Srinivasa Ramanujan: Going Strong at 125, Part II

Krishnaswami Alladi, Editor

Ramanujan passport photo.

The 125th anniversary of the birth of Srinivasa Ramanujan was on December 22, 2012. To mark the occasion, the Notices *is publishing a feature article of which this is the second and final installment. This installment contains pieces by Ken Ono, Kannan Soundararajan, Robert Vaughan, and Ole Warnaar on various aspects of Ramanujan's work. The first installment appeared in the December 2012 issue, and contained an introductory piece by Krishnaswami Alladi plus pieces by George Andrews, Bruce Berndt, and Jonathan Borwein on Ramanujan's work.*

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Modular Forms, Partitions, and Mock Theta Functions

Ramanujan's work on modular forms, partitions, and mock theta functions is rich with tantalizing examples of deeper mathematical structures. Indeed, his tau-function, his partition congruences, and his mock theta functions are prototypes that have helped to shape the modern mathematical landscape.

Ramanujan was fascinated by the coefficients of the function

$$(1) \quad \Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n := q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

$$= q - 24q^2 + 252q^3 - 1472q^4 + \cdots,$$

where $q := e^{2\pi i z}$ and $\text{Im}(z) > 0$. This function is a weight 12 *modular form*. In other words, $\Delta(z)$ is a function on the upper half of the complex plane such that

$$\Delta\left(\frac{az + b}{cz + d}\right) = (cz + d)^{12}\Delta(z)$$

for every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. He conjectured (see p. 153 of [12]) that

$$\tau(nm) = \tau(n)\tau(m),$$

for every pair of coprime positive integers n and m, and that

$$\tau(p)\tau(p^s) = \tau(p^{s+1}) + p^{11}\tau(p^{s-1}),$$

for primes p and positive integers s. Mordell proved these conjectures, and in the 1930s Hecke later

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developed a framework for the theory of modular forms and L-functions in which such properties play a fundamental role.

Ramanujan was also very interested in the size of the numbers $\tau(n)$, and for primes p he conjectured (see pp. 153–154 of [12]), but could not prove, that

$$|\tau(p)| \le 2p^{\frac{11}{2}}.$$

This speculation is the first example of the Ramanujan-Petersson Conjectures, among the deepest problems in the analytic theory of automorphic forms. This inequality was triumphantly confirmed by Deligne [8] as a corollary to his proof of the Weil Conjectures, work which won him a Fields Medal in 1978. Although Ramanujan did not anticipate the Weil Conjectures (the Riemann Hypothesis for varieties over finite fields), he correctly anticipated the importance of optimally bounding coefficients of modular forms.

Ramanujan is also well known for his work on the congruence properties of his tau-numbers. For example, he proved that (see page 159 of [12])

$$(2) \qquad \tau(n) \equiv \sum_{d \mid n} d^{11} \pmod{691}.$$

About forty years ago, Serre [13] and Swinnerton-Dyer [14] wrote beautiful papers interpreting such congruences in terms of certain two-dimensional ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the Galois group of the algebraic closure of $\mathbb{Q}$. Deligne had just proved that such representations encode the coefficients of certain modular forms as "traces of the images of Frobenius elements". Serre and Swinnerton-Dyer interpreted Ramanujan's tau-congruences, such as (2), as the first nontrivial examples of certain exceptional representations. For the prime $\ell = 691$, there is a (residual) Galois representation

$$\rho_{\Delta, 691} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{Z}/691\mathbb{Z})$$

which, for primes $p \ne 691$, satisfies

$$\rho_{\Delta, 691}(\mathrm{Frob}(p)) = \begin{pmatrix} 1 & * \\ 0 & p^{11} \end{pmatrix},$$

where $\mathrm{Frob}(p) \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denotes the "Frobenius element at p". Congruence (2) is then implied by Deligne's prescription that, for primes $p \ne 691$, one has

$$\tau(p) = \mathrm{Tr}(\rho_{\Delta, 691}(\mathrm{Frob}(p))) \equiv 1 + p^{11} \pmod{691}.$$

The theory of modular ℓ-adic Galois representations has subsequently flourished and famously is the "language" of Wiles's proof of Fermat's Last Theorem.

Ramanujan's work on tau-congruences is intimately related to his work on partition congruences. A *partition* of the natural number n is any nonincreasing sequence of natural numbers whose sum

is n, and the number of partitions of n is denoted by $p(n)$. The generating function for $p(n)$ has two convenient representations:

$$(3)$$
$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2 (1-q^2)^2 \cdots (1-q^n)^2}.$$

The infinite product representation is analogous to the infinite product for $\Delta(z)$ in (1), which explains the important role that modular forms play in the study of $p(n)$, while the other representation as a "basic hypergeometric series" foreshadows Ramanujan's last work: his discovery of "mock theta functions".

In groundbreaking work, Ramanujan proved that

$$p(5n + 4) \equiv 0 \pmod{5},$$
$$p(7n + 5) \equiv 0 \pmod{7},$$
$$p(11n + 6) \equiv 0 \pmod{11}.$$

He also conjectured corresponding congruences, known as Ramanujan's congruences, modulo powers of 5, 7, and 11. Extending his ideas, Atkin and Watson proved [3], [15]:

If $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1 \pmod{\delta}$,

then $p(\delta n + \lambda) \equiv 0 \pmod{5^a 7^{\lfloor \frac{b}{2} \rfloor + 1} 11^c}$.

These congruences have inspired much further work.

In the 1940s Dyson, seeking a combinatorial explanation for the congruences, defined [9] the *rank* of a partition to be the largest summand minus the number of summands. He conjectured that the partitions of $5n + 4$ (resp. $7n + 5$) are divided into 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7), thereby providing a combinatorial explanation for the congruences mod 5 and 7. This conjecture was proved in the 1950s by Atkin and Swinnerton-Dyer [5]. Dyson was unable to offer a combinatorial explanation for the mod 11 congruence, and he conjectured the existence of a statistic, which he referred to as the "crank", which would suitably explain the congruences mod 5, 7, and 11. In the 1980s Andrews and Garvan [2] finally found the elusive crank.

In addition to finding a combinatorial explanation for Ramanujan's original congruences, researchers have looked for further congruences. In the 1960s Atkin [4] found congruences, though not so systematic, with modulus 13, 17, 19, 23, 29, and 31. About ten years ago the author and Ahlgren [1], [10] found that there are partition

The lovely colonial style architecture is evident in the Town High School in Kumbakonam that Ramanujan attended.

congruences modulo every integer Q relatively prime to 6. For example, we have

$$p(59^4 \cdot 13n + 111247) \equiv 0 \pmod{13}.$$

These more recent results arise from the fact that (3) is essentially a modular form. This fact, combined with deep work of Shimura, makes it possible to apply the methods of Deligne and Serre, which explained Ramanujan's tau-congruences, to the partition numbers.

During his last year of life, when he was seeking a return to good health in south India, Ramanujan discovered functions he called *mock theta functions*. In his last letter to Hardy, dated January 12, 1920, Ramanujan shared hints (see p. 220 of [6]) of his last theory. The letter, roughly four typewritten pages, consists of formulas for seventeen strange power series and a discussion of their asymptotics. It contained no proofs of any kind. By changing signs in (3), we obtain a typical example of a mock theta function:

$$f(q) = \sum_{n=0}^{\infty} a_f(n)q^n$$
$$:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

For followers of Ramanujan, the main questions were: What did Ramanujan mean by a mock theta function, and what roles do these functions play in mathematics? These questions would remain open for eighty years. In the meantime, such researchers as Andrews, Choi, Dragonette, Dyson, Gordon, Hickerson, McIntosh, Selberg, and Watson, to name a few, proved beautiful theorems about these strange q-series. Then in 2002, in his Ph.D. thesis [17] written under Zagier, Zwegers made sense of the mock theta functions. He discovered how to "complete" them by adding a nonholomorphic function, a so-called "period integral", to obtain a *real analytic modular form*.

In the case of $f(q)$ and its companion $\omega(q)$, Zwegers defined the vector-valued functions

$$F(z) = (F_0(z),\ F_1(z),\ F_2(z))^T$$
$$:= (q^{-\frac{1}{24}}f(q),\ 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}),\ 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}}))^T,$$
$$G(z) = (G_0(z),\ G_1(z),\ G_2(z))^T$$
$$:= 2i\sqrt{3}\int_{-\bar{z}}^{i\infty} \frac{(g_1(\tau),\ g_0(\tau),\ -g_2(\tau))^T}{\sqrt{-i(\tau+z)}}\,d\tau,$$

where the $g_i(z)$ are theta functions, and he proved that $H(z) := F(z) - G(z)$ satisfies

$$H(z+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(z)$$

and

$$H(-1/z) = \sqrt{-iz}\cdot\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(z),$$

where $\zeta_n := e^{\frac{2\pi i}{n}}$, making it a *vector-valued real analytic modular form*.

All of Ramanujan's mock theta functions turn out to be holomorphic parts of special weight $1/2$ real analytic modular forms, which Bruinier and Funke [7] call *weak harmonic Maass forms*. Loosely speaking, a *weight k harmonic Maass form* is a smooth function $M(z)$ on $\mathbb{H}$ which transforms as does a weight k modular form and which also satisfies $\Delta_k(M) = 0$. Here the *hyperbolic Laplacian* Δ_k, where $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$, is given by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Since modular forms appear prominently in mathematics, I think that one expects these functions to have far-reaching implications. One might expect the functions themselves to play many roles. This has already turned out to be the case. These forms have appeared prominently in the following subjects [11], [16]: partitions and q-series, moonshine, Donaldson invariants, probability theory, Borcherds products and elliptic curves, and many others.

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Hardy and Ramanujan and inspired a great deal of work over the years by many authors (notably Pillai [20], Erdős [8], Sathe [25], and Selberg [26]), culminating in the definitive work of Hildebrand and Tenenbaum [15].

The Hardy-Ramanujan theorem, furthered by the insight of Erdős, led to a host of problems (see, for example, [13]) designed to understand the multiplicative structure of a random integer—or as Granville [4] has termed it, "the anatomy of (random) integers". Interestingly, the anatomy of a random integer is strikingly similar to the anatomy of random polynomials over finite fields and to the anatomy of the cycle decompositions of random permutations; see [2] and the forthcoming graphic novel(!) [12] for accounts of these similarities. Moreover, understanding the anatomy of integers has proven valuable in many problems of algorithmic number theory, for example, in the development of factorization algorithms such as the quadratic sieve [21].

Out of the many beautiful theorems in the area, let us mention the striking recent work of Ford [10] in understanding the distribution of integers having a divisor of a given size. Ford's work resolved a longstanding problem of Erdős: How many distinct integers are there in the $N \times N$ multiplication table? The Hardy-Ramanujan theorem, Erdős observed, gives a quick proof that the number of distinct integers in the multiplication table is $o(N^2)$. A typical integer of size N has about $\log \log N$ prime factors, and so the product of two such integers has about $2 \log \log N$ prime factors. Thus a typical element appearing in the multiplication table has $2 \log \log N$ prime factors, whereas a typical integer of size N^2 has only about $\log \log (N^2) = \log \log N + \log 2$ prime factors. In other words, elements in the multiplication table are quite atypical and thus few in number. Refining a result of Tenenbaum [28], Ford's work establishes that true order of magnitude for the number of integers in the multiplication table is $N^2/((\log N)^{\delta}(\log \log N)^{\frac{3}{2}})$, with δ being the esoteric constant $1 - (1 + \log \log 2)/\log 2 = 0.086\ldots$. In the study of random permutations, there arises an analogous problem: Given n and $1 \leq k \leq n - 1$, the permutation group S_n acts naturally on the k-element subsets of $\{1, \ldots, n\}$. Now given a random element in S_n, what is the probability that there exists some k-element subset left fixed by this permutation? Partial results on this problem were established by Luczak and Pyber [19] and have interesting applications [5]. Using Ford's work on integers as a guide to the anatomy of permutations, one can obtain a more refined understanding of this question [6].

We now turn to Ramanujan's work on "highly composite numbers". These are numbers n such that $d(n) > d(k)$ for all $1 \leq k < n$, where $d(n)$ counts the number of divisors of n. Ramanujan described (see paper 15 in [22]) the anatomy of such highly composite integers and also made a careful study of the extremal order of $d(n)$. Ramanujan was also interested in finding large values of the iterated functions $d(d(n))$ (whose exact order is determined in the surprisingly recent paper [3]) and $d(d(d(n)))$. Closely related to highly composite numbers are the *superabundant* numbers: these are the integers n for which $\sigma(n)/n$ is larger than $\sigma(k)/k$ for all $1 \leq k \leq n - 1$, and where $\sigma(n)$ denotes the sum of the divisors of n. Superabundant numbers were studied extensively by Alaoglu and Erdős [1], who were directly motivated by Ramanujan's work. Interestingly, Ramanujan himself had studied these classes of numbers, in notes related to his long paper on highly composite numbers, but owing to space considerations these were not included in the published work. These manuscript notes were discovered as part of Ramanujan's "Lost Notebook" [23] and were published with annotations by Nicolas and Robin in [24]. While interesting, these problems seem far from the mainstream, and so it may come as a surprise that the Riemann Hypothesis is equivalent to the estimate $\sigma(n) \leq H_n + \exp(H_n) \log H_n$ (where H_n is the "harmonic number" $H_n = \sum_{j=1}^{n} 1/j$), and the difficult cases of this inequality are precisely when n is a superabundant number; see Lagarias [18] for a charming account of this connection.

We end by mentioning an unpublished and largely unknown fragment of Ramanujan on a central topic in the anatomy of integers, namely, on "smooth" (or "friable") numbers. A number is called y-smooth if all its prime factors are at most y. Smooth numbers appear prominently in algorithmic number theory (e.g., in factoring algorithms) and are also useful in seemingly unrelated questions such as Waring's problem; see [16] and [11] for excellent surveys on these integers without large prime factors. A basic question is to estimate $\Psi(x, y)$, which counts the number of y-smooth integers below x; a precise understanding of $\Psi(x, y)$ uniformly in a wide range is equivalent to the Riemann Hypothesis [14]. The first published work on smooth numbers in 1930 was by Dickman, who showed that if $x = y^u$, $u > 0$ is considered fixed, and y tends to infinity, then $\Psi(x, y) \sim \rho(u)x$, where ρ (the Dickman function) is defined as the unique continuous solution to the differential-difference equation $u\rho'(u) = -\rho(u - 1)$ for $u > 1$, and with initial condition $\rho(u) = 1$ for $0 \leq u \leq 1$. Given the significance of smooth numbers, it may be interesting to record that Ramanujan appears to have made a study of this problem. On page 337 of [23] one finds formulas for $\Psi(y^u, y)$ in the range $1 \leq u \leq 5$, which give inclusion-exclusion type

expressions for $\rho(u)$ in terms of iterated integrals and with a clear indication of how the pattern is to be continued.

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R. C. Vaughan

The Hardy–Littlewood–Ramanujan Method

In modern analytic number theory there are three very powerful standard techniques that are central to many investigations. One is the use of the theory of Dirichlet series, including the Riemann zeta function and Dirichlet L-functions; a second is the application of sieve theory in all its manifestations; and the third is the subject of this survey. It has its genesis in two papers of Ramanujan. The most famous of these is the celebrated paper [Hardy, G. H. & Ramanujan, S. 1918] (there are also announcements in [Hardy, G. H. & Ramanujan, S. 1917a] and [Hardy, G. H. & Ramanujan, S. 1917b]), which is concerned mostly with the partition function, but the paper [Ramanujan, S. 1918] also describes the same fundamental idea. This is that interesting number theoretic functions have representations, or close approximations, as infinite series indexed by the positive rational numbers—that is, the a/q with $1 \le a \le q$ and $(a, q) = 1$—and moreover that such expressions can be obtained by consideration of generating functions that have the unit circle as a natural boundary and the terms in the aforementioned series arise from the neighborhood of singularities

at $e(a/q)$. A very simple example of this is the interesting formula

$$(1) \qquad \sigma(n) = \frac{\pi^2}{6} n \sum_{q=1}^{\infty} q^{-2} c_q(n),$$

which first appears in the second of these papers. Here $c_q(n)$ is the eponymous sum

$$(2) \qquad c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e(an/q).$$

In addition to the partition function, Hardy and Ramanujan also observe that the same ideas can be applied to the number of representations of a number as a sum of a fixed number of squares.

One might speculate how this could have developed if Ramanujan had not been struck down by illness. Fortunately, Hardy and Littlewood realized the very great importance and generality of the underlying conceptions and were able to develop them so as to apply to the Goldbach and Waring, and cognate, problems. There was also the realization of the predictive possibilities of the method in those cases where the arguments could not be otherwise pursued to a satisfactory conclusion. Later this led, through further important ideas of I. M. Vinogradov and Davenport, in the 1930s to the essential resolution of the ternary Goldbach problem (namely, that every large odd integer is the sum of three primes) and substantial progress in the known upper bounds for $G(k)$ (the smallest s such that every large natural number is the sum of at most s kth powers) in Waring's problem (see [Vaughan, R. C. 1997]). Davenport, along with Birch and Lewis, embarked on the question of applying the method to nondiagonal questions, for example, the nontrivial representation of zero by cubic forms (see [Davenport, H. 1963] and [Davenport, H. 1962, 2005]). [Davenport, H. & Lewis, D. J. 1963] had also established the Artin conjecture concerning the local-to-global principle for diagonal forms for all but a finite set of degrees. When I was a postgraduate student in the late 1960s, I became interested in the method. However, I was warned by a number of distinguished mathematicians that the method was already played out and it would be better to work on other things! Nevertheless, in [Vaughan, R. C. 1977] I did manage to use the method to deal with the cases not dealt with by Davenport and Lewis, and in a different direction [Montgomery, H. L. & Vaughan, R. C. 1975]   we were able to establish that there is a positive number δ such that the number $E(x)$ of even numbers not exceeding x which are not the sum of two primes satisfies

$$E(x) \ll x^{1-\delta}.$$

In the last twenty-five years there has been a huge explosion of activity. This note is too short to do more than mention a few highlights, with the hope that the reader may be pointed in an interesting direction. On Waring's problem the development of efficient differencing, building on the earlier work, especially that of Davenport, led to significant further progress on the upper bound for $G(k)$, and there is a survey of this and some other connected results in   [Vaughan, R. C. & Wooley, T. D. 2002].   Very recently, the sensational work of [Wooley, T. D. to appear] now means that the asymptotic formula for the number of representations of a natural number as the sum of s kth powers is known to hold when $s \geq 2k^2 - 2k - 8$. On questions involving forms there has been impressive progress on applications of the method on a number of fronts, especially by [Schmidt, W. M. 1979] and   [Wooley, T. D. 1997]   on   small   zeros; [Heath–Brown, D. R. 1983],   [Hooley,   C.,   1988, 1991, 1994], and [Hooley, C., 1988, 1991, 1994] on cubic forms; [Heath-Brown, D. R. & Browning, T. D. 2009] on cubic polynomials; and [Brüdern, J & Wooley, T. D. 1998] and [Brüdern, J & Wooley, T. D. 2003] on a variety of topics.

In a different direction, [Heath-Brown, D. R. 1981] has ingeniously combined the circle method with sieve theory, and this was taken up by [Brüdern, J. 1988] in his thesis and has been exploited extensively by [Brüdern, J. 1995], [Brüdern, J. & Kawada, K. 2002], [Kawada, K. 1997], and [Kawada, K. 2005]. This has spawned a whole new industry of ongoing work on sums of powers of primes and almost primes.

In the original series of papers "On some problems of partitio numerorum" by Hardy and Littlewood there was an unpublished manuscript on small differences between consecutive primes, and it was this manuscript which, when combined with developments in the large sieve, led to the work of [Bombieri, E. & Davenport, H. 1966] and the later refinements of [Pil'tjaĭ, G. Z. 1972], [Huxley, M. N. 1977], and [Maier, H. 1985] on

$$(3) \qquad \delta = \liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log p_n}.$$

The later proof by [Goldston, D. A., Pintz, J. & Yıldırım 2009] that $\delta = 0$ rests mostly on sieve theory, yet the motivation for the shape of some of the main terms comes from heuristics arising from the Hardy-Littlewood method. In many ways the motivation for the large sieve also comes from the Hardy–Littlewood method, and perhaps this can be seen most clearly in [Gallagher, P. X. 1967]'s proof of the large sieve inequality.

An unexpected development was the adaptation of the method to deal with questions involving

arithmetic progressions in general sets. This began with the seminal paper of [Roth, K. F. 1953] dealing with sets having no three elements in arithmetic progression and led to [Gowers, W. T. 1998]'s proof of Szemerédi's theorem and the theorem of [Green, B. J. & Tao, T. 2008] that the primes contain arbitrarily long arithmetic progressions. This is another very active area; see, for example, [Green, B. J. 2007] or [Tao, T. & Vu, Van, 2006]. A cognate area concerns the mean square distribution of sets of integers in arithmetic progressions. The prototype is the more precise version of the Barban-Davenport-Halberstam theorem concerning the set of primes due to [Montgomery, H. L. 1970] and refined by [Hooley, C. 1975]. Montgomery used a variant of the Hardy-Littlewood method, and more recently this was revived by [Goldston, D. A. & Vaughan, R. C. 1996] for the set of primes and by [Vaughan, R. C. 1998] to treat general dense sets. This has been taken a step further by [Brüdern, J. 2009], which extends the ideas of the latter paper to treat general binary additive problems.

Another aspect of the method is that the idea of large peaks near "rational points" $e(a/q)$ with q relatively small (the major arcs) and lesser peaks near $e(a/q)$ with q relatively large (the minor arcs) has suggested divisions of arguments in, apparently, otherwise quite unrelated topics. An important example of this is the work of Bombieri, Iwaniec, Huxley, and Watt on exponential sums which led to the currently best-known bounds for the Riemann zeta function on the $\frac{1}{2}$-line, and the error terms in the Dirichlet divisor problem and Gauss's estimation of the number of lattice points in a large disc centered at the origin. For an exposition of this, see [Huxley, M. N. 1996].

For standard accounts of the Hardy-Littlewood method in its more classical forms, see [Davenport, H. 1962, 2005] and [Vaughan, R. C. 1997]. The foreword to the second edition of [Davenport, H. 1962, 2005] gives a good survey of a number of the applications as they stood in 2005.

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Ramanujan's $_1\psi_1$ Summation

Notation. It is impossible to give an account of the $_1\psi_1$ summation without introducing some q-series notation. To keep the presentation as simple as possible, we assume that $0 < q < 1$. Suppressing q-dependence, we define two q-shifted factorials: $(a)_\infty := \prod_{k=0}^\infty (1 - aq^k)$ and $(a)_z := (a)_\infty/(aq^z)_\infty$ for $z \in \mathbb{C}$. Note that $1/(q)_n = 0$ if n is a negative integer. For $x \in \mathbb{C} - \{0, -1, \dots\}$, the q-gamma function is defined as $\Gamma_q(x) := (q)_{x-1}/(1-q)^{x-1}$.

Ramanujan recorded his now famous $_1\psi_1$ summation as item 17 of Chapter 16 in the second of his three notebooks [13, p. 32], [46]. It was brought to the attention of the wider mathematical community in 1940 by Hardy, who included it in his twelfth and final lecture on Ramanujan's work [31]. Hardy remarked that the result constituted "a remarkable formula with many parameters." Instead of presenting the $_1\psi_1$ sum as given by Ramanujan and Hardy, we will state its modern form:

$$(1) \quad \sum_{n=-\infty}^\infty \frac{(a)_n}{(b)_n} z^n = \frac{(az)_\infty (q/az)_\infty (b/a)_\infty (q)_\infty}{(z)_\infty (b/az)_\infty (q/a)_\infty (b)_\infty},$$
$$|b/a| < |z| < 1,$$

where it is understood that $a, q/b \notin \{q, q^2, \dots\}$. Characteristically, Ramanujan did not provide a proof of (1). Neither did Hardy, who, however, remarked that it could be "deduced from one which is familiar and probably goes back to Euler." The result to which Hardy was referring is another famous identity—known as the q-binomial theorem—corresponding to (1) with $b = q$: $\sum_{n=0}^\infty z^n(a)_n/(q)_n = (az)_\infty/(z)_\infty$ and valid for $|z| < 1$. Although not actually due to Euler, the q-binomial theorem is certainly classic. It seems to have appeared first and without proof (for $a = q^{-N}$) in Rothe's 1811 book *Systematisches Lehrbuch der Arithmetik*, and in the 1840s many mathematicians of note, such as Cauchy (1843), Eisenstein (1846), Heine (1847), and Jacobi (1847), published proofs. The first proof of the $_1\psi_1$ sum is due to Hahn in 1949 [30] and, as hinted by Hardy, uses the q-binomial theorem. After Hahn, a large number of alternative proofs of (1) were found, including one probabilistic and three combinatorial proofs [2], [3], [5], [16], [17], [19],

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[20], [18], [23], [24], [32], [34], [35], [44], [48], [53], [50]. The proof from the book, which again relies on the q-binomial theorem, was discovered by Ismail [32] and is short enough to include here. Assuming $|z| < 1$ and $|b| < \min\{1, |az|\}$, both sides are analytic functions of b. Moreover, they coincide when $b = q^{k+1}$ with $k = 0, 1, 2, \ldots$ by the q-binomial theorem with $a \mapsto aq^{-k}$. Since 0 is the accumulation point of this sequence of b's, the proof is complete.

Apart from the q-binomial theorem, the $_1\psi_1$ sum generalizes another classic identity, known as the Jacobi triple-product identity: $\sum_{n=-\infty}^{\infty}(-1)^n z^n q^{\binom{n}{2}} = (z)_\infty (q/z)_\infty (q)_\infty =: \theta(z)$. This result plays a central role in the theory of theta and elliptic functions.

The $_1\psi_1$ Sum as Discrete Beta Integral

As pointed out by Askey [8], [9], the $_1\psi_1$ summation may be viewed as a discrete analogue of Euler's beta integral. First, define the Jackson or q-integral $\int_0^{c\cdot\infty} f(t)\mathrm{d}_q t := (1-q)\sum_{n=-\infty}^{\infty} f(cq^n)cq^n$. Replacing $(a,b,z) \mapsto (-c, -cq^{\alpha+\beta}, q^\alpha)$ in (1) then gives

$$(2) \quad \int_0^{c\cdot\infty} \frac{t^{\alpha-1}}{(-t)_{\alpha+\beta}}\,\mathrm{d}_q t = c^\alpha\,\frac{\theta(-cq^\alpha)}{\theta(-c)}\,\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)},$$

where $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$. For real, positive c the limit $q \to 1$ can be taken, resulting in the beta integral modulo the substitution $t \mapsto t/(1-t)$. Askey further noted in [8] that the specialization $(\alpha, \beta) \mapsto (x, 1-x)$ in (2) (so that $0 < \mathrm{Re}(x) < 1$) may be viewed as a q-analogue of Euler's reflection formula.

Simple Applications of the $_1\psi_1$ Sum

There are numerous easy applications of the $_1\psi_1$ sum. For example, Jacobi's well-known four- and six-square theorems as well as a number of similar results readily follow from (1); see e.g., [1], [14], [15], [21], [22], [25]. To give a flavor of how the $_1\psi_1$ implies these types of results, we shall sketch a proof of the four-square theorem. Let $r_s(n)$ be the number of representations of n as the sum of s squares. The generating function $R_s(q) := \sum_{n\geq 0} r_s(n)(-q)^n$ is given by $\left(\sum_{m=-\infty}^{\infty}(-1)^m q^{m^2}\right)^s$. By the triple-product identity this is also $\left((q)_\infty/(-q)_\infty\right)^s$. Any identity that allows the extraction of the coefficient of $(-q)^n$ results in an explicit formula for $r_s(n)$. Back to (1): replace $(b,z) \mapsto (aq, b)$ and multiply both sides by $(1-b)/(1-ab)$. By the geometric series this yields

$$(3) \quad 1 + \frac{(1-a)(1-b)}{1-ab}\sum_{k,n=1}^{\infty} q^{kn}(a^k b^n - a^{-k} b^{-n})$$
$$= \frac{(abq)_\infty (q/ab)_\infty (q)_\infty^2}{(aq)_\infty (q/a)_\infty (bq)_\infty (q/b)_\infty},$$

which may also be found in Kronecker's 1881 paper "Zur Theorie der elliptischen Functionen". For $a, b \to -1$ the right side gives $R_4(q)$, whereas the left side becomes

$$1 - 8\sum_{m=1}^{\infty} q^m \sum_{\substack{n,k=1 \\ nk=m}}^{\infty} n(-1)^{n+k}$$
$$= 1 + 8\sum_{m=1}^{\infty} (-q)^m \sum_{\substack{d\geq 1 \\ 4\nmid d\mid m}} d.$$

Hence $r_4(n) = 8\sum_{d\geq 1;\ 4\nmid d\mid n} d$. This result of Jacobi implies Lagrange's theorem that every positive integer is a sum of four squares. By taking $a, b^2 \to -1$ in (3) the reader will have little trouble showing that $r_2(n) = 4(d_1(n) - d_3(n))$, with $d_k(n)$ the number of divisors of n of the form $4m + k$. This is a result of Gauss and Lagrange which implies Fermat's two-square theorem.

Other simple but important applications of the $_1\psi_1$ sum concern orthogonal polynomials. In [11] it was employed by Askey and Wilson to compute a special case—corresponding to the continuous q-Jacobi polynomials—of the Askey-Wilson integral, and in [10] Askey gave an elementary proof of the full Askey-Wilson integral using the $_1\psi_1$ sum. The sum also implies the norm evaluation of the weight functions of the q-Laguerre polynomials [45]. These form a family of orthogonal polynomials with discrete measure μ on $[0, c\cdot\infty)$ given by $\mathrm{d}_q\mu(t) = t^\alpha/(-t)_\infty\,\mathrm{d}_q t$. The normalization $\int \mathrm{d}_q\mu(t)$ thus follows from the q-beta integral (2) in the limit of large β.

Generalizations in One Dimension

There exist several generalizations of Ramanujan's sum containing one additional parameter. In his work on partial theta functions, Andrews [4] obtained a generalization in which each product of four infinite products on the right-hand side is replaced by six such products. Another example is the curious identity of Guo and Schlosser, which is no longer hypergeometric in nature [27]:

$$\sum_{k=-\infty}^{\infty} \frac{(a)_k(1 - ac_k q^k)(c_k q)_\infty (b/ac_k)_\infty}{(b)_k(1 - azq^k)(ac_k)_\infty (q/ac_k)_\infty}\,c_k^k$$
$$= \frac{1}{(1-z)}\,\frac{(q)_\infty (b/a)_\infty}{(q/a)_\infty (b)_\infty},$$

where $c_k := z(1 - aczq^k)/(1 - azq^k)$ and $|b/ac| < |z| < 1$. For $c = 1$ this is (1).

As discovered by Schlosser [49], a quite different extension of the $_1\psi_1$ sum arises by considering noncommutative variables. Let R be a unital Banach algebra with identity 1, central elements b and q, and norm $\|\cdot\|$. Write a^{-1} for the inverse of an invertible element $a \in R$. Let $\prod_{i=m}^{n} a_i$ stand for 1

if $n = m - 1$, $a_m \cdots a_n$ if $n \geq m$ and $a_{m-1}^{-1} \cdots a_{n+1}^{-1}$ if $n < m - 1$, and define

$$\begin{pmatrix} a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{pmatrix} ; z \begin{pmatrix} \end{pmatrix}_k^{\pm}$$
$$:= \prod_i \left[z \prod_{s=1}^r (1 - a_s q^{i-1})(1 - b_s q^{i-1})^{-1} \right],$$

where $k \in \mathbb{Z} \cup \{\infty\}$, $a_1, \ldots, a_r, b_1, \ldots, b_r \in R$, $\prod_i = \prod_{i=1}^k$ in the $+$ case and $\prod_i = \prod_{i=k}^1$ in the $-$ case. Subject to $\max\{\|q\|, \|z\|, \|ba^{-1}z^{-1}\|\} < 1$, the following noncommutative $_1\psi_1$ sum holds:

$$\sum_{k=-\infty}^{\infty} \begin{pmatrix} a \\ b \end{pmatrix} ; z \begin{pmatrix} \end{pmatrix}_k^+$$
$$= \begin{pmatrix} za \\ z \end{pmatrix} ; 1 \begin{pmatrix} \end{pmatrix}_{\infty}^- \begin{pmatrix} qa^{-1}z^{-1} \\ qza^{-1}z^{-1} \end{pmatrix} ; 1 \begin{pmatrix} \end{pmatrix}_{\infty}^+ \begin{pmatrix} bza^{-1}z^{-1}, q \\ ba^{-1}z^{-1}, b \end{pmatrix} ; 1 \begin{pmatrix} \end{pmatrix}_{\infty}^-.$$

Higher-Dimensional Generalizations

Various authors have generalized (1) to multiple $_1\psi_1$ sums. Below we state a generalization due to Gustafson and Milne [28], [41] which is labelled by the A-type root system. Similar such $_1\psi_1$ sums are given in [6], [7], [29], [43], [47]. More involved multiple $_1\psi_1$ sums with a Schur or Macdonald polynomial argument can be found in [12], [36], [42], [52]. For $r = (r_1, \ldots, r_n) \in \mathbb{Z}^n$ denote $|r| := r_1 + \cdots + r_n$. Then

$$\sum_{r \in \mathbb{Z}^n} z^{|r|} \prod_{1 \leq i < j \leq n} \frac{x_i q^{r_i} - x_j q^{r_j}}{x_i - x_j} \prod_{i,j=1}^n \frac{(a_j x_{ij})_{r_i}}{(b_j x_{ij})_{r_i}}$$
$$= \frac{(az)_{\infty}(q/az)_{\infty}}{(z)_{\infty}(b/az)_{\infty}} \prod_{i,j=1}^n \frac{(b_j x_{ij}/a_i)_{\infty}(q x_{ij})_{\infty}}{(q x_{ij}/a_i)_{\infty}(b_j x_{ij})_{\infty}},$$

where $a := a_1 \cdots a_n$, $b := q^{1-n} b_1 \cdots b_n$, $x_{ij} := x_i/x_j$ and $|b/a| < |z| < 1$. Milne first proved this for $b_1 = \cdots = b_n$ [41], and shortly thereafter Gustafson established the full result [28]. We have already seen that the $_1\psi_1$ sum implies the Jacobi triple-product identity. The latter is the $A_1^{(1)}$ case of Macdonald's generalized Weyl denominator identities for affine root systems [38]. To obtain further Macdonald identities from the Gustafson-Milne sum, one replaces $z \to z/a$ before letting $a_1, \ldots, a_n \to \infty$ and $b_1, \ldots, b_n \to 0$. Extracting the coefficient of z^0 (on the right this requires the triple-product identity) results in the Macdonald identity for $A_{n-1}^{(1)}$.

Higher-dimensional generalizations of a special case of the $_1\psi_1$ sum can be given for all affine root systems. A full description is beyond our scope here, so we will sketch only the simplest case. The reader is referred to [26], [38], [39], [40] for the full details. In [39] Macdonald gave the following multivariable extension of the product formula for the Poincaré polynomial of a Coxeter group:

(4)
$$W(t) := \sum_{w \in W} \prod_{\alpha \in R^+} \frac{1 - t_\alpha e^{w(\alpha)}}{1 - e^{w(\alpha)}} = \prod_{\alpha \in R^+} \frac{1 - t_\alpha t^{\mathrm{ht}(\alpha)}}{1 - t^{\mathrm{ht}(\alpha)}}.$$

Here R is a reduced, irreducible finite root system in a Euclidean space V, R^+ the set of positive roots, W the Weyl group, and t_α for $\alpha \in R^+$ a set of formal variables constant along Weyl orbits. The symbol $t^{\mathrm{ht}(\alpha)}$ stands for $\prod_{\beta \in R^+} t_\beta^{(\beta, \alpha)/\|\alpha\|^2}$ with $(\cdot, \cdot)$ the W-invariant positive definite bilinear form on V. If all t_α are set to t, then $t^{\mathrm{ht}(\alpha)} = t^{\mathrm{ht}(\alpha)}$ with $\mathrm{ht}(\alpha)$ the usual height function on R, in which case $W(t)$ reduces to the classical Poincaré polynomial $W(t)$. Now let S be a reduced, irreducible affine root system of type $S = S(R)$ [38]. In analogy with the finite case, assume that t_a for $a \in S$ is constant along orbits of the affine Weyl group W of S. Then Macdonald generalized (4) to [40]

(5)
$$\sum_{w \in W} \prod_{a \in S^+} \frac{1 - t_a e^{w(a)}}{1 - e^{w(a)}}$$
$$= \prod_{\alpha \in R^+} \frac{(t_\alpha t^{\mathrm{ht}(\alpha)})_{\infty}(t^{\mathrm{ht}(\alpha)} q^{\chi(\alpha \in B)}/t_\alpha)_{\infty}}{(t^{\mathrm{ht}(\alpha)})_{\infty}^2},$$

where B is a base for R. The parameter q on the right is fixed by $q = \prod_{a \in B(S)} \exp(n_a a)$, where $B(S)$ is a basis for S and the n_a are the labels of the extended Dynkin diagrams given in [38]. If R is simply-laced, then $t_a = t$. In the case of $S(R) = A_1^{(1)}$, $q = \exp(a_0 + a_1)$ so that after replacing $\exp(a_1)$ by x we obtain the $_1\psi_1$ sum (1) with $(a, b, z) \to (x/t, tx, t)$. This is not the end of the story concerning root systems and the $_1\psi_1$ sum. Identity (5) can be rewritten as [40]

(6)
$$\sum_{\gamma \in Q^\vee} \prod_{\alpha \in R} \frac{(q e^\alpha)_{(\alpha, \gamma)}}{(t_\alpha q e^\alpha)_{(\alpha, \gamma)}}$$
$$= \prod_{\alpha \in R^+} \frac{(t_\alpha t^{\mathrm{ht}(\alpha)} q)_{\infty}(t^{\mathrm{ht}(\alpha)} q^{\chi(\alpha \in B)}/t_\alpha)_{\infty}}{(t^{\mathrm{ht}(\alpha)} q)_{\infty}(t^{\mathrm{ht}(\alpha)})_{\infty}}$$
$$\times \frac{(q e^\alpha)_{\infty}(q e^{-\alpha})_{\infty}}{(t_\alpha q e^\alpha)_{\infty}(t_\alpha q e^{-\alpha})_{\infty}},$$

where $Q^\vee$ is the coroot lattice. Interestingly, for $t_\alpha = t$ this was also found by Fishel, Grojnowski and Teleman [26] by computing the generating function of the q-weighted Euler characteristics of certain Dolbeault cohomologies. For $R = A_{n-1}$, $Q^\vee = Q = \sum_{i=1}^n r_i \epsilon_i$ with $|r| = 0$, $R = \{\epsilon_i - \epsilon_j : 1 \leq i \neq j \leq n\}$, and $t^{\mathrm{ht}(\epsilon_i - \epsilon_j)} = t^{j-i}$. By fairly elementary manipulations, the identity (6) may then be transformed into the multiple $_1\psi_1$ sum

$$\sum_{r \in \mathbb{Z}^n} z^{|r|} \frac{(a)_{|r|}}{(b)_{|r|}} \prod_{1 \leq i < j \leq n} \frac{x_i q^{r_i} - x_j q^{r_j}}{x_i - x_j} \frac{(t^{-1} x_{ij})_{r_i - r_j}}{(t q x_{ij})_{r_i - r_j}} t^{r_i - r_j} q^{-r_j}$$

$$= \frac{(az)_\infty (q/az)_\infty (b/a)_\infty (tq)_\infty}{(z)_\infty (b/az)_\infty (q/a)_\infty (b)_\infty} \prod_{i=1}^{n-1} \frac{(t^{i+1} q)_\infty}{(t^i)_\infty} \prod_{i,j=1}^{n} \frac{(q x_{ij})_\infty}{(t q x_{ij})_\infty},$$

for $|b/a| < |z| < 1$ and $|t| < 1$. This is the only result in this survey that is new.

We finally remark that all higher-dimensional $_1\psi_1$ sums admit representations as discrete Selberg-type integrals. The most important such integrals are due to Aomoto [6], [7] and Ito [33], and are closely related to (5). Further examples may be found in [37], [51].

Photo courtesy of Krishnaswami Alladi.

The museum at SASTRA University in Kumbakonam, where several manuscripts, letters, papers, documents, photographs, etc., relating to Ramanujan are displayed.

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