REMARKS ON THE PAPER "SKEW PIERI RULES FOR HALL–LITTLEWOOD FUNCTIONS" BY KONVALINKA AND LAUVE

S. OLE WARNAAR

Abstract. In a recent paper Konvalinka and Lauve proved several skew Pieri rules for Hall–Littlewood polynomials. In this note we show that q-analogues of these rules are encoded in a q-binomial theorem for Macdonald polynomials due to Lascoux and the author.

1. The Konvalinka–Lauve formulas and their q-analogues

We refer the reader to [15] for definitions concerning Hall–Littlewood and Macdonald polynomials.

Let $P_{\lambda/\mu} = P_{\lambda/\mu}(X;t)$ and $Q_{\lambda/\mu} = Q_{\lambda/\mu}(X;t)$ be the skew Hall–Littlewood polynomials, $e_r = P_{(1^r)}$ the rth elementary symmetric function, h_r the rth complete symmetric function and $q_r = Q_{(r)}$. Then the ordinary Pieri formulas for Hall–Littlewood polynomials are given by [15]

$$P_\mu e_r = \sum_\lambda \mathrm{vs}_{\lambda/\mu}(t) P_\lambda \tag{1.1a}$$

$$P_\mu q_r = \sum_\lambda \mathrm{hs}_{\lambda/\mu}(t) P_\lambda, \tag{1.1b}$$

where the sums on the right are over partitions λ such that $|\lambda| = |\mu| + r$. The Pieri coefficient $\mathrm{vs}_{\lambda/\mu}(t)$ is given by [15, p. 215, (3.2)]

$$\mathrm{vs}_{\lambda/\mu}(t) = \prod_{i\geq 1} \begin{bmatrix} \lambda'_i - \lambda'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_t, \tag{1.2}$$

so that $\mathrm{vs}_{\lambda/\mu}(t)$ is zero unless $\mu \subseteq \lambda$ with $\lambda - \mu$ a vertical strip. Similarly, $\mathrm{hs}_{\lambda/\mu}(t)$ vanishes unless $\mu \subseteq \lambda$ with $\lambda - \mu$ a horizontal strip, in which case [15, p. 218, (3.10)]

$$\mathrm{hs}_{\lambda/\mu}(t) = \prod_{\substack{\lambda'_i = \mu'_i + 1 \\ \lambda'_{i+1} = \mu'_{i+1}}} \big(1 - t^{\lambda'_i - \lambda'_{i+1}}\big). \tag{1.3}$$

To express the skew Pieri formulas, Konvalinka and Lauve [9] (see also [8]) introduced a third Pieri coefficient

$$\mathrm{sk}_{\lambda/\mu}(t) := t^{n(\lambda/\mu)} \prod_{i\geq 1} \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_t, \tag{1.4}$$

where $n(\lambda/\mu) := \sum_{i\geq 1} \binom{\lambda'_i - \mu'_i}{2}$. Note that $\mathrm{sk}_{\lambda/\mu}(t) = 0$ if $\mu \not\subseteq \lambda$.

2010 *Mathematics Subject Classification.* 33D52,05E05.

Work supported by the Australian Research Council.

It seems Konvalinka and Lauve have been unaware that the above function has appeared in the literature before. Indeed, in exactly the above form and denoted as $g_\mu^\lambda(t)$, it was used by Kirillov to prove the Pieri rule [7, Lemma 4.1]

$$P_\mu h_r = \sum_\lambda \mathrm{sk}_{\lambda/\mu}(t) P_\lambda. \tag{1.5}$$

Moreover, $\mathrm{sk}_{\lambda/\mu}(t)$ arose in [20, Equation (4.3)] as a formula for the modified Hall–Littlewood polynomial $Q'_{\lambda/\mu}(1) = Q_{\lambda/\mu}(1,t,t^2,\dots)$—a result first stated in [12, Theorem 3.1], albeit in the not-so-easily-recognisable form

$$Q'_{\lambda/\mu}(1) = \begin{cases} t^{n(\lambda/\mu)} \displaystyle\prod_{i=1}^{l(\mu)} \frac{1-t^{\lambda'_{\mu_i}-i+1}}{(t;t)_{\mu'_i-\mu'_{i+1}}} & \text{for } \mu \subseteq \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

In a more general form pertaining to Macdonald polynomials it also appeared in [18, p. 173, Remark 2] and [19, Proposition 3.2], see (1.8) below. Prior to the above-mentioned papers $\mathrm{sk}_{\lambda/\mu}(t)$ appeared in the theory of abelian p-groups:

$$\mathrm{sk}_{\lambda/\mu}(t) = t^{n(\lambda)-n(\mu)} \alpha_\lambda(\mu; t^{-1}),$$

where $\alpha_\lambda(\mu;p)$ is the number of subgroups of type μ in a finite abelian p-group of type λ, [2–4, 21].

Theorem 1.1 (Konvalinka–Lauve [9, Theorems 2–4]). *For partitions $\nu \subseteq \mu$,*

$$P_{\mu/\nu} e_r = \sum_{\lambda,\eta} (-1)^{|\nu-\eta|} \mathrm{vs}_{\lambda/\mu}(t)\, \mathrm{sk}_{\nu/\eta}(t) P_{\lambda/\eta} \tag{1.6a}$$

$$P_{\mu/\nu} h_r = \sum_{\lambda,\eta} (-1)^{|\nu-\eta|} \mathrm{sk}_{\lambda/\mu}(t)\, \mathrm{vs}_{\nu/\eta}(t) P_{\lambda/\eta} \tag{1.6b}$$

$$P_{\mu/\nu} q_r = \sum_{\lambda,\eta,\omega} (-1)^{|\nu-\omega|} t^{|\omega-\eta|}\, \mathrm{hs}_{\lambda/\mu}(t)\, \mathrm{vs}_{\nu/\omega}(t)\, \mathrm{sk}_{\omega/\eta}(t) P_{\lambda/\eta}, \tag{1.6c}$$

where each of the multiple sums is subject to the restriction $|\lambda|+|\eta| = |\mu|+|\nu|+r$.

For $\nu = 0$ the first and third skew Pieri formulas reduce to (1.1a) and (1.1b) respectively, whereas the second formula simplifies to (1.5) (see also [9, Theorem 1]). Theorem 1.1 for $t=0$ gives the skew Pieri rules for Schur functions due to Assaf and McNamara [1] who, more generally, conjectured a skew Littlewood–Richardson rule. The identities (1.6a) and (1.6b) were first conjectured by Konvalinka in [8]. The subsequent proof of the theorem by Konvalinka and Lauve combines Hopf algebraic techniques in the spirit of the proof of the Assaf–McNamara conjecture [10] with intricate manipulations involving t-binomial coefficients.

The aim of this note is to point out that all of the skew Pieri formulas (1.6a)–(1.6c) are implied by a generalised q-binomial theorem for Macdonald polynomials and, consequently, have simple q-analogues.

From here on let $P_{\lambda/\mu} = P_{\lambda/\mu}(X;q,t)$ and $Q_{\lambda/\mu} = Q_{\lambda/\mu}(X;q,t)$ denote skew Macdonald polynomials. Let f be an arbitrary symmetric function. Adopting plethystic or λ-ring notation, see e.g., [5, 11], we define $f\big((a-b)/(1-t)\big)$ in terms of the power sums with positive index r as

$$p_r\Big(\frac{a-b}{1-t}\Big) = \frac{a^r-b^r}{1-t^r}.$$

In other words, $p_r\big((a-b)/(1-t)\big)=a^r\epsilon_{b/a,t}(p_r)$ with $\epsilon_{u,r}$ Macdonald's evaluation homomorphism [15, p. 338, (6.16)]. Equivalently, in terms of complete symmetric functions,

$$h_r\Big(\frac{a-b}{1-t}\Big)=[z^r]\,\frac{(bz;t)_\infty}{(az;t)_\infty}.$$

We now define the following five Pieri coefficients for Macdonald polynomials:

$$\mathrm{vs}_{\lambda/\mu}(q,t):=\psi'_{\lambda/\mu}(q,t)=(-1)^{|\lambda-\mu|}Q_{\lambda/\mu}\Big(\frac{q-1}{1-t}\Big) \tag{1.7a}$$

$$\mathrm{hs}_{\lambda/\mu}(q;t):=\varphi_{\lambda/\mu}(q,t)=Q_{\lambda/\mu}(1) \tag{1.7b}$$

$$\mathrm{sk}_{\lambda/\mu}(q,t):=Q_{\lambda/\mu}\Big(\frac{1-q}{1-t}\Big) \tag{1.7c}$$

$$\widehat{\mathrm{sk}}_{\lambda/\mu}(q,t):=Q_{\lambda/\mu}\Big(\frac{1-q/t}{1-t}\Big) \tag{1.7d}$$

$$\mathrm{ks}_{\lambda/\mu}(q,t):=Q_{\lambda/\mu}(-1), \tag{1.7e}$$

where $\psi'_{\lambda/\mu}(q,t)$ and $\varphi_{\lambda/\mu}(q,t)$ is notation used by Macdonald, and where the -1 in $Q_{\lambda/\mu}(-1)$ is a plethystic -1, i.e., applied to the power sum p_r of positive index r it gives the number -1. The Pieri coefficients $\mathrm{vs}_{\lambda/\mu}(q,t)$ and $\mathrm{hs}_{\lambda/\mu}(q,t)$ have nice factorised forms generalising (1.2) and (1.3), see [16, pp. 336–342]. So does $\widehat{\mathrm{sk}}_{\lambda/\mu}(q,t)$ [18, p. 173, Remark 2], [19, Proposition 3.2]:

$$\widehat{\mathrm{sk}}_{\lambda/\mu}(q,t)=\begin{cases} t^{n(\lambda)-n(\mu)}\displaystyle\prod_{i,j=1}^{l(\lambda)}\frac{(qt^{j-i-1};q)_{\lambda_i-\mu_j}(qt^{j-i};q)_{\mu_i-\mu_j}}{(qt^{j-i-1};q)_{\mu_i-\mu_j}(qt^{j-i};q)_{\lambda_i-\mu_j}} & \text{for } \mu\subseteq\lambda, \\ 0 & \text{otherwise,}\end{cases} \tag{1.8}$$

where $(a;q)_k:=(a;q)_\infty/(aq^k;q)_\infty$ for all $k\in\mathbb{Z}$. We leave it to the reader to verify that the above right-hand side for $q=0$ reduces to the right-hand side of (1.4). The remaining two Pieri coefficients do not factor into binomials. For example

$$\mathrm{sk}_{(2,1)/(1,0)}(q,t)=\frac{1-q-q^2+t+qt-q^2t}{1-q^2t}$$
$$\mathrm{ks}_{(2,1)/(1,0)}(q,t)=\frac{(1-t)(1+q-t+qt-t^2-qt^2)}{(1-q)(1-q^2t)}.$$

Of course, $\mathrm{sk}_{\lambda/\mu}(0,t)=\mathrm{sk}_{\lambda/\mu}(t)$ so it does factorise in the classical limit. This is however not the case for $\mathrm{ks}_{\lambda/\mu}(0,t)$, and

$$\mathrm{ks}_{(2,1)/(1,0)}(0,t)=(1-t)(1-t-t^2).$$

Let $g_r=g_r(X;q,t)=Q_{(r)}(X;q,t)$, so that $g_r(X;0,t)=q_r(X;t)$. Then the following q-analogue of Theorem 1.1 holds.

Theorem 1.2. *For partitions $\nu \subseteq \mu$,*

$$P_{\mu/\nu} e_r = \sum_{\lambda,\eta} (-1)^{|\nu-\eta|} \operatorname{vs}_{\lambda/\mu}(q,t) \operatorname{sk}_{\nu/\eta}(q,t) P_{\lambda/\eta} \tag{1.9a}$$

$$P_{\mu/\nu} h_r = \sum_{\lambda,\eta} (-1)^{|\nu-\eta|} \operatorname{sk}_{\lambda/\mu}(q,t) \operatorname{vs}_{\nu/\eta}(q,t) P_{\lambda/\eta} \tag{1.9b}$$

$$P_{\mu/\nu} g_r = \sum_{\lambda,\eta} \operatorname{hs}_{\lambda/\mu}(q,t) \operatorname{ks}_{\nu/\eta}(q,t) P_{\lambda/\eta} \tag{1.9c}$$

$$= \sum_{\lambda,\eta,\omega} (-1)^{|\nu-\omega|} t^{|\omega-\eta|} \operatorname{hs}_{\lambda/\mu}(q,t) \operatorname{vs}_{\nu/\omega}(q,t) \widehat{\operatorname{sk}}_{\omega/\eta}(q,t) P_{\lambda/\eta}, \tag{1.9d}$$

where each of the multiple sums is subject to the restriction $|\lambda| + |\eta| = |\mu| + |\nu| + r$.

2. The q-binomial theorem for Macdonald polynomials

In [14, Equation (2.11)] Lascoux and the author proved the following q-binomial theorem for Macdonald polynomials:

$$\sum_{\lambda} Q_{\lambda/\nu}\Big(\frac{a-b}{1-t}\Big) P_{\lambda/\mu}(X) = \bigg(\prod_{x\in X} \frac{(bx;q)_\infty}{(ax;q)_\infty}\bigg) \sum_{\lambda} Q_{\mu/\lambda}\Big(\frac{a-b}{1-t}\Big) P_{\nu/\lambda}(X). \tag{2.1}$$

For $\mu = \nu = 0$ and $(a,b) \mapsto (1,a)$ this is the well-known Kaneko–Macdonald q-binomial theorem [6,16]

$$\sum_{\lambda} t^{n(\lambda)} \frac{(a)_\lambda}{c'_\lambda} P_\lambda(X) = \prod_{x\in X} \frac{(ax;q)_\infty}{(x;q)_\infty}, \tag{2.2}$$

where we have used that [15, p. 338, (6.17)]

$$Q_\lambda\Big(\frac{1-a}{1-t}\Big) = t^{n(\lambda)} \frac{(a)_\lambda}{c'_\lambda}.$$

Here $(a)_\lambda = (a;q,t)_\lambda := \prod_{i\geq 1} (at^{1-i};q)_{\lambda_i}$ and $c'_\lambda = c'_\lambda(q,t)$ is the generalised hook polynomial $c'_\lambda = \prod_{s\in\lambda}\big(1 - q^{a(s)+1} t^{l(s)}\big)$ with $a(s)$ and $l(s)$ the arm-length and leg-length of the square $s \in \lambda$.

To show that (2.1) encodes the skew Pieri formulas (1.9a)–(1.9d) we first consider the $\mu = 0$ case

$$\sum_{\lambda} Q_{\lambda/\nu}\Big(\frac{a-b}{1-t}\Big) P_\lambda(X) = P_\nu(X) \prod_{x\in X} \frac{(bx;q)_\infty}{(ax;q)_\infty}. \tag{2.3}$$

If we multiply this by $Q_{\nu/\mu}\big((b-a)/(1-t)\big)$ and sum over ν using (2.3) with $(\lambda,\nu,a,b) \mapsto (\nu,\mu,b,a)$ we obtain

$$\sum_{\lambda,\nu} Q_{\lambda/\nu}\Big(\frac{a-b}{1-t}\Big) Q_{\nu/\mu}\Big(\frac{b-a}{1-t}\Big) P_\lambda(X) = P_\mu(X).$$

This implies the orthogonality relation (implicit in [17] and given in its more general nonsymmetric form in [13, Equation (6.5)])

$$\sum_{\nu} Q_{\lambda/\nu}\Big(\frac{a-b}{1-t}\Big) Q_{\nu/\mu}\Big(\frac{b-a}{1-t}\Big) = \delta_{\lambda\mu}. \tag{2.4}$$

Thanks to (2.4), identity (2.1) is equivalent to

$$\sum_{\lambda,\eta} Q_{\nu/\eta}\Big(\frac{a-b}{1-t}\Big) Q_{\lambda/\mu}\Big(\frac{b-a}{1-t}\Big) P_{\lambda/\eta}(X) = P_{\mu/\nu}(X) \prod_{x\in X} \frac{(ax;q)_\infty}{(bx;q)_\infty}.$$

There are now three special cases to consider. First, if $b=aq$ then

$$P_{\mu/\nu}(X) \prod_{x\in X}(1-ax) = \sum_{\lambda,\eta} a^{|\lambda-\mu|+|\nu-\eta|} Q_{\lambda/\mu}\Big(\frac{q-1}{1-t}\Big) Q_{\nu/\eta}\Big(\frac{1-q}{1-t}\Big) P_{\lambda/\eta}(X).$$

Equating coefficients of $(-a)^r$ and using definition (1.7) yields (1.9a). Next, if $a=bq$

$$P_{\mu/\nu}(X) \prod_{x\in X} \frac{1}{1-bx} = \sum_{\lambda,\eta} b^{|\lambda-\mu|+|\nu-\eta|} Q_{\lambda/\mu}\Big(\frac{1-q}{1-t}\Big) Q_{\nu/\eta}\Big(\frac{q-1}{1-t}\Big) P_{\lambda/\eta}(X).$$

Equating coefficients of b^r and again using (1.7) yields (1.9b). Finally, if $a=bt$

$$P_{\mu/\nu}(X) \prod_{x\in X} \frac{(btx;q)_\infty}{(bx;q)_\infty} = \sum_{\lambda,\eta} b^{|\lambda-\mu|+|\nu-\eta|} Q_{\lambda/\mu}(1) Q_{\nu/\eta}(-1) P_{\lambda/\eta}(X),$$

Equating coefficients of b^r and using (1.7) gives (1.9c). To show that (1.9c) and (1.9d) are equivalent, we recall Rains' q-Pfaff–Saalschütz summation for Macdonald polynomials [17, Corollary 4.9]:

$$\sum_\nu \frac{(a)_\nu}{(c)_\nu} Q_{\lambda/\nu}\Big(\frac{a-b}{1-t}\Big) Q_{\nu/\mu}\Big(\frac{b-c}{1-t}\Big) = \frac{(a)_\mu (b)_\lambda}{(b)_\mu (c)_\lambda} Q_{\lambda/\mu}\Big(\frac{a-c}{1-t}\Big), \tag{2.5}$$

which for $c=a$ is (2.4). Setting $b=a/q$ and $c=a/t$ and using (1.7) yields

$$\mathrm{ks}_{\lambda/\mu}(q,t) = (t/q)^{|\lambda-\mu|} \frac{(a/q)_\mu (a/t)_\lambda}{(a)_\mu (a/q)_\lambda} \sum_\nu (-1)^{|\lambda-\nu|} \frac{(a)_\nu}{(a/t)_\nu}\, \mathrm{vs}_{\lambda/\nu}(q,t)\, \widehat{\mathrm{sk}}_{\nu/\mu}(q,t).$$

Taking the $a\to\infty$ limit this further simplifies to

$$\mathrm{ks}_{\lambda/\mu}(q,t) = \sum_\nu (-1)^{|\lambda-\nu|} t^{|\nu-\mu|}\, \mathrm{vs}_{\lambda/\nu}(q,t)\, \widehat{\mathrm{sk}}_{\nu/\mu}(q,t),$$

which proves the equality between (1.9c) and (1.9d).

To conclude let us mention that all other identities of [9] admit simple q-analogues. For example, if we take (2.5) and specialise $b=a/q$ and $c=at$ then

$$\sum_\mu \frac{(a)_\mu}{(at)_\mu} (-1)^{|\lambda-\mu|}\, \mathrm{vs}_{\lambda/\mu}(q,t) Q_{\mu/\nu}\Big(\frac{1-qt}{1-t}\Big) = \frac{(a)_\nu (a/q)_\lambda}{(a/q)_\nu (at)_\lambda}\, q^{|\lambda-\nu|}\, \mathrm{hs}_{\lambda/\nu}(q,t).$$

Letting $a\to\infty$ this reduces to

$$\sum_\mu (-t)^{|\lambda-\mu|}\, \mathrm{vs}_{\lambda/\mu}(q,t) Q_{\mu/\nu}\Big(\frac{1-qt}{1-t}\Big) = \mathrm{hs}_{\lambda/\nu}(q,t).$$

For $q=0$ this is [9, Lemma 5]

$$\sum_\mu (-t)^{|\lambda-\mu|}\, \mathrm{vs}_{\lambda/\mu}(t)\, \mathrm{sk}_{\mu/\nu}(t) = \mathrm{hs}_{\lambda/\nu}(t).$$

Similarly, according to [13, Equation (6.23)]

$$\sum_\nu t^{n(\nu)} \frac{(a)_\nu}{c'_\nu} f^\lambda_{\mu\nu}(q,t) = Q_{\lambda/\mu}\Big(\frac{1-a}{1-t}\Big). \tag{2.6}$$

For $a=q=0$ this is [7, Corollary 4.2], [9, Corollary 6]

$$\sum_{\nu} t^{n(\nu)} f^{\lambda}_{\mu\nu}(t) = \mathrm{sk}_{\lambda/\mu}(t).$$

Finally, to obtain a q-analogue of [9, Theorem 7] we have to work a little harder. First note that

$$\begin{aligned} P_\nu(X)e_m(X)\sum_{r=0}^{\infty} h_r(X) &= \sum_{\mu} \mathrm{sk}_{\mu/\nu}(q,t)P_\mu(X)e_m(X) \\ &= \sum_{\mu}\sum_{\substack{\lambda \\ |\lambda-\mu|=m}} \mathrm{vs}_{\lambda/\mu}(q,t)\,\mathrm{sk}_{\mu/\nu}(q,t)P_\lambda(X). \end{aligned} \tag{2.7}$$

To compute this in a different way, observe that if we set $a=q$ in (2.2) then

$$\sum_{\lambda} t^{n(\lambda)} \frac{(q)_\lambda}{c'_\lambda}\, P_\lambda(X) = \prod_{x\in X}\frac{1}{1-x} = \sum_{r=0}^{\infty} h_r(X).$$

Using this as well as $e_m = P_{(1^m)}$ we get

$$P_\nu(X)e_m(X)\sum_{r=0}^{\infty} h_r(X) = \sum_{\eta} t^{n(\eta)}\frac{(q)_\eta}{c'_\eta}\, P_\nu(X)P_\eta(X)P_{(1^m)}(X).$$

By a double use of $P_\mu P_\nu = f^{\lambda}_{\mu\nu}P_\lambda$ this leads to

$$\begin{aligned} P_\nu(X)e_m(X)\sum_{r=0}^{\infty} h_r(X) &= \sum_{\eta} t^{n(\eta)}\frac{(q)_\eta}{c'_\eta}\, P_\nu(X)P_\eta(X)P_{(1^m)}(X) \\ &= \sum_{\mu,\eta} t^{n(\eta)}\frac{(q)_\eta}{c'_\eta}\, f^{\mu}_{\eta,(1^m)}(q,t)P_\mu(X)P_\nu(X) \\ &= \sum_{\lambda,\mu,\eta} t^{n(\eta)}\frac{(q)_\eta}{c'_\eta}\, f^{\mu}_{\eta,(1^m)}(q,t)f^{\lambda}_{\mu\nu}(q,t)P_\lambda(X) \\ &= \sum_{\lambda,\mu} \mathrm{sk}_{\mu/(1^m)}(q,t)f^{\lambda}_{\mu\nu}(q,t)P_\lambda(X), \end{aligned} \tag{2.8}$$

where the final equality follows from the $a=q$ case of (2.6). Equating coefficients of $P_\lambda(X)$ in (2.7) and (2.8) yields

$$\sum_{\substack{\mu \\ |\lambda-\mu|=m}} \mathrm{vs}_{\lambda/\mu}(q,t)\,\mathrm{sk}_{\mu/\nu}(q,t) = \sum_{\mu} \mathrm{sk}_{\mu/(1^m)}(q,t)f^{\lambda}_{\mu\nu}(q,t).$$

By (1.4),

$$\mathrm{sk}_{\lambda/(1^m)}(0,t) = \mathrm{sk}_{\lambda/(1^m)}(t) = t^{n(\lambda/(1^m))}\begin{bmatrix}\lambda'_1\\ m\end{bmatrix}_t = t^{n(\lambda)-\binom{m}{2}}\begin{bmatrix}\lambda'_1\\ m\end{bmatrix}_{t^{-1}},$$

so that for $q=0$ we obtain [9, Theorem 7]

$$\sum_{\substack{\mu \\ |\lambda-\mu|=m}} \mathrm{vs}_{\lambda/\mu}(t)\,\mathrm{sk}_{\mu/\nu}(t) = \sum_{\mu} t^{n(\lambda)-\binom{m}{2}} f^{\lambda}_{\mu\nu}(t)\begin{bmatrix}\lambda'_1\\ m\end{bmatrix}_{t^{-1}}.$$

Acknowledgements. I thank Matjaž Konvalinka and Aaron Lauve for helpful discussions.

References

1. S. H. Assaf and P. R. W. McNamara, *A Pieri rule for skew shapes*, J. Combin. Theory Ser. A **118** (2011), 277–290. 1
2. L. M. Butler, *Subgroup Lattices and Symmetric Functions*, Mem. Amer. Math. Soc. **112** (1994), no. 539. 1
3. S. Delsarte, *Fonctions de Möbius sur les groupes abelian finis*, Ann. Math. **49** (1948), 600–609. 1
4. P. E. Djubjuk, *On the number of subgroups of a finite abelian group*, Izv. Akad. Nauk USSR, Ser. Mat. **12** (1948), 351–378. 1
5. J. Haglund, The q,t-Catalan Numbers and the Space of Diagonal Harmonics, Univ. Lecture Ser., vol. 41, Amer. Math. Soc., Providence, RI, 2008. 1
6. J. Kaneko, *q-Selberg integrals and Macdonald polynomials*, Ann. Sci. Ecole Norm. Sup. (4) 29 (1996), 583–637. 2
7. A. N. Kirillov, *New combinatorial formula for modified Hall–Littlewood polynomials*, in *q-Series from a Contemporary Perspective*, pp. 283–333, Contemp. Math. Vol. 254, AMS, Providance, RI, 2000. 1, 2
8. M. Konvalinka, *Skew quantum Murnaghan–Nakayama rule*, J. Algebraic Combin. **35** (2012), 519–545. 1, 1
9. M. Konvalinka and A. Lauve, *Skew Pieri rules for Hall–Littlewood functions*, J. Algebraic Combin., to appear. 1, 1.1, 1, 2, 2, 2
10. T. Lam, A. Lauve and F. Sottile, *Skew Littlewood–Richardson rules from Hopf algebras*, Int. Math. Res. Not. IMRN **2011**, 1205–1219. 1
11. A. Lascoux, *Symmetric Functions and Combinatorial Operators on Polynomials*, CBMS Regional Conference Series in Math. Vol. 99, AMS, Providance, RI, 2003. 1
12. A. Lascoux, *Adding* 1 *to the argument of a Hall–Littlewood polynomial*, Sém. Lothar. Combin. **54** (2005/07) Art. B54n, 17 pp. 1
13. A. Lascoux, E. M. Rains and S. O. Warnaar, *Nonsymmetric interpolation Macdonald polynomials and* $\mathfrak{gl}_n$ *basic hypergeometric series*, Transform. Groups **14** (2009), 613–647. 2, 2
14. A. Lascoux and S. O. Warnaar, *Branching Rules for symmetric Macdonald polynomials and* $\mathfrak{sl}_n$ *basic hypergeometric series*, Adv. in Applied Math. **46** (2011), 424–456. 2
15. I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, Clarendon Press, Oxford, 1995. 1, 1, 1, 1, 2
16. I. G. Macdonald, *Hypergeometric series II*, unpublished manuscript. 1, 2
17. E. M. Rains, BC_n*-symmetric polynomials*, Transform. Groups **10** (2005), 63–132. 2, 2
18. E. M. Rains, BC_n*-symmetric abelian functions*, Duke Math. J. **135** (2006), 99–180. 1, 1
19. S. O. Warnaar, *q-Selberg integrals and Macdonald polynomials*, Ramanujan J. **10** (2005), 237–268. 1, 1
20. S. O. Warnaar and W. Zudilin, *Dedekind's* η*-function and Rogers–Ramanujan identities*, Bull. Lond. Math. Soc. **44** (2012), 1–11. 1
21. Y. Yeh, *On prime power Abelian groups*, Bull. Amer. Math. Soc. **54** (1948), 323–327. 1

School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia