+j} q^{T_i+T_j+T_k} \begin{bmatrix} L-i \\ j \end{bmatrix} \begin{bmatrix} L-j \\ k \end{bmatrix} \begin{bmatrix} L-k \\ i \end{bmatrix} \tag{2}$$

and

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j+1)/2} \begin{bmatrix} 2L-j \\ L+j \end{bmatrix} = 1. \tag{3}$$

Here $T_n = n(n+1)/2$ is a triangular number and $\begin{bmatrix} n \\ k \end{bmatrix}$ a q-binomial coefficient defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

for $0 \leq k \leq n$ and zero otherwise, with $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$.

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Identity (1), which is [2, Eq. (1.12)] and which was first stated in [1], is a polynomial analogue of Jacobi's triple product identity. Identity (2), which is [2, Eq. (1.15)], is a polynomial analogue of Lebesgue's identity and (3), which is [4, Eq. (1.28)], is a polynomial analogue of Euler's pentagonal number theorem and member of an infinite family of such identities obtained in [4].

The first explicit proof of (1) was given by Berkovich and Riese [5] who showed that both sides satisfy a highly nontrivial fourth order recurrence relation. The proof of (1) given by Alladi and Berkovich uses transformation formulas for basic hypergeometric series. In particular they show that the right side of (1) can be transformed into the left side using, consecutively, Heine's ${}_2\phi_1$ transformation, the q-Chu–Vandermonde sum and the Sears–Carlitz transformation between ${}_3\phi_2$ and ${}_5\phi_4$ series. In this note we present a simple, one-page proof of (1) that only requires elementary summations and no transformations. As a bonus we find that (1) is the $c=0$ instance of

$$\sum_{n=-L}^{L}\sum_{m=-L}^{n}(-1)^{n+m}z^m q^{T_n}\frac{(cq;q)_{L-m}}{(cq;q)_{L-n}(cq;q)_{n-m}} = \sum_{i,j,k\geq 0}(-1)^k z^{i-j}q^{T_i+T_j+T_k}\frac{1-c}{1-cq^j}\begin{bmatrix}L-i\\ j\end{bmatrix}\begin{bmatrix}L-j\\ k\end{bmatrix}\begin{bmatrix}L-k\\ i\end{bmatrix} \tag{4}$$

and of

$$\sum_{n=0}^{L}\sum_{m=-n}^{n}(-1)^{n+m}z^m q^{T_n}\frac{(cq;q)_{L-|m|}}{(cq;q)_{L-n}(cq;q)_{n-|m|}} = \sum_{i,j,k\geq 0}(-1)^k z^{i-j}q^{T_i+T_j+T_k}\frac{1-c}{1-cq^{\min(i,j)}}\begin{bmatrix}L-i\\ j\end{bmatrix}\begin{bmatrix}L-j\\ k\end{bmatrix}\begin{bmatrix}L-k\\ i\end{bmatrix}. \tag{5}$$

Letting c tend to infinity and performing the sum over m leads to the following two variants of (1):

$$\sum_{n=0}^{2L}\frac{(-1)^n z^{-L}q^{n(2L-n+1)}+z^{L-n+1}}{q^n+z}\,q^{T_{L-n}} = \sum_{i,j,k\geq 0}(-1)^k z^{i-j}q^{T_i+T_j+T_k-j}\begin{bmatrix}L-i\\ j\end{bmatrix}\begin{bmatrix}L-j\\ k\end{bmatrix}\begin{bmatrix}L-k\\ i\end{bmatrix}$$

and

$$\sum_{n=0}^{L}\left\{\frac{(-1)^n q^{(n+1)(L-n)}+z^{n+1}}{q^{L-n}+z}+\frac{(-1)^{n+1}q^{n(L-n)}+z^{-n}}{1+zq^{L-n}}\right\}q^{T_n} = \sum_{i,j,k\geq 0}(-1)^k z^{i-j}q^{T_i+T_j+T_k-\min(i,j)}\begin{bmatrix}L-k\\ i\end{bmatrix}\begin{bmatrix}L-j\\ k\end{bmatrix}\begin{bmatrix}L-i\\ j\end{bmatrix}.$$

The proof of (2) as found by Alladi and Berkovich [2] is combinatorial, based on a bounded version of the Göllnitz's big partition theorem. They however ask for a q-hypergeometric proof of (2), or more precisely, for a q-hypergeometric proof of the closely related [2, Eq. (6.14)]

$$\sum_{j\geq 0}(z^2q^2;q^2)_j\, q^{\binom{L-2j}{2}}\begin{bmatrix}L+1\\2j+1\end{bmatrix} = \sum_{i,j,k\geq 0}(-1)^j z^{i+j}q^{T_i+T_j+T_k}\begin{bmatrix}L-i\\j\end{bmatrix}\begin{bmatrix}L-j\\k\end{bmatrix}\begin{bmatrix}L-k\\i\end{bmatrix}. \tag{6}$$

Here we present such a proof of (2) and (6).

Identity (3) was discovered and proved by Berkovich and Garvan by finitizing Dyson's proof of Euler's identity using the rank of a partition. Again it is posed as a problem to find a q-hypergeometric proof. Here we show that (3) is a simple consequence of a known cubic summation formula. This same cubic sum may be applied to also yield

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j-1)/2}\begin{bmatrix}2L-j+1\\L+j\end{bmatrix}=1. \tag{7}$$

2. Proofs

In the following we adopt the notation of [7] for basic hypergeometric series, writing

$$\begin{aligned}{}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\b_1,\dots,b_r\end{matrix};q,z\right] &= {}_{r+1}\phi_r(a_1,a_2,\dots,a_{r+1};b_1,\dots,b_r;q,z)\\ &=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_k}{(q,b_1,\dots,b_r;q)_k}\,z^k,\end{aligned}$$

where $(a_1,a_2,\dots,a_k;q)_n=(a_1;q)_n\cdots(a_k;q)_n$ and $|q|,|z|<1$ if the sum is nonterminating. Whenever a basic hypergeometric identity occurs with a term q^{-n}, n is assumed to be a nonnegative integer.

2.1. **Proof of** (1)**.** Using $(1+z^{2n+1})/(1+z)=\sum_{m=0}^{2n}(-z)^m$, interchanging sums and shifting $n\to n+|m|$, the left side of (1) can be written as

$$\mathrm{LHS}(1)=\sum_{m=-L}^{L}\sum_{n=0}^{L-|m|}(-1)^n z^m q^{T_{n+|m|}}. \tag{8}$$

If on the right of (1) we eliminate i in favour of m by $i=m+j$ we can use (8) to equate coefficients of z^m in (1). This leads to an identity, denoted $(*)$, which should hold for all m and L such that $|m|\leq L$. After dividing both sides by $q^{T_{|m|}}$, $(*)$ can easily be seen to be the $b=q^{|m|+1}$ and $N=L-|m|$

instance of

$$(9)\qquad \sum_{n=0}^{N}(-1)^n b^n q^{\binom{n}{2}} = \sum_{j=0}^{\lfloor N/2\rfloor}\sum_{k=0}^{N-j} \frac{(-1)^k b^j q^{j^2+T_k}(q,b;q)_{N-j}(b;q)_{N-k}}{(q,b;q)_j(q,b;q)_{N-j-k}(q;q)_k(q;q)_{N-2j}}.$$

We now take the right-hand side and make the variable change $k\to N-j-k$. Then

$$\begin{aligned}
\mathrm{RHS}(9) &= \sum_{j=0}^{\lfloor N/2\rfloor}\frac{(-1)^{N-j}b^j q^{j^2+T_{N-j}}(b;q)_{N-j}}{(q;q)_j(q;q)_{N-2j}}\,{}_2\phi_1\left[\begin{matrix} bq^j, q^{-N+j} \\ b\end{matrix};q,1\right] \\
&= \sum_{j=0}^{\lfloor N/2\rfloor} b^j q^{j^2}(bq^j;q)_{N-2j}\begin{bmatrix} N-j \\ j\end{bmatrix} \\
&= \sum_{j=0}^{\lfloor N/2\rfloor}\sum_{n=0}^{N-2j}(-1)^n b^{j+n} q^{j(j+n)+\binom{n}{2}}\begin{bmatrix} N-j \\ j\end{bmatrix}\begin{bmatrix} N-2j \\ n\end{bmatrix} \\
&= \sum_{n=0}^{N}(-1)^n b^n q^{\binom{n}{2}}\begin{bmatrix} N \\ n\end{bmatrix}{}_2\phi_1\left[\begin{matrix} q^{-N+n}, q^{-n} \\ q^{-N}\end{matrix};q,q\right] \\
&= \sum_{n=0}^{N}(-1)^n b^n q^{\binom{n}{2}}\begin{bmatrix} N \\ n\end{bmatrix}\frac{(q^{-n};q)_n}{(q^{-N};q)_n}q^{-(N-n)n} = \mathrm{LHS}(9).
\end{aligned}$$

Here the second equality follows from

$$(10)\qquad {}_2\phi_1(aq^m,q^{-n};a;q,1) = \frac{(q^{-n};q)_n}{(a;q)_m}\qquad 0\le m\le n,$$

the third equality follows from the q-binomial theorem [7, Eq. (II.4)]

$$(11)\qquad {}_1\phi_0(q^{-n};—;q,z) = (zq^{-n};q)_n,$$

the fourth equality follows from the shift $n\to n-j$ followed by an interchange of sums, and the second-last equality follows from the q-Chu–Vandermonde sum [7, Eq. (II.6)]

$$(12)\qquad {}_2\phi_1(a,q^{-n};c;q,q) = \frac{(c/a;q)_n}{(c;q)_n}\,a^n.$$

To prove (10), multiply both sides by the denominator on the right to get

$$\sum_{k=0}^{n}\frac{(aq^k;q)_m(q^{-n};q)_k}{(q;q)_k} = (q^{-n};q)_n.$$

Expanding $(aq^k;q)_m$ by (11) one can, again by (11), perform the sum over k to find

$$\sum_{j=0}^{m}\frac{(q^{-m};q)_j(q^{j-n};q)_n a^j}{(q;q)_j} = (q^{-n};q)_n.$$

Since $(q^{j-n};q)_n=0$ for $1\le j\le n$ and since $0\le j\le m\le n$ the only contributing term in the sum corresponds to $j=0$.

By replacing the q-Chu–Vandermonde sum (12) with the more general q-Saalschütz sum [7, Eq. (II.12)], the above proof generalizes to yield

$$\sum_{n=0}^{N}(-1)^n b^n q^{\binom{n}{2}} \frac{(cq;q)_N}{(cq;q)_n(cq;q)_{N-n}} \\ = \sum_{j=0}^{\lfloor N/2\rfloor}\sum_{k=0}^{N-j} \frac{(-1)^k b^j q^{j^2+T_k}(c;q)_j(q,b;q)_{N-j}(b;q)_{N-k}}{(q,b,cq;q)_j(q,b;q)_{N-j-k}(q;q)_k(q;q)_{N-2j}},$$

which for $c=0$ reduces to (9). Taking $b=q^{m+1}$ ($b=q^{|m|+1}$) and $N=L-m$ ($N=L-|m|$), multiplying both sides by b^m and summing m from $-L$ to L gives (4) ((5)).

2.2. **Proof of** (2) **and** (6)**.** First consider the left-hand side of (2). Shifting $i\to i+j$ this becomes

$$\text{(13)}\quad \begin{aligned} \mathrm{LHS}(2) &= \sum_{j=0}^{\lfloor L/2\rfloor}(-1)^j z^{2j} q^{j(j+1)} \begin{bmatrix} L-j \\ j\end{bmatrix} {}_1\phi_0(q^{-L+2j};—;q,-q^{L-j+1}) \\ &= \sum_{j=0}^{\lfloor L/2\rfloor}(-1)^j z^{2j} q^{j(j+1)} \begin{bmatrix} L-j \\ j\end{bmatrix} \frac{(-q;q)_{L-j}}{(-q;q)_j}, \end{aligned}$$

where the second equality follows from (11).

Next consider the right-hand side of (2). By reshuffling the terms that make up the three q-binomial coefficients, it readily follows that the summand $S_{L;i,j,k}$ on the right satisfies $S_{L;i,j,k}=(-1)^{i+j}S_{L;j,i,k}$. Hence all contributions to the sum arising from $i+j$ being odd cancel, and we may add the restriction "$i+j$ even" to the sum. By replacing $j\to 2j-i$ we can then extract the coefficient of z^{2j}. Equating this with the coefficient of z^{2j} arising from (13) results in the identity

$$\text{(14)}\quad \sum_{i,k\geq 0}\frac{(-1)^{i-j}q^{(i-j)^2+T_k}}{(-q;q)_{L-2j}} \begin{bmatrix} L-i \\ 2j-i\end{bmatrix}\begin{bmatrix} L+i-2j \\ k\end{bmatrix}\begin{bmatrix} L-k \\ i\end{bmatrix} = \begin{bmatrix} L-j \\ j\end{bmatrix}_{q^2}$$

for $0 \le j \le \lfloor L/2 \rfloor$. This is easily proved as follows:

$$\begin{aligned}
\mathrm{LHS}(14) &= \begin{bmatrix} L \\ 2j \end{bmatrix} \sum_{i=0}^{2j} \frac{(-1)^{i-j} q^{(i-j)^2}}{(-q;q)_{L-2j}} \begin{bmatrix} 2j \\ i \end{bmatrix} {}_2\phi_1 \left[\begin{matrix} q^{-L+i}, q^{-L-i+2j} \\ q^{-L} \end{matrix} ; q, -q^{L-2j+1} \right] \\
&= \begin{bmatrix} L \\ 2j \end{bmatrix} \sum_{i=0}^{2j} (-1)^{i-j} q^{(i-j)^2} \begin{bmatrix} 2j \\ i \end{bmatrix} {}_2\phi_1 \left[\begin{matrix} q^{-i}, q^{-2j+i} \\ q^{-L} \end{matrix} ; q, -q \right] \\
&= (-1)^j q^{j^2} \begin{bmatrix} L \\ 2j \end{bmatrix} \sum_{k=0}^{j} \frac{(q^{-2j};q)_{2k}\, q^k}{(q, q^{-L};q)_k} \sum_{i=0}^{2j-2k} \frac{q^{T_i} (q^{-2j+2k};q)_i}{(q;q)_i} \\
&= (-1)^j q^{j^2} \begin{bmatrix} L \\ 2j \end{bmatrix} \sum_{k=0}^{j} \frac{(q^{-2j};q)_{2k}\, q^k}{(q, q^{-L};q)_k} (q^{1-2j+2k};q^2)_{j-k} \\
&= (q;q^2)_j \begin{bmatrix} L \\ 2j \end{bmatrix} {}_2\phi_1 \left[\begin{matrix} q^{-j}, -q^{-j} \\ q^{-L} \end{matrix} ; q, q \right] \\
&= (q;q^2)_j \begin{bmatrix} L \\ 2j \end{bmatrix} \frac{(-q^{-L+j};q)_j}{(q^{-L};q)_j} (-1)^j q^{-j^2} = \mathrm{RHS}(14).
\end{aligned}$$

Here the second equality follows from Heine's transformation [7, Eq. (III.3)]

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(abz/c;q)_\infty}{(z;q)_\infty} \, {}_2\phi_1(c/a, c/b; c; q, abz/c), $$

the third equality follows from an interchange of sums and the shift $i \to i+k$, the fourth equality follows by specializing $a = q^{-2j+2k}$ in Lebesgue's identity [3, Cor. 2.7]

$$ \sum_{i=0}^{\infty} \frac{q^{T_i}(a;q)_i}{(q;q)_i} = \frac{(aq;q^2)_\infty}{(q;q^2)_\infty}, $$

and the second-last equality follows from (12).

To now also prove (6) we only need to show its left-hand side equals the left-hand side of (2). To achieve this we expand $(z^2q^2;q^2)_j$ using (11), expressing the left side of (6) as a double sum over j and k. Interchanging the order of these sums and shifting $j \to j+k$ leads to

$$\begin{aligned}
\mathrm{LHS}(6) = \sum_{k=0}^{\lfloor L/2 \rfloor} (-1)^k z^{2k} q^{k(k+1)+\binom{L-2k}{2}} &\begin{bmatrix} L+1 \\ 2k+1 \end{bmatrix} \\
&\times {}_2\phi_1 \left[\begin{matrix} q^{-L+2k}, q^{1-L+2k} \\ q^{2k+3} \end{matrix} ; q^2, q^2 \right].
\end{aligned}$$

The ${}_2\phi_1$ can be evaluated by (12) with $n = \lfloor L/2 \rfloor - k$ yielding

$$ \frac{(q^2;q^2)_{L-k} (q;q)_{2k+1}}{(q;q)_{L+1} (q^2;q^2)_k} \, q^{-\binom{L-2k}{2}}. $$

Hence

$$\mathrm{LHS}(6)=\sum_{k=0}^{\lfloor L/2\rfloor}(-1)^k z^{2k}q^{k(k+1)}\frac{(q^2;q^2)_{L-k}}{(q^2;q^2)_k(q;q)_{L-2k}}.$$

Since this is identical to the expression for the left-hand side of (2) found in (13) we are done.

2.3. **Proof of** (3) **and** (7)**.** From [7, Eq. (3.8.19); $c\to bq^{-n-1}$] or [9, Cor. 4.13; $p\to 0$] we get the following cubic summation

$$\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1-aq^{4k}}{1-a}\frac{(a,aq^{n+1};q^3)_k}{(q,q^{-n};q)_k}\frac{(q^{-n};q)_{2k}}{(aq^{n+1};q)_{2k}}\frac{(c,d;q)_k}{(aq^3/c,aq^3/d;q^3)_k}q^k$$
$$=\begin{cases}\dfrac{(aq^3,q^{2-n}/c,q^{2-n}/d;q^3)_{\lfloor n/3\rfloor}}{(aq^3/c,aq^3/d,q^{2-n}/cd;q^3)_{\lfloor n/3\rfloor}} & n\not\equiv 2 \pmod 3\\ 0 & n\equiv 2\pmod 3\end{cases}$$

for $cd=aq^{n+1}$. Replacing $a\to a^2$ followed by $c\to ac$ and $d\to ad$, and then letting a tend to zero yields

$$\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(q^{-n};q)_{2k}}{(q,q^{-n};q)_k}q^k=\begin{cases}(-1)^{\lfloor n/3\rfloor}q^{-n(n-1)/6} & n\not\equiv 2\pmod 3\\ 0 & n\equiv 2\pmod 3.\end{cases}\tag{15}$$

Taking $n=3L$, replacing $k\to j+L$ and making some simplifications gives (3). In much the same way does $n=3L+1$ lead to (7). Making the same replacements for a,b and c but letting a tend to infinity instead of zero results in

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n-k\\k\end{bmatrix}=\begin{cases}(-1)^{\lfloor n/3\rfloor}q^{n(n-1)/6} & n\not\equiv 2\pmod 3\\ 0 & n\equiv 2\pmod 3\end{cases}\tag{16}$$

which is equivalent to (15) as can be seen by replacing $q\to 1/q$. The identity (16) was proven previously by Ekhad and Zeilberger [6] and by Krattenthaler [8], the latter proof being essentially the one presented here.

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Department of Mathematics and Statistics, The University of Melbourne, Vic 3010, Australia
E-mail address: warnaar@ms.unimelb.edu.au