DEDEKIND'S η-FUNCTION AND ROGERS–RAMANUJAN IDENTITIES

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Abstract. We prove a q-series identity that generalises Macdonald's $\mathrm{A}_{2n}^{(2)}$ η-function identity and the Rogers–Ramanujan identities. We conjecture our result to generalise even further to also include the Andrews–Gordon identities.

1. Introduction

In 1972 Macdonald published his seminal paper [27] in which he extended Weyl's denominator formula for classical reduced root systems to root systems of affine type. These identities, which include the Jacobi triple product identity and the quintuple product identity as the special cases $\mathrm{A}_1^{(1)}$ and $\mathrm{A}_2^{(2)}$, are now commonly known as the Macdonald identities. Through the procedure of specialisation the Macdonald identities imply identities for powers of the Dedekind η-function

$$\eta(\tau)=q^{1/24}\prod_{j=1}^{\infty}(1-q^j),$$

where $q=\exp(2\pi\mathrm{i}\tau)$ and $\mathrm{Im}(\tau)>0$. For example, the specialisation [27, p. 138, (6)(c)] of the $\mathrm{A}_{2n}^{(2)}$ Macdonald identity corresponds the following beautiful generalisation of the Euler pentagonal number theorem:

$$\eta(\tau)^{2n^2-n}=\sum \xi(\boldsymbol{v}/\boldsymbol{\rho})(-1)^{|\boldsymbol{v}|-|\boldsymbol{\rho}|}q^{\|\boldsymbol{v}\|^2/(2(2n+1))}. \tag{1.1}$$

Here $\boldsymbol{v}=(v_1,\dots,v_n)$, $\boldsymbol{\rho}=(1/2,3/2,\dots,n-1/2)$, $|\boldsymbol{v}|=v_1+\dots+v_n$, $\|\boldsymbol{v}\|^2=\boldsymbol{v}\cdot\boldsymbol{v}=v_1^2+\dots+v_n^2$,

$$\xi(\boldsymbol{v}/\boldsymbol{w})=\prod_{1\le i<j\le n}\frac{v_i^2-v_j^2}{w_i^2-w_j^2}$$

and the sum on the right of (1.1) is over $\boldsymbol{v}\in(\mathbb{Z}/2)^n$ such that $v_i\equiv\rho_i \pmod{2n+1}$.

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Another famous family of combinatorial identities are the formulae of Rogers and Ramanujan [2, 32]

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m} = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q)_\infty} \tag{1.2a}$$

$$\sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q)_m} = \frac{(q, q^4, q^5; q^5)_\infty}{(q)_\infty} \tag{1.2b}$$

and their generalisations to arbitrary odd moduli due to Andrews and Gordon [1, 11]

$$\sum_{m_1,\dots,m_{k-1}} \frac{q^{M_1^2+\dots+M_{k-1}^2+M_p+\dots+M_{k-1}}}{(q)_{m_1}\cdots(q)_{m_{k-1}}} = \frac{(q^p, q^{2k-p+1}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}, \tag{1.3}$$

where $1 \leq p \leq k$ and $M_i = m_i + \dots + m_{k-1}$. In (1.2) and (1.3) we employ the standard q-notation

$$(a)_m = (a;q)_m = \prod_{i=1}^{m} (1 - aq^{i-1})$$

and

$$(a_1, \dots, a_k)_m = (a_1, \dots, a_k; q)_m = (a_1;q)_m \cdots (a_k;q)_m$$

for $m \in \mathbb{N} \cup \{\infty\}$ (with the convention that $\mathbb{N} = \{0, 1, 2, \dots\}$).

In this paper we link Macdonald's identity (1.1) to the Rogers–Ramanujan and Andrews–Gordon identities (1.2) and (1.3). More specifically, we present a family of q-series identities depending on positive integers k, n and $p \in \{1, k\}$ such that

(1) For $k = 1$ we recover Macdonald's $\mathrm{A}_{2n}^{(2)}$ identity (1.1) for the Dedekind η-function.
(2) For $n = 1$ and $k = 2$ we recover, modulo the Jacobi triple product identity, the Rogers–Ramanujan identities (1.2).
(3) For $n = 1$ and general k we recover the $p = k$ and $p = 1$ instances of the Andrews–Gordon identities (1.3).
(4) For general n and $k \to \infty$ we recover the A_{2n-1} case of an identity of Hua related to representations of quivers.

The fact that the Rogers–Ramanujan identities have a close connection with affine root systems or, more generally, affine Kac–Moody algebras is not new, and well-known are the interpretations of (1.2) and (1.3) in terms of standard modules of $\mathrm{A}_1^{(1)}$, see e.g., [23, 24, 25, 26, 30] and as characters corresponding to certain non-unitary Virasoro modules, see e.g., [7, 31]. For our generalisation of the Rogers–Ramanujan and Andrews–Gordon identities, however, it is crucial to interpret the right-hand sides of (1.2) and (1.3) as of type type $\mathrm{A}_2^{(2)}$, not $\mathrm{A}_1^{(1)}$.

Before stating our main results we observe that, by an appeal to Jacobi's triple product identity [10, (II.28)], the right-hand side of (1.3) may be rewritten as

$$\frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}(-1)^j q^{(2k+1)\binom{j}{2}+pj} = \frac{q^{-(2k-2p+1)^2/(8(2k+1))}}{(q)_\infty}\sum(-1)^{v-k+p-1/2}q^{v^2/(2(2k+1))},$$

where on the right the sum is over $v\in\mathbb{Z}/2$ such that $v\equiv k-p+1/2 \pmod{2k+1}$. Comparing this sum with that in (1.1) it takes little imagination to make the following conjecture.

For $1\leq a,b\leq N-1$, let C_{ab} be the Cartan integers of the Lie algebra A_{N-1}, i.e., $C_{aa}=2$, $C_{a,a\pm1}=-1$ and $C_{ab}=0$ otherwise. By abuse of notation, for $\boldsymbol{w}=(w_1,\dots,w_n)$ and a a scalar, set $\boldsymbol{w}+a=(w_1+a,\dots,w_n+a)$ so that, in particular,

$$\|\boldsymbol{w}+a\|^2=\|\boldsymbol{w}\|^2+2a|\boldsymbol{w}|+na^2 \quad\text{and}\quad |\boldsymbol{w}+a|=|\boldsymbol{w}|+na.$$

Conjecture 1.1. *For k,n positive integers, $N=2n$ and $p\in\{1,k\}$,*

$$\sum\frac{q^{\frac12\sum_{a,b=1}^{N-1}\sum_{i=1}^{k-1}C_{ab}M_i^{(a)}M_i^{(b)}+\sum_{a=1}^{N-1}\sum_{i=p}^{k-1}(-1)^aM_i^{(a)}}}{\prod_{a=1}^{N-1}\prod_{i=1}^{k-1}(q)_{m_i^{(a)}}} = \frac{1}{(q)_\infty^{2n^2-n}}\sum\xi(\boldsymbol{v}/\boldsymbol{\rho})(-1)^{|\boldsymbol{v}|-|\boldsymbol{\rho}+k-p|}q^{\frac{\|\boldsymbol{v}\|^2-\|\boldsymbol{\rho}+k-p\|^2}{2(2k+2n-1)}}, \tag{1.4}$$

where the sum on the left is over $m_i^{(a)}\in\mathbb{N}$ (for all $1\leq a\leq N-1$ and $1\leq i\leq k-1$) and the sum on the right is over $\boldsymbol{v}\in(\mathbb{Z}/2)^n$ such that $v_i\equiv\rho_i+k-p \pmod{2k+2n-1}$. The integers $M_i^{(a)}$ are defined as $M_i^{(a)}=m_i^{(a)}+\dots+m_{k-1}^{(a)}$, i.e., $m_i^{(a)}=M_i^{(a)}-M_{i+1}^{(a)}$ for $1\leq i\leq k-2$ and $m_{k-1}^{(a)}=M_{k-1}^{(a)}$.

Theorem 1.2 (Generalised Rogers–Ramanujan identities)**.** *Conjecture 1.1 is true for $k=2$. That is, for n a positive integer and $N=2n$,*

$$\sum_{\boldsymbol{m}\in\mathbb{N}^{N-1}}\frac{q^{\frac12\boldsymbol{m}C\boldsymbol{m}^t}}{(q)_{\boldsymbol{m}}}=\frac{1}{(q)_\infty^{2n^2-n}}\sum\xi(\boldsymbol{v}/\boldsymbol{\rho})(-1)^{|\boldsymbol{v}|-|\boldsymbol{\rho}|}q^{\frac{\|\boldsymbol{v}\|^2-\|\boldsymbol{\rho}\|^2}{2(2n+3)}}, \tag{1.5}$$

where the sum on the right is over $\boldsymbol{v} \in (\mathbb{Z}/2)^n$ such that $v_i \equiv \rho_i \pmod{2n+3}$, and

$$\sum_{\boldsymbol{m}\in\mathbb{N}^{N-1}} \frac{q^{\frac{1}{2}\boldsymbol{m}C\boldsymbol{m}^t+|\boldsymbol{m}|_-}}{(q)_{\boldsymbol{m}}} = \frac{1}{(q)_\infty^{2n^2-n}} \sum \xi(\boldsymbol{v}/\boldsymbol{\rho})(-1)^{|\boldsymbol{v}|-|\boldsymbol{\rho}+1|} q^{\frac{\|\boldsymbol{v}\|^2-\|\boldsymbol{\rho}+1\|^2}{2(2n+3)}},$$

where the sum on the right is over $\boldsymbol{v} \in (\mathbb{Z}/2)^n$ such that $v_i \equiv \rho_i + 1 \pmod{2n+3}$. In the above $(q)_{\boldsymbol{m}} = (q)_{m_1} \dots (q)_{m_{N-1}}$, C is the Cartan matrix of A_{N-1}, i.e.,

$$\frac{1}{2}\boldsymbol{m}C\boldsymbol{m}^t = \sum_{i=1}^{N-1} m_i^2 - \sum_{i=1}^{N-2} m_i m_{i+1},$$

and, for $\boldsymbol{m} \in \mathbb{N}^{N-1}$,

$$|\boldsymbol{m}|_- = \sum_{i=1}^{N-1} (-1)^{i-1} m_i.$$

Analogous to the above theorem, the left-hand side of (1.4) can be expressed without the use of indices by introducing the square matrix B of dimension $d := (N-1)(k-1)$ given by the Kronecker product of the Cartan matrix C of A_{N-1} and the $(k-1)\times(k-1)$ matrix T^{-1} with entries $(T^{-1})_{ij} = \min\{i,j\}$:

$$B_{ai,bj} = (C \otimes T^{-1})_{ai,bj} = C_{ab} \min\{i,j\}.$$

For example, the $k = p$ instance of (1.4) generalises (1.5) to

$$\sum_{\boldsymbol{m}\in\mathbb{N}^d} \frac{q^{\frac{1}{2}\boldsymbol{m}B\boldsymbol{m}^t}}{(q)_{\boldsymbol{m}}} = \frac{1}{(q)_\infty^{2n^2-n}} \sum \xi(\boldsymbol{v}/\boldsymbol{\rho})(-1)^{|\boldsymbol{v}|-|\boldsymbol{\rho}|} q^{\frac{\|\boldsymbol{v}\|^2-\|\boldsymbol{\rho}\|^2}{2(2k+2n-1)}}. \tag{1.6}$$

We should remark that the expression on the left-hand side of (1.6) has a representation theoretic interpretation due to Feigin and Stoyanovsky [8]. Their interpretation in fact holds for $B = C \otimes T^{-1}$ where C is a Cartan matrix of any semi-simple simply laced Lie algebra $\mathfrak{g}$. Let $\hat{\mathfrak{g}}$ be the (nontwisted) affine counterpart of $\mathfrak{g}$ and V_l the level-l vacuum integrable highest weight module of $\mathfrak{g}$ with vacuum vector v. Then W_l is the space $W_l = U(\hat{\mathfrak{n}}_+) \cdot v_0 \subset V_l$, with U the universal enveloping algebra and $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ the Cartan decomposition of $\mathfrak{g}$. The Feigin–Stoyanovsky formula states that

$$\mathrm{Tr}(q^{L_0})|_{W_{k-1}} = \sum_{\boldsymbol{m}\in\mathbb{N}^d} \frac{q^{\frac{1}{2}\boldsymbol{m}B\boldsymbol{m}^t}}{(q)_{\boldsymbol{m}}},$$

where $d = (k-1)\mathrm{rank}(\mathfrak{g})$ and L_0 the energy operator.

There is another elegant expression for the left-hand side of (1.4) using notation from the theory of Hall–Littlewood polynomials, see [28, 34]. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition and λ' its conjugate. Then the q-function

$$b_\lambda(q) = \prod_{i\geq 1}(q)_{\lambda'_i - \lambda'_{i+1}}$$

features in the Cauchy identity for the Hall–Littlewood polynomials $P_\lambda(\boldsymbol{x}) = P_\lambda(\boldsymbol{x};q)$ and in the principal specialisation formula on an infinite alphabet:

$$P_\lambda(1, q, q^2, \dots) = \frac{q^{n(\lambda)}}{b_\lambda(q)},$$

where

$$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i = \sum_{i\geq 1}\binom{\lambda'_i}{2}. \tag{1.7}$$

If we further denote $(\lambda|\mu) = \sum_{i\geq 1}\lambda'_i\mu'_i$ then the left-hand side of (1.4) for $p = k$ or $p = 1$ corresponds to

$$\sum q^{\frac{1}{2}\sum_{a,b=1}^{N-1} C_{ab}(\lambda^{(a)}|\lambda^{(b)})} \prod_{a=1}^{N-1} \frac{z_a^{|\lambda^{(a)}|}}{b_{\lambda^{(a)}}(q)} \tag{1.8}$$

summed over partitions $\lambda^{(1)}, \dots, \lambda^{(N-1)}$ such that $\lambda_1^{(1)}, \dots, \lambda_1^{(N-1)} \leq k-1$, i.e., such that the largest parts of the $\lambda^{(i)}$ do not exceed $k-1$. In (1.8) $z_i = 1$ for all i if $p = k$ and $z_{2i-1} = q$ and $z_{2i} = q^{-1}$ for all i if $p = 1$.

The rewriting (1.8) of the left-hand side of (1.4) shows that not only the $k = 1$ case of (1.4) is known, but also the limiting case $k \to \infty$ (with $p = k$). Indeed, in [13] Hua derived a combinatorial identity related to one of Kac's conjectures [15] (recently proved in [12]) concerning the number of isomorphism classes of absolutely indecomposable representations of quivers over $\mathbb{F}_q$. For the finite quiver A_{N-1} (N not necessarily even) Hua's identity (corrected in [9]) is

$$\sum_{\lambda^{(1)},\dots,\lambda^{(N-1)}} q^{\frac{1}{2}\sum_{a,b=1}^{N-1} C_{ab}(\lambda^{(a)}|\lambda^{(b)})} \prod_{a=1}^{N-1} \frac{z_a^{|\lambda^{(a)}|}}{b_{\lambda^{(a)}}(q)} = \prod_{\alpha\in R_+}\frac{1}{(z^\alpha q)_\infty},$$

where R_+ is the set of positive roots of A_{N-1}:

$$R_+ = \{\alpha_i + \alpha_{i+1} + \dots + \alpha_j : \ 1 \leq i \leq j \leq N-1\},$$

and $z^{\alpha_i+\alpha_{i+1}+\dots+\alpha_j} = z_i z_{i+1}\cdots z_j$.

2. Further conjectures

At first sight it may seem surprising that the left-hand side of (1.4) features the root system A_{2n-1} instead of, more simply, A_n. However, this is not that unexpected in view of the following theorem due to Feigin and Stoyanovsky [8] ($n=1$) and Stoyanovsky [33] ($n>1$).

Theorem 2.1. *For k,n positive integers and $N=2n+1$*

$$\sum \frac{q^{\frac{1}{2}\sum_{a,b=1}^{N-1}\sum_{i=1}^{k-1} C_{ab} M_i^{(a)} M_i^{(b)}}}{\prod_{a=1}^{N-1}\prod_{i=1}^{k-1}(q)_{m_i^{(a)}}} = \frac{1}{(q)_\infty^{2n^2+n}} \sum \chi(\boldsymbol{v}/\boldsymbol{\rho}^*) q^{\frac{\|\boldsymbol{v}\|^2-\|\boldsymbol{\rho}^*\|^2}{4(k+n)}}, \tag{2.1}$$

where $\boldsymbol{\rho}^*=(1,2,\dots,n)$,

$$\chi(\boldsymbol{v}/\boldsymbol{w}) = \prod_{i=1}^n \frac{v_i}{w_i} \prod_{1\le i<j\le n} \frac{v_i^2-v_j^2}{w_i^2-w_j^2} \tag{2.2}$$

and the sum on the right is over $\boldsymbol{v}\in\mathbb{Z}^n$ such that $v_i\equiv\rho_i^* \pmod{2k+2n}$.

The Feigin–Stoyanovsky theorem generalises Macdonald's $\mathrm{C}_n^{(1)}$ η-function identity [27, p. 136, (6)]:

$$\eta(\tau)^{2n^2+n} = \sum \chi(\boldsymbol{v}/\boldsymbol{\rho}^*) q^{\|\boldsymbol{v}\|^2/(4(n+1))},$$

which, for $n=1$, is equivalent to Jacobi's well-known

$$(q)_\infty^3 = \sum_{m=0}^\infty (-1)^m (2m+1) q^{\binom{m+1}{2}}.$$

Equation (2.1) is the odd N counterpart of the $k=p$ case of the conjectured equation (1.4). We propose the following odd N analogue of (1.4) for $k=1$.

Conjecture 2.2. *For k,n positive integers and $N=2n+1$*

$$\sum \frac{q^{\frac{1}{2}\sum_{a,b=1}^{N-1}\sum_{i=1}^{k-1} C_{ab} M_i^{(a)} M_i^{(b)} + \sum_{a=1}^{N-1}\sum_{i=1}^{k-1}(-1)^a M_i^{(a)}}}{\prod_{a=1}^{N-1}\prod_{i=1}^{k-1}(q)_{m_i^{(a)}}} = \frac{1}{(q)_\infty^{2n^2+n}} \sum \chi(\boldsymbol{v}/\boldsymbol{\rho}^*) q^{\frac{\|\boldsymbol{v}\|^2-\|\rho^*+k-1\|^2}{4(k+n)}},$$

where the sum on the right is over $\boldsymbol{v}\in\mathbb{Z}^n$ such that $v_i\equiv\rho_i^*+k-1$ $\pmod{2k+2n}$.

Theorem 2.3. *Conjecture 2.2 is true for $k=2$, i.e., for n a positive integer and $N=2n+1$,*

$$\sum_{\boldsymbol{m}\in\mathbb{N}^{N-1}} \frac{q^{\frac{1}{2}\boldsymbol{m}C\boldsymbol{m}^t+|\boldsymbol{m}|_-}}{(q)_{\boldsymbol{m}}} = \frac{1}{(q)_\infty^{2n^2+n}} \sum \chi(\boldsymbol{v}/\boldsymbol{\rho}^*) q^{\frac{\|\boldsymbol{v}\|^2-\|\boldsymbol{\rho}^*+1\|^2}{4(n+2)}},$$

where the sum on the right is over $\boldsymbol{v}\in\mathbb{Z}^n$ such that $v_i\equiv\rho_i^*+1 \pmod{2n+4}$.

Of course, by the Feigin–Stoyanovsky theorem we also have

$$\sum_{\boldsymbol{m}\in\mathbb{N}^{N-1}} \frac{q^{\frac{1}{2}\boldsymbol{m}C\boldsymbol{m}^t}}{(q)_{\boldsymbol{m}}} = \frac{1}{(q)_\infty^{2n^2+n}} \sum \chi(\boldsymbol{v}/\boldsymbol{\rho}^*) q^{\frac{\|\boldsymbol{v}\|^2-\|\boldsymbol{\rho}^*\|^2}{4(n+2)}},$$

where the sum on the right is over $\boldsymbol{v}\in\mathbb{Z}^n$ such that $v_i\equiv\rho_i^* \pmod{2n+4}$.

There is a well-known even-modulus counterpart of the Andrews–Gordon identities (1.3) due to Bressoud [3]:

$$\sum_{m_1,\dots,m_{k-1}} \frac{q^{M_1^2+\cdots+M_{k-1}^2+M_p+\cdots+M_{k-1}}}{(q)_{m_1}\cdots(q)_{m_{k-2}}(q^2;q^2)_{m_{k-1}}} = \frac{(q^p,q^{2k-p},q^{2k};q^{2k})_\infty}{(q)_\infty}, \tag{2.3}$$

where $k>1$ and, again, $1\le p\le k$ and $M_i=m_i+\cdots+m_{k-1}$. It will be convenient to interpret $1/(q^2;q^2)_{m_0}$ as $(q)_\infty/(q^2;q^2)_\infty$ so that (2.3) is true for all positive integers k.

Our next conjecture unifies Bressoud's identity for $p=k$ with Macdonald's η-function identity for $\mathrm{A}_{2n-1}^{(2)}$ [27, p. 136, (6)(b)]:

$$\frac{\eta(\tau)^{2n^2+n-1}}{\eta(2\tau)^{2n-1}} = \sum \xi(\boldsymbol{v}/\boldsymbol{\rho}^\star)(-1)^{(|\boldsymbol{v}|-|\boldsymbol{\rho}^\star|)/(2n)} q^{\|\boldsymbol{v}\|^2/(4n)}, \tag{2.4}$$

where $\boldsymbol{\rho}^\star=(0,1,\dots,n-1)$ and $\boldsymbol{v}$ is summed over $\mathbb{Z}^n$ such that $v_i\equiv\rho_i^\star \pmod{2n}$, and a second η-function identity for $\mathrm{A}_{2n}^{(2)}$ [27, p. 138, (6)(a)]:

$$\frac{\eta(\tau)^{2n^2+3n}}{\eta(2\tau)^{2n}} = \sum \chi(\boldsymbol{v}/\boldsymbol{\rho}) q^{\|\boldsymbol{v}\|^2/(2(2n+1))}, \tag{2.5}$$

where the sum is over $\boldsymbol{v}\in\mathbb{Z}^n$ such that $v_i\equiv\rho_i \pmod{2n+1}$.

Conjecture 2.4. *For k, N positive integers and $n = \lfloor N/2 \rfloor$,*

$$\sum \frac{q^{\frac{1}{2}\sum_{a,b=1}^{N-1}\sum_{i=1}^{k-1} C_{ab} M_i^{(a)} M_i^{(b)}}}{\prod_{a=1}^{N-1}\bigl(\prod_{i=1}^{k-2}(q)_{m_i^{(a)}}\bigr)(q^2;q^2)_{m_{k-1}^{(a)}}}$$

$$= \frac{1}{(q)_\infty^{N(N-1)/2}} \sum \xi(\boldsymbol{v}/\boldsymbol{\rho}^\star)(-1)^{\frac{|\boldsymbol{v}|-|\boldsymbol{\rho}^\star|}{2k+N-2}} q^{\frac{\|\boldsymbol{v}\|^2-\|\boldsymbol{\rho}^\star\|^2}{2(2k+N-2)}} \tag{2.6a}$$

$$= \frac{1}{(q)_\infty^{N(N-1)/2}} \sum \chi(\boldsymbol{v}/\boldsymbol{\rho}) q^{\frac{\|\boldsymbol{v}\|^2-\|\boldsymbol{\rho}\|^2}{2(2k+N-2)}}, \tag{2.6b}$$

where (2.6a) *applies for even N, in which case the sum is over $\boldsymbol{v} \in \mathbb{Z}^n$ such that $v_i \equiv \rho_i^\star \pmod{2k+N-2}$ and* (2.6b) *applies for odd N, in which case the sum is over $\boldsymbol{v} \in \mathbb{Z}^n$ such that $v_i \equiv \rho_i \pmod{2k+N-2}$.*

As before, to recover (2.4) and (2.5) as the $k=1$ case of (2.6) we have to interpret $1/(q^2;q^2)_{m_0^{(a)}}$ as $(q)_\infty/(q^2;q^2)_\infty$.

3. Dilogarithm identities

To provide further support for Conjecture 1.1, we show below that a standard asymptotic analysis applied to (1.4) implies an identity for the Rogers dilogarithm due to Kirillov.

We begin by recalling the definition of the Rogers dilogarithm function

$$\mathrm{L}(x) = -\frac{1}{2}\int_0^x \biggl(\frac{\log(1-t)}{t} + \frac{\log t}{1-t}\biggr)\mathrm{d}t, \qquad x \in [0,1].$$

Note in particular that $\mathrm{L}(1) = \pi^2/6$.

In [18] Kirillov proved the following A_{N-1} type dilogarithm identity

$$\frac{1}{\mathrm{L}(1)}\sum_{a=1}^{N-1}\sum_{i=1}^{K-1} \mathrm{L}\left(\frac{\sin\bigl(\frac{a\pi}{K+N-1}\bigr)\sin\bigl(\frac{(N-a)\pi}{K+N-1}\bigr)}{\sin\bigl(\frac{(i+a)\pi}{K+N-1}\bigr)\sin\bigl(\frac{(i+N-a)\pi}{K+N-1}\bigr)}\right) = \frac{(N^2-1)(K-1)}{K+N-1}. \tag{3.1}$$

If we denote the summand on the left by $S(K,N;a,i)$ then

$$S(2k,N;a,i) = S(2k,N;a,2k-i-1) \quad \text{for } 1 \le i \le k-1$$

and $S(2k,N;a,2k-1) = \mathrm{L}(1)$. Hence

$$\frac{1}{\mathrm{L}(1)}\sum_{a=1}^{N-1}\sum_{i=1}^{k-1} \mathrm{L}\left(\frac{\sin\bigl(\frac{a\pi}{2k+N-1}\bigr)\sin\bigl(\frac{(N-a)\pi}{2k+N-1}\bigr)}{\sin\bigl(\frac{(i+a)\pi}{2k+N-1}\bigr)\sin\bigl(\frac{(i+N-a)\pi}{2k+N-1}\bigr)}\right) = \frac{N(N-1)(k-1)}{2k+N-1}. \tag{3.2}$$

We now recall the following result from [19].

Lemma 3.1. *Let B be a $d\times d$ symmetric, positive definite, rational matrix and let*

$$\sum_{i=0}^{\infty} a_i q^i = \sum_{\boldsymbol{m}\in\mathbb{N}^d} \frac{q^{\frac{1}{2}\boldsymbol{m}B\boldsymbol{m}^t}}{(q)_{\boldsymbol{m}}}.$$

Then

$$\lim_{m\to\infty} \frac{\log^2 a_m}{4m} = \sum_{i=1}^{d} \mathrm{L}(x_i),$$

where the x_i for $1\le i\le d$ are the solutions of

$$x_i = \prod_{j=1}^{d} (1-x_j)^{B_{ij}}$$

such that $x_i\in(0,1)$ for all i.

If we apply the above lemma to the expression on the left-hand side of (1.6) we are led to the system of equations

$$f_i^{(a)} = \prod_{b=1}^{N-1}\prod_{j=1}^{k-1} (1-f_i^{(b)})^{C_{ab}\min\{i,j\}},$$

for $1\le a\le N-1$ and $1\le i\le k-1$. It is readily verified that this is solved by

$$f_i^{(a)} = \frac{\sin\bigl(\frac{a\pi}{2k+N-1}\bigr)\sin\bigl(\frac{(N-a)\pi}{2k+N-1}\bigr)}{\sin\bigl(\frac{(i+a)\pi}{2k+N-1}\bigr)\sin\bigl(\frac{(i+N-a)\pi}{2k+N-1}\bigr)}.$$

Hence, denoting the q-series on either side of (1.6) by $\sum_{i\ge 0} a_i q^i$, we find that

$$\frac{1}{\mathrm{L}(1)}\lim_{m\to\infty}\frac{\log^2 a_m}{4m} = \mathrm{LHS}(3.2).$$

The right-hand side of (1.6) is a specialised standard module of $\mathrm{A}_{2n}^{(2)}$ [14, 16]. Exploiting its modular properties [17] we obtain

$$\frac{1}{\mathrm{L}(1)}\lim_{m\to\infty}\frac{\log^2 a_m}{4m} = \mathrm{RHS}(3.2)$$

(recall that $N=2n$), leading to (3.2).

In much the same way an asymptotic analysis of Theorem 2.1 gives (3.2) for odd N. The asymptotics of Conjecture 2.4, on the other hand, can be related to the odd K case of (3.1).

4. Proof of Theorems 1.2 and 2.3

The proof given below uses the theory of Hall–Littlewood polynomials, and for notation and definitions pertaining to these functions we refer the reader to [28, 34].

For $\boldsymbol{x} = (x_1, \dots, x_n)$ and μ a partition of length $l(\mu) \le n$, let $Q'_\mu(\boldsymbol{x}) = Q'_\mu(\boldsymbol{x}; q)$ be the modified Hall–Littlewood polynomial [22]

$$Q'_\mu(\boldsymbol{x}) = \sum_\lambda K_{\lambda\mu}(q) s_\lambda(\boldsymbol{x}),$$

where the $K_{\lambda\mu}(q)$ are the Kostka–Foulkes polynomials and $s_\lambda(\boldsymbol{x})$ the Schur functions. In the ring of symmetric functions, the polynomials Q'_μ form the adjoint basis of P_λ with respect to the Hall inner product. They may also be viewed in λ-ring notation [20] as $Q'_\mu(\boldsymbol{x}) = b_\mu(q) P_\mu(\boldsymbol{x}/(1-q))$.

Theorem 4.1. *Let C be the A_n Cartan matrix and $|r|_- = r_1 - r_2 + r_3 - \cdots$. Then*

$$\sum_{m=0}^\infty \frac{q^m}{(q)_m} Q'_{(2^m)}(1^n) = \sum_{\boldsymbol{r} \in \mathbb{N}^n} \frac{q^{\frac{1}{2}\boldsymbol{r}C\boldsymbol{r}^t}}{(q)_{\boldsymbol{r}}} \tag{4.1a}$$

and

$$\sum_{m=0}^\infty \frac{q^{2m}}{(q)_m} Q'_{(2^m)}(\underbrace{1, q^{-1}, 1, q^{-1}, \dots}_{n \text{ terms}}) = \sum_{\boldsymbol{r} \in \mathbb{N}^n} \frac{q^{\frac{1}{2}\boldsymbol{r}C\boldsymbol{r}^t + |\boldsymbol{r}|_-}}{(q)_{\boldsymbol{r}}}. \tag{4.1b}$$

Our proof requires a generalisation of [28]

$$Q'_\lambda(1) = q^{n(\lambda)} \tag{4.2}$$

due to Lascoux. Extend (1.7) to skew shapes by

$$n(\lambda/\mu) = \sum_{i \ge 1} \binom{\lambda'_i - \mu'_i}{2}.$$

Theorem 4.2 ([21, Theorem 3.1])**.** *For $\mu \subseteq \lambda$*

$$Q'_{\lambda/\mu}(1) = \frac{q^{n(\lambda/\mu)}}{b_\mu(q)} \prod_{i=1}^{l(\mu)} (1 - q^{\lambda'_{\mu_i} - i + 1}),$$

and $Q'_{\lambda/\mu}(1) = 0$ otherwise.

Before we show how this implies Theorem 4.1 we note that the above may be rewritten as

$$Q'_{\lambda/\mu}(1) = q^{n(\lambda/\mu)} \prod_{i \ge 1} \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}, \tag{4.3}$$

where $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]$ is a q-binomial coefficient. A classical result in the theory of abelian p-groups states that if $\alpha_\lambda(\mu;p)$ is the number of subgroups of type μ in a finite abelian p-group of type λ, then [4, 5, 6, 35]

$$q^{n(\lambda)-n(\mu)}\alpha_\lambda(\mu;q^{-1}) = q^{n(\lambda/\mu)} \prod_{i\geq 1} \begin{bmatrix} \lambda_i' - \mu_{i+1}' \\ \lambda_i' - \mu_i' \end{bmatrix}.$$

This obviously implies that

$$Q'_{\lambda/\mu}(1) = q^{n(\lambda)-n(\mu)}\alpha_\lambda(\mu;q^{-1}),$$

a result we failed to find in the literature.

Proof of Theorems 4.1. For $n=1$ (4.1) is trivial since, by (4.2),

$$Q'_{(2^m)}(1) = q^{m^2-m}.$$

To prove (4.1) for general n we apply

$$Q'_\lambda(\boldsymbol{x}) = \sum \prod_{i=1}^{n} x_i^{|\mu^{(i-1)}-\mu^{(i)}|} Q'_{\mu^{(i-1)}/\mu^{(i)}}(1),$$

where the sum is over

$$0 = \mu^{(n)} \subseteq \cdots \subseteq \mu^{(1)} \subseteq \mu^{(0)} = \lambda. \tag{4.4}$$

Hence

$$\mathrm{LHS}(4.1\mathrm{a}) = \sum_{m=0}^{\infty} \sum \frac{q^m}{(q)_m} \prod_{i=1}^{n} Q'_{\mu^{(i-1)}/\mu^{(i)}}(1)$$

and

$$\mathrm{LHS}(4.1\mathrm{b}) = \sum_{m=0}^{\infty} \sum \frac{q^{2m}}{(q)_m} \prod_{i=1}^{n} q^{(-1)^i|\mu^{(i)}|} Q'_{\mu^{(i-1)}/\mu^{(i)}}(1),$$

where the inner sums on the right are over (4.4) with $\lambda = (2^m)$. If we now change the above double-sums to a sum over $k_1,\dots,k_{n-1}$ and $\boldsymbol{r} = (r_1,\dots,r_n)$ by setting

$$\mu^{(i)} = (1^{k_i} 2^{r_{i+1}+\cdots+r_n-k_i-\cdots-k_{n-1}}), \quad 0 \leq i \leq n,$$

(where $k_0 = k_n := 0$), and insert (4.3), we arrive at

$$\mathrm{LHS}(4.1\mathrm{a}) = \sum_{\boldsymbol{r}\in\mathbb{N}^n}^{\infty} \frac{q^{\|\boldsymbol{r}\|^2}}{(q)_{r_1}} \prod_{i=1}^{n-1} \sum_{k_i=0}^{\min\{r_i,r_{i+1}\}} \frac{q^{k_i(k_i-r_i-r_{i+1})}}{(q)_{r_{i+1}-k_i}} \begin{bmatrix} r_i \\ k_i \end{bmatrix}$$

and

$$\mathrm{LHS}(4.1\mathrm{b}) = \sum_{\boldsymbol{r}\in\mathbb{N}^n}^{\infty} \frac{q^{\|\boldsymbol{r}\|^2+|\boldsymbol{r}|_-}}{(q)_{r_1}} \prod_{i=1}^{n-1} \sum_{k_i=0}^{\min\{r_i,r_{i+1}\}} \frac{q^{k_i(k_i-r_i-r_{i+1})}}{(q)_{r_{i+1}-k_i}} \begin{bmatrix} r_i \\ k_i \end{bmatrix}.$$

By the q-Chu–Vandermonde sum [10, (II.6)] the sum over k_i yields

$$\frac{q^{-r_i r_{i+1}}}{(q)_{r_{i+1}}},$$

thus proving (4.1). □

Theorem 4.1 combined with Milne's C_n analogue of the Rogers–Selberg identity implies Theorems 1.2 and 2.3 as outlined below.

Let $\Delta(\boldsymbol{x})$ be the C_n Vandermonde product

$$\Delta(\boldsymbol{x}) = \prod_{1\le i<j\le n} (1-x_i/x_j) \prod_{1\le i\le j\le n} (1-x_i x_j)$$

and, for $\boldsymbol{u}\in\mathbb{N}^n$, let $n(\boldsymbol{u}) = \sum_{i=1}^n (i-1)u_i$.

Theorem 4.3 ([29, Corollary 2.21])**.**

$$\begin{aligned} \sum_{\boldsymbol{u}\in\mathbb{N}^n} \frac{\Delta(\boldsymbol{x}q^{\boldsymbol{u}})}{\Delta(\boldsymbol{x})} \prod_{i,j=1}^n \frac{(x_i x_j)_{u_i}}{(q x_i/x_j)_{u_i}} \\ \times (-1)^{n|\boldsymbol{u}|} q^{n(\boldsymbol{u})+\frac{1}{2}(n+4)\|\boldsymbol{u}\|^2-\frac{1}{2}n|\boldsymbol{u}|} \prod_{i=1}^n x_i^{(n+4)u_i-|\boldsymbol{u}|} \\ = \prod_{i=1}^n (q x_i^2)_\infty \prod_{1\le i<j\le n} (q x_i x_j)_\infty \sum_{m=0}^\infty \frac{q^m}{(q)_m} Q'_{(2^m)}(\boldsymbol{x}). \end{aligned}$$

For $\boldsymbol{u}\in\mathbb{N}^n$ denote by $\boldsymbol{u}^+$ the unique partition in the $\mathfrak{S}_n$ orbit of $\boldsymbol{u}$. For example, if $\boldsymbol{u}=(2,0,3,2,0)$ then $\boldsymbol{u}^+=(3,2,2,0,0)=(3,2,2)$. Similarly, let W_n be the Weyl group of C_n, i.e., $W_n=(\mathbb{Z}_2)^n \rtimes \mathfrak{S}_n$, acting on $\boldsymbol{v}\in\mathbb{Z}^n$ by permutation and sign-reversal of its components. Then $\boldsymbol{v}^*$ denotes the unique partition in the W_n orbit of $\boldsymbol{v}$. For example, if $\boldsymbol{v}=(-2,0,-3,2,0)$ then $\boldsymbol{v}^*=(3,2,2,0,0)=(3,2,2)$.

Let us now denote the summand on the left of Theorem 4.3 by $L_n(\boldsymbol{u},\boldsymbol{x})$ and let us denote the right-hand sides of (1.4) and (2.1) for $k=p=2$ by $R_{2n-1}(\boldsymbol{v})$ and $R_{2n}(\boldsymbol{v})$, respectively. Then, for λ a partition of length at most n,

$$\lim_{\boldsymbol{x}\to(1^n)} \sum_{\substack{\boldsymbol{u}\in\mathbb{N}^n \\ \boldsymbol{u}^+=\lambda}} L_n(\boldsymbol{u},\boldsymbol{x}) = \chi\big(l(\lambda)\le\lfloor(n+1)/2\rfloor\big) \sum_{\substack{\boldsymbol{v}\in\mathbb{Z}^n \\ \boldsymbol{v}^*=\lambda}} R_n(\boldsymbol{v}).$$

Here χ is the truth function (and not the character (2.2)). As an immediate consequence of the above we find that

$$\lim_{\boldsymbol{x}\to(1^n)} \sum_{\boldsymbol{u}\in\mathbb{N}^n} L_n(\boldsymbol{u},\boldsymbol{x}) = \sum_{\boldsymbol{v}\in\mathbb{Z}^n} R_n(\boldsymbol{v}).$$

By Theorem 4.3

$$\begin{aligned}
&\lim_{\boldsymbol{x}\to(1^n)} \sum_{\boldsymbol{u}\in\mathbb{N}^n} L_n(\boldsymbol{u},\boldsymbol{x}) \\
&\qquad = \lim_{\boldsymbol{x}\to(1^n)} \prod_{i=1}^n (qx_i^2)_\infty \prod_{1\le i<j\le n} (qx_ix_j)_\infty \sum_{m=0}^\infty \frac{q^m}{(q)_m}\, Q'_{(2^m)}(\boldsymbol{x}) \\
&\qquad = (q)_\infty^{n(n+1)/2} \sum_{m=0}^\infty \frac{q^m}{(q)_m}\, Q'_{(2^m)}(1^n).
\end{aligned}$$

Thanks to (4.1a) this proves the $k=p=2$ instances of (1.4) and (2.1).

In much the same way one obtains (1.4) and (2.1) with $k=2$ and $p=1$ by taking the $\boldsymbol{x}\to(q^{1/2},q^{-1/2},q^{1/2},\dots)$ limit in Theorem 4.3.

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