PARTIAL THETA FUNCTIONS. I. BEYOND THE LOST NOTEBOOK

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Abstract. It is shown how many of the partial theta function identities in Ramanujan's lost notebook can be generalized to infinite families of such identities. Key in our construction is the Bailey lemma and a new generalization of the Jacobi triple product identity. By computing residues around the poles of our identities we find a surprising connection between partial theta functions identities and Garrett–Ismail–Stanton-type extensions of multisum Rogers–Ramanujan identities.

1. Introduction

G. E. Andrews' discovery in 1976 of Ramanujan's lost notebook [19] can probably be regarded as one of the most exciting finds ever in mathematics. The lost notebook, which was hidden in a box containing papers from the late G. N. Watson's estate, is a handwritten manuscript of well over a hundred pages of hardly decipherable but very beautiful identities. The first formula given by Andrews in his *An introduction to Ramanujan's "Lost" notebook* [5] is the following q-series transformation [19, p. 37]:

$$\sum_{n=0}^{\infty} \frac{q^n}{(1-a)\prod_{j=1}^{n}(1-aq^j)(1-q^j/a)} = \sum_{n=0}^{\infty}(-1)^n a^{3n} q^{n(3n+1)/2}(1-a^2q^{2n+1}) + \frac{\sum_{n=0}^{\infty}(-1)^n a^{2n+1} q^{n(n+1)/2}}{\prod_{j=1}^{\infty}(1-aq^{j-1})(1-q^j/a)} \tag{1.1}$$

on which Andrews puts the adjective "marvelous".

Characteristic of the above identity is that it contains a partial theta product $(1-a)\prod_{j=1}^{n}(1-aq^j)(1-q^j/a)$ and a partial theta sum $\sum_{n=0}^{\infty}(-a^2)^n q^{n(n+1)/2}$. Here it should be noted that complete theta products and sums are connected by Jacobi's famous triple product identity [16, Eq. (2.28)]

$$\sum_{n=-\infty}^{\infty}(-1)^n a^n q^{n(n-1)/2} = \prod_{n=1}^{\infty}(1-aq^{n-1})(1-q^n/a)(1-q^n). \tag{1.2}$$

There are many more identities for "partial theta functions" in the lost notebook. With the standard notation $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$ and $(a_1,\dots,a_k;q)_n =$

2000 *Mathematics Subject Classification.* Primary 05A30, 33D15, 33D90.

Work supported by the Australian Research Council.

$(a_1;q)_n\cdots(a_k;q)_n$, we state the following further examples [19, p. 12; p. 4; p. 29]

$$\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_n q^n}{(a;q)_{n+1}(q/a;q)_n}=\sum_{n=0}^{\infty}a^n q^{n(n+1)}+\sum_{n=0}^{\infty}\frac{a^{3n+1}q^{n(3n+2)}(1-aq^{2n+1})}{(a,q/a;q)_\infty} \tag{1.3}$$

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^n}{(a;q)_{n+1}(q/a;q)_n}=\sum_{n=0}^{\infty}a^n q^{n(n+1)/2}+\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}q^{n(n+1)}}{(-q,a,q/a;q)_\infty} \tag{1.4}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(q;q^2)_n q^{2n}}{(a;q^2)_{n+1}(q^2/a;q^2)_n}\\ &=\sum_{n=0}^{\infty}a^n q^{n(n+1)/2}+\sum_{n=0}^{\infty}\frac{(-1)^n a^{3n+1}q^{n(3n+2)}(1+aq^{2n+1})}{(-q;q)_\infty(a,q^2/a;q^2)_\infty}.\end{aligned} \tag{1.5}$$

As was typical of Ramanujan, there are no proofs of any of the partial theta function formulae in the lost notebook, making it impossible to determine how Ramanujan actually discovered them. Proofs of (1.1) and (1.3)–(1.5) (and of many more of Ramanujan's partial theta function identities) were found by Andrews [6, Eq. (1.1), (1.2), (3.16) and (3.14)]. Equation (1.1) was also proved by Andrews in [5] and by Fine [14, Eqs. (7.2) and (7.5)]. Andrews' proofs are at times quite intricate and rely heavily on standard and some not-so-standard identities for basic hypergeometric series. This perhaps partially explains why Ramanujan's partial theta function identities, though beautiful and deep, have remained rather isolated and have not become as widely appreciated and studied as, for example, Ramanujan's mock theta functions.

The aim of this paper is to show that Ramanujan's partial theta function identities are just the tip of the iceberg and that there is actually a lot of hidden structure to (1.1) and (1.3)–(1.5). For example, the identity (1.3) is the first in an infinite series of identities; the next identity in this series being closely related to another of Ramanujan's discoveries, that of the Rogers–Ramanujan identities [4]. Specifically, we claim

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(q^{n+1};q)_n q^n}{(a;q)_{n+1}(q/a;q)_n}\sum_{r=0}^{n}q^{r(r+1)}\begin{bmatrix}n\\r\end{bmatrix}_q\\ &=\sum_{n=0}^{\infty}a^n q^{2n(n+1)}+\sum_{i=1}^{4}(-1)^{i+1}a^i q^{\binom{i}{2}}\frac{(q^i,q^{5-i},q^5;q^5)_\infty}{(q,a,q/a;q)_\infty}\sum_{n=0}^{\infty}a^{5n}q^{2n(5n+2i)},\end{aligned}$$

where the q-binomial coefficients or Gaussian polynomials are defined by

$$\begin{bmatrix}n\\m\end{bmatrix}_q=\begin{cases}\dfrac{(q;q)_n}{(q;q)_m(q;q)_{n-m}} & m\in\{0,1,\dots,n\}\\ 0 & \text{otherwise.}\end{cases}$$

More generally we have the following theorem.

Theorem 1.1. *For k a positive integer, $\kappa = 2k+1$ and $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$ there holds*

$$\sum_{n=0}^{\infty} \frac{(q;q)_{2n} q^n}{(a;q)_{n+1}(q/a;q)_n} \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}}$$
$$= \sum_{n=0}^{\infty} a^n q^{kn(n+1)} + \sum_{i=1}^{\kappa-1} (-1)^{i+1} a^i q^{\binom{i}{2}} \frac{(q^i, q^{\kappa-i}, q^{\kappa}; q^{\kappa})_\infty}{(q, a, q/a; q)_\infty} \sum_{n=0}^{\infty} a^{\kappa n} q^{kn(\kappa n + 2i)}.$$

Here we define $(a;q)_n$ for all integers n as

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \tag{1.6}$$

so that, in particular, $1/(q;q)_n = 0$ for negative n. Observe that for $i \in \{1,2\}$ and $k=1$ the triple product $(q^i, q^{\kappa-i}, q^{\kappa}; q^{\kappa})_\infty$ becomes $(q,q^2,q^3;q^3)_\infty = (q;q)_\infty$. Since also $(q;q)_{2n}/(q;q)_n = (q^{n+1};q)_n$ one indeed finds (1.3) as the $k=1$ case of Theorem 1.1.

Our next theorem embeds (1.4) in an infinite family.

Theorem 1.2. *For k a positive integer, $\kappa = 2k$ and $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$ there holds*

$$\sum_{n=0}^{\infty} \frac{(q;q)_{2n} q^n}{(a;q)_{n+1}(q/a;q)_n} \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q^2;q^2)_{n_{k-1}}}$$
$$= \sum_{n=0}^{\infty} a^n q^{(\kappa-1)\binom{n+1}{2}}$$
$$+ \sum_{i=1}^{\kappa-1} (-1)^{i+1} a^i q^{\binom{i}{2}} \frac{(q^i, q^{\kappa-i}, q^{\kappa}; q^{\kappa})_\infty}{(q, a, q/a; q)_\infty} \sum_{n=0}^{\infty} (-1)^n a^{\kappa n} q^{(\kappa-1)(kn+i)n}.$$

To extend (1.5) we have to rewrite the term $(q;q^2)_n$ in the summand on the left as $(q^2;q^2)_{2n}/(q^2;q^2)_n(-q;q)_{2n}$.

Theorem 1.3. *For k a positive integer, $\kappa = 2k-1/2$ and $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$ there holds*

$$\sum_{n=0}^{\infty} \frac{(q;q)_{2n} q^n}{(a;q)_{n+1}(q/a;q)_n} \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}(-q^{1/2};q^{1/2})_{2n_{k-1}}}$$
$$= \sum_{n=0}^{\infty} a^n q^{(\kappa-1)\binom{n+1}{2}} + \sum_{i=1}^{2k-1} (-1)^{i+1} a^i q^{\binom{i}{2}} \frac{(q^i, q^{\kappa-i}, q^{\kappa}; q^{\kappa})_\infty}{(q, a, q/a; q)_\infty}$$
$$\times \sum_{n=0}^{\infty} (-1)^n a^{2\kappa n} q^{2(\kappa-1)(\kappa n+i)n} \bigl(1 + a^{2\kappa-2i} q^{2(\kappa-1)(\kappa-i)(2n+1)}\bigr).$$

Finally, the generalization of (1.1) to an infinite series is more complicated, involving a quintuple instead of triple product.

Theorem 1.4. *For k a positive integer, $\kappa=3k-1$ and $N_j=n_j+n_{j+1}+\cdots+n_{k-1}$ there holds*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(q;q)_{2n}q^n}{(a;q)_{n+1}(q/a;q)_n}\sum_{n_1,\dots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q;q)_{2n_{k-1}}}\\
&\quad=\sum_{n=0}^{\infty}(-1)^n a^{3n}q^{(2\kappa-3)(3n+1)n/2}\bigl(1-a^2q^{(2\kappa-3)(2n+1)}\bigr)\\
&\qquad+\sum_{i=1}^{\kappa-1}(-1)^{i+1}a^iq^{\binom{i}{2}}\frac{(q^i,q^{2\kappa-i},q^{2\kappa};q^{2\kappa})_\infty(q^{2\kappa-2i},q^{2\kappa+2i};q^{4\kappa})_\infty}{(q,a,q/a;q)_\infty}\\
&\qquad\qquad\times\Bigl[1-\sum_{n=1}^{\infty}a^{2\kappa n-2i}q^{(2\kappa-3)(\kappa n-i)n}\bigl(1-a^{2i}q^{2(2\kappa-3)in}\bigr)\Bigr].
\end{aligned}$$

To see this generalizes (1.1), note that $(q^i,q^{2\kappa-i},q^{2\kappa};q^{2\kappa})_\infty(q^{2\kappa-2i},q^{2\kappa+2i};q^{4\kappa})_\infty$ for $k=i=1$ becomes $(q,q^3,q^4;q^4)_\infty(q^2,q^6;q^8)_\infty=(q,q^2,q^3,q^4;q^4)_\infty=(q;q)_\infty$.

Several further results similar to Theorems 1.1–1.4, but reducing for $k=1$ to partial theta function identities not in the lost notebook, will also be proved in this paper. Apart from some deep but known results from the theory of q-series, the following generalized triple product identity will be crucial in our derivation of partial theta function identities.

Theorem 1.5. *There holds*

$$1+\sum_{n=1}^{\infty}(-1)^nq^{n(n-1)/2}(a^n+b^n)=(q,a,b;q)_\infty\sum_{n=0}^{\infty}\frac{(ab/q;q)_{2n}q^n}{(q,a,b,ab;q)_n}. \tag{1.7}$$

We believe this to be a very beautiful formula. Note in particular that for $b=q/a$ one recovers the Jacobi triple product identity (1.2). Indeed, making this specialization only the $n=0$ term contributes to the sum on the right, whereas the left simplifies to $\sum_{n=-\infty}^{\infty}a^nq^{n(n-1)/2}$. Another nice specialization occurs when $b=-a$. Substituting this and replacing a by $(aq)^{1/2}$ gives the transformation

$$1+2\sum_{n=1}^{\infty}a^nq^{2n^2}=(q;q)_\infty(aq;q^2)_\infty\sum_{n=0}^{\infty}\frac{(-a;q)_{2n}q^n}{(q,-aq;q)_n(aq;q^2)_n}.$$

The remainder of this paper is organized as follows. The next section contains a proof of the key identity (1.7). In section 3 it is shown how (1.7) can be applied to give a very general partial theta function identity. As examples Ramanujan's identities (1.3) and (1.4) are obtained. Section 4 is devoted to the proof of Theorems 1.1–1.4 and related identities. In sections 5 and 6 we exploit the fact that all of the partial theta function identities exhibit poles at $a=q^N$. Calculating the residues around these simple poles yields new identities which turn out to be Rogers–Ramanujan identities of the Garrett–Ismail–Stanton type. We conclude the paper in section 7 with a brief discussion of the possibilities and limitations of our approach to partial theta function identities. The proofs of several polynomial identities needed in the main text can be found in an appendix.

2. Proof of Theorem 1.5

As a first step we use (1.6) and $(a;q)_{2n}=(a;q)_n(aq^n;q)_n$ to put (1.7) in the form

$$\sum_{n=0}^{\infty}\frac{(aq^n,bq^n;q)_\infty(abq^{n-1};q)_nq^n}{(1-abq^{n-1})(q;q)_n} \\ =\frac{1}{(1-ab/q)(q;q)_\infty}\Big\{1+\sum_{n=1}^{\infty}(-1)^nq^{\binom{n}{2}}(a^n+b^n)\Big\}. \tag{2.1}$$

Expanding the left side using the q-binomial theorem [4, Eq. (3.3.6)]

$$\sum_{k=0}^{n}(-z)^kq^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q=(z;q)_n \tag{2.2}$$

and its limiting q-exponential sum [16, Eq. (II.2)]

$$\sum_{k=0}^{\infty}\frac{(-z)^kq^{\binom{k}{2}}}{(q;q)_k}=(z;q)_\infty \tag{2.3}$$

gives the fivefold sum

$$\mathrm{LHS}(2.1)=\sum_{n,i,j,k,l=0}^{\infty}\frac{(-1)^{i+j+k}a^{i+k+l}b^{j+k+l}q^{\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+n(i+j+k+l+1)-k-l}}{(q;q)_i(q;q)_j(q;q)_k(q;q)_{n-k}}.$$

Shifting $n\to n+k$, then summing over n using [16, Eq. (II.1)]

$$\sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n}=\frac{1}{(z;q)_\infty}, \tag{2.4}$$

and finally shifting $i\to i-k$ and $j\to j-k$ gives

$$\mathrm{LHS}(2.1)=\sum_{\substack{i,j,k,l=0\\k\le\min\{i,j\}}}^{\infty}\frac{(-1)^{i+j+k}a^{i+l}b^{j+l}q^{\binom{i}{2}+\binom{j}{2}+\binom{k+1}{2}+l(k-1)}(q;q)_{i+j+l-k}}{(q;q)_\infty(q;q)_{i-k}(q;q)_{j-k}(q;q)_k}.$$

Here the condition on the sum over k is added to avoid possible ambiguity for $i+j+l-k<0$. Employing the standard q-hypergeometric notation [16]

$$ {}_{r+1}\phi_r\left[\begin{matrix}a_1,\dots,a_{r+1}\\b_1,\dots,b_r\end{matrix};q,z\right]={}_{r+1}\phi_r(a_1,\dots,a_{r+1};b_1,\dots,b_r;q,z) \\ =\sum_{n=0}^{\infty}\frac{(a_1,\dots,a_{r+1};q)_n}{(q,b_1,\dots,b_r;q)_n}z^n$$

we can carry out the sum over k by the q-Chu–Vandermonde sum [16, (II.6)]

$$ {}_2\phi_1(a,q^{-n};c;q,q)=\frac{(c/a;q)_n}{(c;q)_n}a^n, \tag{2.5}$$

with $a=q^{-i}$, $n=j$ and $c=q^{-i-j-l}$. This yields

$$\mathrm{LHS}(2.1)=\sum_{i,j,l=0}^{\infty}\frac{(-1)^{i+j}a^{i+l}b^{j+l}q^{\binom{i}{2}+\binom{j}{2}-l}(q;q)_{i+l}(q;q)_{j+l}}{(q;q)_\infty(q;q)_i(q;q)_j(q;q)_l}.$$

After the changes $i \to i-l$ and $j \to j-l$ the sum over l can again be performed by (2.5), now with $a = q^{-i}$, $n = j$ and $c = 0$. Hence

$$\mathrm{LHS}(2.1) = \frac{1}{(q;q)_\infty} \sum_{i,j=0}^{\infty} (-1)^{i+j} a^i b^j q^{\binom{i}{2}+\binom{j}{2}-ij}.$$

Equating this with the right-hand side of (2.1) we are left to show that

$$(1-ab/q) \sum_{i,j=0}^{\infty} (-1)^{i+j} a^i b^j q^{\binom{i}{2}+\binom{j}{2}-ij} = 1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} (a^n + b^n). \tag{2.6}$$

Making the change $i \to i+j$ on the left and taking care to respect the ranges of summation gives

$$\begin{aligned}
\mathrm{LHS}(2.6) &= (1-ab/q) \sum_{i=-\infty}^{\infty} \sum_{j=\max\{0,-i\}}^{\infty} (-1)^i a^i (ab/q)^j q^{\binom{i}{2}} \\
&= \sum_{i=-\infty}^{\infty} (-1)^i a^i (ab/q)^{\max\{0,-i\}} q^{\binom{i}{2}} \\
&= 1 + \sum_{i=1}^{\infty} (-1)^i a^i q^{\binom{i}{2}} + \sum_{i=-\infty}^{-1} (-1)^i b^{-i} q^{\binom{i+1}{2}} = \mathrm{RHS}(2.6).
\end{aligned}$$

3. A general partial theta function identity

In this section we will show how (1.7) can be applied to yield a very general identity for partial theta functions from which Ramanujan's identities of the introduction easily follow. First we replace $a \to aq^{r+1}$ and $b \to bq^r$ in (1.7). Using (1.6), $(aq^k;q)_n = (a;q)_{n+k}/(a;q)_k$ and shifting $n \to n-r$ on the right this can be written as

$$\begin{aligned}
1 &+ \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} \big((aq^{r+1})^n + (bq^r)^n \big) \\
&= q^{-r} (q,a,b;q)_\infty \frac{1-abq^{2r}}{1-ab} \sum_{n=r}^{\infty} \frac{(ab;q)_{2n} q^n}{(q;q)_{n-r} (abq;q)_{n+r} (a;q)_{n+1} (b;q)_n}.
\end{aligned}$$

By the use of the triple product identity (1.2) it follows that

$$\begin{aligned}
(-1)^r q^{\binom{r+1}{2}} &\Big\{ 1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} \big((aq^{r+1})^n + (bq^r)^n \big) \Big\} \\
&= (q/b)^r (q,b,q/b;q)_\infty + \sum_{n=1}^{\infty} (-1)^{n+r} \Big\{ a^n q^{\binom{n+r+1}{2}} - (q/b)^n q^{\binom{n-r}{2}} \Big\}.
\end{aligned}$$

Consequently there holds

$$\begin{aligned}
(q/b)^r (q,b,q/b;q)_\infty &+ \sum_{n=1}^{\infty} (-1)^{n+r} \Big\{ a^n q^{\binom{n+r+1}{2}} - (q/b)^n q^{\binom{n-r}{2}} \Big\} \\
&= (-1)^r q^{\binom{r}{2}} (q,a,b;q)_\infty \frac{1-abq^{2r}}{1-ab} \sum_{n=r}^{\infty} \frac{(ab;q)_{2n} q^n}{(q;q)_{n-r} (abq;q)_{n+r} (a;q)_{n+1} (b;q)_n}.
\end{aligned}$$

Next we multiply both sides by f_r and sum r over the nonnegative integers leading to

$$\sum_{n=0}^{\infty}\frac{(ab;q)_{2n}q^n}{(a;q)_{n+1}(b;q)_n}\sum_{r=0}^{n}\frac{(-1)^r q^{\binom{r}{2}}f_r(1-abq^{2r})}{(q;q)_{n-r}(ab;q)_{n+r+1}}-\frac{(q/b;q)_\infty}{(a;q)_\infty}\sum_{r=0}^{\infty}(q/b)^r f_r \\ =\frac{1}{(q,a,b;q)_\infty}\sum_{n=1}^{\infty}\Big\{a^n\sum_{r=0}^{\infty}f_r+(q/b)^n\sum_{r=-\infty}^{-1}f_{-r-1}\Big\}(-1)^{n+r}q^{\binom{n+r+1}{2}}. \tag{3.1}$$

Of course one either has to impose conditions on f_r to ensure convergence of all sums or one has to view this identity as a formal power series in q. Nearly all our applications of this general result assume the relation $b=q/a$.

Proposition 3.1. *Provided all sums converge there holds*

$$\sum_{n=0}^{\infty}\frac{(q;q)_{2n}q^n}{(a;q)_{n+1}(q/a;q)_n}\sum_{r=0}^{n}\frac{(-1)^r q^{\binom{r}{2}}f_r(1-q^{2r+1})}{(q;q)_{n-r}(q;q)_{n+r+1}}-\sum_{r=0}^{\infty}a^r f_r \\ =\frac{1}{(q,a,q/a;q)_\infty}\sum_{n=1}^{\infty}\Big\{\sum_{r=0}^{\infty}f_r+\sum_{r=-\infty}^{-1}f_{-r-1}\Big\}(-1)^{n+r}a^n q^{\binom{n+r+1}{2}}. \tag{3.2}$$

All that remains to be done to turn this into a Ramanujan-type partial theta function identity is to appropriately choose f_r such that the sum

$$\beta_n=\sum_{r=0}^{n}\frac{(-1)^r q^{\binom{r}{2}}f_r(1-q^{2r+1})}{(q;q)_{n-r}(q;q)_{n+r+1}} \tag{3.3}$$

can be carried out explicitly. The most general such f_r appears to be

$$f_r=\frac{(b,c;q)_r}{(q^2/b,q^2/c;q)_r}\Big(\frac{q^2}{bc}\Big)^r. \tag{3.4}$$

Then, by the $a=q$ case of Rogers' q-Dougall sum [16, Eq. (II.21)]

$${}_6\phi_5\left[\begin{matrix}a,qa^{1/2},-qa^{1/2},b,c,q^{-n}\\ a^{1/2},-a^{1/2},aq/b,aq/c,aq^{n+1}\end{matrix};q,\frac{aq^{n+1}}{bc}\right]=\frac{(aq,aq/bc;q)_n}{(aq/b,aq/c;q)_n}, \tag{3.5}$$

it follows that

$$\beta_n=\frac{(q^2/bc;q)_n}{(q,q^2/b,q^2/c;q)_n}. \tag{3.6}$$

Inserted into Proposition 3.1 this leads to

$$\sum_{n=0}^{\infty}\frac{(q^{n+1},q^2/bc;q)_n q^n}{(a;q)_{n+1}(q/a,q^2/b,q^2/c;q)_n}-\sum_{r=0}^{\infty}\frac{(b,c;q)_r}{(q^2/b,q^2/c;q)_r}\Big(\frac{aq^2}{bc}\Big)^r \\ =\sum_{n=1}^{\infty}\frac{(-1)^n a^n q^{\binom{n+1}{2}}}{(q,a,q/a;q)_\infty}\sum_{r=-\infty}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}(b,c;q)_r}{(q^2/b,q^2/c;q)_r}\Big(\frac{q^{n+3}}{bc}\Big)^r, \tag{3.7}$$

where we have used the symmetry $f_{-r-1}=f_r$ as follows from

$$(a;q)_{-n}=\frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}.$$

The partial theta function identities (1.3) and (1.4) of Ramanujan arise as special limiting cases of this identity. First, when b and c tend to infinity the sum over r on the right simplifies to

$$\begin{aligned}\sum_{r=-\infty}^{\infty}(-1)^r q^{3\binom{r}{2}}q^{(n+3)r}&=(q^{n+3},q^{-n},q^3;q^3)_\infty\\&=(q;q)_\infty(-1)^{\lfloor (n+2)/3\rfloor}q^{-(n+1)(n+2)/6}\chi(n\not\equiv 0\pmod 3),\end{aligned}$$

where we have used the triple product identity (1.2) and

$$(q^{i+mn},q^{m-mn-i};q^m)_\infty=(q^i,q^{m-i};q^m)_\infty(-1)^n q^{-ni-\binom{n}{2}m}, \tag{3.8}$$

and where $\chi(\text{true})=1$, $\chi(\text{false})=0$ and $\lfloor x\rfloor$ is the integer part of x. Also letting $b,c\to\infty$ in the other terms in (3.7) we obtain Ramanujan's (1.3). This solves a problem of Andrews who remarked in [6]: "The primary reason that our proof is so complicated is that we have been unable to prove any generalization of $(1.2)_R$". Here $(1.2)_R$ is our (1.3) and the type of generalization Andrews alludes to is not a generalization like Theorem 1.1 but a generalization involving additional free parameters.

In much the same way as we obtained (1.3) one finds (1.4) after taking $b=-q$ and $c\to\infty$ in (3.7).

The q-Dougall sum (3.5) can also be used to derive quadratic and cubic analogues of (3.7). Specifically, from (3.5) it follows that

$$f_{2r}=\frac{(q,b;q^2)_r}{(q^2,q^3/b;q^2)_r}\Big(\frac{q^2}{b}\Big)^r,\quad f_{2r+1}=0\quad\text{and}\quad \beta_n=\frac{(q^2/b;q^2)_n}{(q,q^2/b;q)_n(q^2;q^2)_n},$$

and

$$f_{3r}=\frac{(q;q^3)_r}{(q^3;q^3)_r q^r},\quad f_{3r+1}=f_{3r+2}=0\quad\text{and}\quad \beta_n=\frac{(q;q^3)_n}{(q;q)_n(q;q)_{2n}}$$

both satisfy (3.3). By Proposition 3.1 we therefore have

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(q,q^2/b;q^2)_n q^n}{(a;q)_{n+1}(q,q/a,q^2/b;q)_n}-\sum_{r=0}^{\infty}\frac{(q,b;q^2)_r}{(q^2,q^3/b;q^2)_r}\Big(\frac{a^2q^2}{b}\Big)^r\\&=\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{n+1}a^n q^{\binom{n-2r}{2}}}{(q,a,q/a;q)_\infty}\frac{(q,b;q^2)_r}{(q^2,q^3/b;q^2)_r}\Big(\frac{q^2}{b}\Big)^r(1-q^{(4r+1)n})\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(q;q^3)_n q^n}{(a;q)_{n+1}(q,q/a;q)_n}-\frac{(a^3q^2;q^3)_\infty}{(a^3q;q^3)_\infty}\\&=\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{n+r+1}a^n q^{r+\binom{n-3r}{2}}}{(q,a,q/a;q)_\infty}\frac{(q;q^3)_r}{(q^3;q^3)_r}(1-q^{(6r+1)n}),\end{aligned}$$

where in the last equation the q-binomial theorem [16, (II.3)] ${}_1\phi_0(a;-;q;z)=(az;q)_\infty/(z;q)_\infty$ has been used to simplify the second term on the left.

Taking $b=1$ in the quadratic transformation leads to a formula that might well have been in the lost notebook,

$$\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_n q^n}{(a;q)_{n+1}(q^2,q/a;q)_n}=1+\frac{a+(1+a)\sum_{n=1}^{\infty}(-1)^n a^n q^{\binom{n+1}{2}}}{(q,a,q/a;q)_\infty}. \tag{3.9}$$

For $a=-1$ this further simplifies to the elegant summation

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^n}{(-q,q^2;q)_n}=2-\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Before we continue to derive all of Ramanujan's identities of the introduction we will slightly change viewpoint and reformulate Proposition 3.1 as a Bailey pair identity.

4. Partial theta functions and the Bailey lemma

Let $\alpha=\{\alpha_n\}_{n=0}^{\infty}$ and $\beta=\{\beta_n\}_{n=0}^{\infty}$. Then the pair of sequences (α,β) is called a Bailey pair relative to a if [11]

$$\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}. \tag{4.1}$$

Comparing this definition with (3.2) and identifying

$$\alpha_n=(-1)^n q^{\binom{n}{2}} f_n(1-q^{2n+1})/(1-q) \tag{4.2}$$

we get the following result.

Corollary 4.1. *For (α,β) a Bailey pair relative to q there holds*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\beta_n(q;q)_{2n}q^n}{(a;q)_{n+1}(q/a;q)_n}&-(1-q)\sum_{n=0}^{\infty}\frac{\alpha_n(-1)^n a^n q^{-\binom{n}{2}}}{1-q^{2n+1}}\\&=\frac{1}{(q^2,a,q/a;q)_\infty}\sum_{r=1}^{\infty}(-1)^{r+1}a^r q^{\binom{r}{2}}\sum_{n=0}^{\infty}\alpha_n q^{(1-r)n}\frac{1-q^{r(2n+1)}}{1-q^{2n+1}},\end{aligned} \tag{4.3}$$

provided all sums converge.

In view of this result we need to find suitable Bailey pairs relative to q. Before we present many such pairs, we recall a special case of Bailey's lemma [8, 18] which states that if (α,β) is a Bailey pair relative to a, then so is the pair (α',β') given by

$$\alpha_n'=a^n q^{n^2}\alpha_n \quad\text{and}\quad \beta_n'=\sum_{r=0}^{n}\frac{a^r q^{r^2}\beta_r}{(q;q)_{n-r}}.$$

Iterating this leads to what is called the Bailey chain. As will be shown shortly, all of our theorems of the introduction arise as Bailey chain identities. To generate new Bailey pairs we will also use the notion of a dual Bailey pair [6]. Let $(\alpha,\beta)=(\alpha(a,q),\beta(a,q))$ be a Bailey pair relative to a. Then the pair (α',β') given by

$$\alpha_n'=a^n q^{n^2}\alpha_n(a^{-1},q^{-1}) \quad\text{and}\quad \beta_n'=a^{-n}q^{-n(n+1)}\beta_n(a^{-1},q^{-1})$$

is again a Bailey pair relative to a. Since in the remainder of this section all Bailey pairs will have $a=q$ we will subsequently drop the phrase "relative to q".

As our first example we prove Theorem 1.1. The required initial Bailey pair is due to Rogers [20] and given as item B(3) in Slater's extensive list of Bailey pairs [23],

$$\alpha_n=(-1)^n q^{n(3n+1)/2}(1-q^{2n+1})/(1-q) \quad\text{and}\quad \beta_n=\frac{1}{(q;q)_n}.$$

Iterating this along the Bailey chain one finds

$$\begin{aligned}\alpha_n^{(k)} &= (-1)^n q^{kn(n+1)+\binom{n}{2}}(1-q^{2n+1})/(1-q)\\ \beta_n^{(k)} &= \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}}\end{aligned}$$

for k a positive integer and $(\alpha^{(1)},\beta^{(1)})=(\alpha,\beta)$. Combining this with Corollary 4.1 and applying the triple product identity (1.2) yields

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\beta_n^{(k)}(q;q)_{2n}q^n}{(a;q)_{n+1}(q/a;q)_n}-\sum_{n=0}^{\infty}a^n q^{kn(n+1)}\\ &\quad=\frac{1}{(q,a,q/a;q)_\infty}\sum_{r=1}^{\infty}(-1)^{r+1}a^r q^{\binom{r}{2}}\sum_{n=-\infty}^{\infty}(-1)^n q^{kn(n+1)+\binom{n+1}{2}-rn}\\ &\quad=\sum_{r=1}^{\infty}(-1)^{r+1}a^r q^{\binom{r}{2}}\frac{(q^r,q^{2k-r+1},q^{2k+1};q^{2k+1})_\infty}{(q,a,q/a;q)_\infty}.\end{aligned}$$

The summand on the right vanishes when $r\equiv 0 \pmod{2k+1}$. Replacing r by $i+(2k+1)n$ with $i\in\{1,\dots,2k\}$ and n a nonnegative integer and using (3.8), we arrive at Theorem 1.1.

The proof of Theorem 1.2 proceeds along the same lines. We begin with the Bailey pair [23, E(3)]

$$\alpha_n=(-1)^n q^{n^2}(1-q^{2n+1})/(1-q) \quad \text{and} \quad \beta_n=\frac{1}{(q^2;q^2)_n}$$

which implies the iterated pair

$$\begin{aligned}\alpha_n^{(k)} &= (-1)^n q^{n(kn+k-1)}(1-q^{2n+1})/(1-q)\\ \beta_n^{(k)} &= \sum_{n_1,\dots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q^2;q^2)_{n_{k-1}}}.\end{aligned}$$

Hence, by Corollary 4.1 and the triple product identity (1.2),

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\beta_n^{(k)}(q;q)_{2n}q^n}{(a;q)_{n+1}(q/a;q)_n}\\ &\qquad=\sum_{n=0}^{\infty}a^n q^{(2k-1)\binom{n+1}{2}}+\sum_{r=1}^{\infty}(-1)^{r+1}a^r q^{\binom{r}{2}}\frac{(q^r,q^{2k-r},q^{2k};q^{2k})_\infty}{(q,a,q/a;q)_\infty}.\end{aligned}$$

Replacing r by $i+2kn$ with $i\in\{1,\dots,2k-1\}$ and applying (3.8) gives Theorem 1.2.

Next we turn to Theorem 1.3, and for the first time a Bailey pair not in Slater's list is needed,

$$\alpha_n=(-1)^n q^{(3n-1)n/4}(1-q^{2n+1})/(1-q) \quad \text{and} \quad \beta_n=\frac{1}{(q^2;q^2)_n(-q^{1/2};q)_n} \tag{4.4}$$

which follows from the polynomial identity

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j-1)/4}{\begin{bmatrix}2n+1\\ n-j\end{bmatrix}}_q=(1-q^{2n+1})(q^{1/2};q)_n \tag{4.5}$$

proved in the appendix. We note that by (4.2) the above choice for α_n is equivalent to taking $f_r = q^{r(r+1)/4}$ in Proposition 3.1. Substituting the iterated Bailey pair

$$\alpha_n^{(k)} = (-1)^n q^{(3n-1)n/4+(k-1)n(n+1)}(1-q^{2n+1})/(1-q)$$
$$\beta_n^{(k)} = \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}(-q^{1/2};q^{1/2})_{2n_{k-1}}}$$

in equation (4.3) gives

$$\sum_{n=0}^{\infty} \frac{\beta_n^{(k)}(q;q)_{2n}q^n}{(a;q)_{n+1}(q/a;q)_n} = \sum_{n=0}^{\infty} a^n q^{(4k-3)(n+1)n/4} + \sum_{r=1}^{\infty}(-1)^{r+1}a^r q^{\binom{r}{2}} \frac{(q^r, q^{2k-r-1/2}, q^{2k-1/2}; q^{2k-1/2})_\infty}{(q,a,q/a;q)_\infty}.$$

Now using that for g_r such that $g_r = 0$ if $r \equiv 0 \pmod{4k-1}$ there holds

$$\sum_{r=1}^{\infty} g_r = \sum_{i=1}^{2k-1}\sum_{n=0}^{\infty}\bigl\{g_{(4k-1)n+i} + g_{(4k-1)(n+1)-i}\bigr\}$$

and applying (3.8) we obtain Theorem 1.3. For $k=1$ this corresponds to (1.5) with $q \to q^{1/2}$ since $(q^i, q^{2k-i-1/2}, q^{2k-1/2}; q^{2k-1/2})_\infty = (q^{1/2};q^{1/2})_\infty$ for $k=i=1$.

Finally we prove Theorem 1.4 which is based on the new Bailey pair

$$\alpha_n = (-1)^{\lfloor (4n+1)/3\rfloor} q^{(2n-1)n/3}\frac{1-q^{2n+1}}{1-q}\chi(n \not\equiv 1 \pmod 3) \tag{4.6}$$
$$\beta_n = \frac{1}{(q;q)_{2n}}$$

as can be extracted from the polynomial identity

$$\sum_{j=-\infty}^{\infty}\Bigl\{q^{j(6j-1)}{\textstyle\begin{bmatrix}2n+1\\ n-3j\end{bmatrix}_q} - q^{(2j+1)(3j+1)}{\textstyle\begin{bmatrix}2n+1\\ n-3j\end{bmatrix}_q}\Bigr\} = 1-q^{2n+1} \tag{4.7}$$

again proved in the appendix. Iteration along the Bailey chain yields

$$\alpha_n^{(k)} = (-1)^{\lfloor (4n+1)/3\rfloor} q^{(2n-1)n/3+(k-1)n(n+1)}\frac{1-q^{2n+1}}{1-q}\chi(n \not\equiv 1 \pmod 3)$$
$$\beta_n^{(k)} = \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q;q)_{2n_{k-1}}}$$

which by Corollary 4.1 implies

$$\sum_{n=0}^{\infty} \frac{\beta_n^{(k)}(q;q)_{2n}q^n}{(a;q)_{n+1}(q/a;q)_n} - \sum_{n=0}^{\infty}(-1)^n a^{3n} q^{n(3n+1)(2\kappa-3)/2}(1-a^2q^{(2\kappa-3)(2n+1)}) = \sum_{r=1}^{\infty}\frac{(-1)^{r+1}a^r q^{\binom{r}{2}}}{(q,a,q/a;q)_\infty}\sum_{n=-\infty}^{\infty} q^{\kappa n(3n+1)}\Bigl\{q^{-3nr} - q^{r(3n+1)}\Bigr\},$$

where $\kappa = 3k-1$. Interestingly it is now the quintuple product identity [16, Exercise 5.6]

$$\sum_{n=-\infty}^{\infty}(z^{3n} - z^{-3n-1})q^{n(3n+1)/2} = (zq, 1/z, q;q)_\infty (z^2q, q/z^2; q^2)_\infty$$

that is needed to transform the sum over n into a product. We thus find that the above right-hand side equals

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}a^r q^{\binom{r}{2}}}{(q,a,q/a;q)_\infty}(q^r,q^{2\kappa-r},q^{2\kappa};q^{2\kappa})_\infty(q^{2\kappa-2r},q^{2\kappa+2r};q^{4\kappa})_\infty. \tag{4.8}$$

To rewrite this further we use that for g_r such that $g_r=0$ if $r\equiv 0 \pmod{\kappa}$ there holds

$$\sum_{r=1}^{\infty} g_r=\sum_{i=1}^{\kappa-1}\Big[g_i+\sum_{n=1}^{\infty}\big(g_{2\kappa n+i}+g_{2\kappa n-i}\big)\Big].$$

Utilizing this and (3.8) expression (4.8) becomes

$$\begin{aligned}\sum_{i=1}^{\kappa-1}(-1)^{i+1}a^i q^{\binom{i}{2}}&\frac{(q^i,q^{2\kappa-i},q^{2\kappa};q^{2\kappa})_\infty(q^{2\kappa-2i},q^{2\kappa+2i};q^{4\kappa})_\infty}{(q,a,q/a;q)_\infty}\\ &\times\Big[1-\sum_{n=1}^{\infty}a^{2\kappa n-2i}q^{(2\kappa-3)(\kappa n-i)n}\big(1-a^{2i}q^{2(2\kappa-3)in}\big)\Big],\end{aligned}$$

concluding the proof of Theorem 1.4.

In the remainder of this section we will prove several further partial theta series identities that do not reduce to identities of Ramanujan when $k=1$. In fact, our first example assumes $k\geq 2$ for reasons of convergence. Calculating the Bailey pair dual to (4.6) gives

$$\begin{aligned}\alpha_n&=(-1)^{\lfloor(4n+1)/3\rfloor}q^{(n-2)n/3}\frac{1-q^{2n+1}}{1-q}\chi(n\not\equiv 1 \pmod 3)\\ \beta_n&=\frac{q^{n(n-1)}}{(q;q)_{2n}}.\end{aligned}$$

Iterating this Bailey pair and copying the previous proof one readily finds the following companion to Theorem 1.4.

Theorem 4.1. *For $k\geq 2$, $\kappa=3k-2$ and $N_j=n_j+n_{j+1}+\cdots+n_{k-1}$,*

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(q;q)_{2n}q^n}{(a;q)_{n+1}(q/a;q)_n}\sum_{n_1,\dots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-2}^2+2N_{k-1}^2+N_1+\cdots+N_{k-2}}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q;q)_{2n_{k-1}}}\\ &=\sum_{n=0}^{\infty}(-1)^n a^{3n}q^{(2\kappa-3)(3n+1)n/2}\big(1-a^2q^{(2\kappa-3)(2n+1)}\big)\\ &\quad+\sum_{i=1}^{\kappa-1}(-1)^{i+1}a^i q^{\binom{i}{2}}\frac{(q^i,q^{2\kappa-i},q^{2\kappa};q^{2\kappa})_\infty(q^{2\kappa-2i},q^{2\kappa+2i};q^{4\kappa})_\infty}{(q,a,q/a;q)_\infty}\\ &\qquad\times\Big[1-\sum_{n=1}^{\infty}a^{2\kappa n-2i}q^{(2\kappa-3)(\kappa n-i)n}\big(1-a^{2i}q^{2(2\kappa-3)in}\big)\Big].\end{aligned}$$

Next we consider the Bailey pair

$$\alpha_n=q^{\binom{n}{2}}(1-q^{2n+1})/(1-q)\quad\text{and}\quad\beta_n=\frac{(-1;q)_n}{(q;q)_n(q^2;q^2)_n} \tag{4.9}$$

which by (4.2) corresponds to $f_n=(-1)^n$. Hence (4.9) follows by taking $b=-c=q$ in (3.4) and (3.6). Inserting the iterated pair into (4.3) gives the next theorem.

Theorem 4.2. *For k a nonnegative integer and $N_j = n_j + n_{j+1} + \cdots + n_k$,*

$$\sum_{n=0}^{\infty} \frac{(q;q)_{2n} q^n}{(a;q)_{n+1}(q/a;q)_n} \sum_{n_1,\dots,n_k=0}^{\infty} \frac{q^{N_1^2+\cdots+N_k^2+N_1+\cdots+N_k}(-1;q)_{n_k}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_k}(q^2;q^2)_{n_k}}$$
$$= \sum_{n=0}^{\infty}(-1)^n a^n q^{kn(n+1)} + \sum_{r=1}^{\infty}\Big\{\sum_{n=0}^{\infty} - \sum_{n=-\infty}^{-1}\Big\} \frac{(-1)^{r+1}a^r q^{kn(n+1)+\binom{r-n}{2}}}{(q,a,q/a;q)_\infty}.$$

Note that the sum over n in the second term on the right takes the form of a false theta series for which there is no product form. This can be traced back to the fact that for $f_n = (-1)^n$ there holds $f_{-n-1} = -f_n$ instead of the usual $f_n = f_{-n-1}$. (Note that for generic b and c (3.4) satisfies $f_{-n-1} = f_n$ but that this is not necessarily so if either b or c assumes the "singular" value q.)

When $k = 0$ the right-hand side trivializes and we get

$$1 + 2\sum_{n=1}^{\infty} \frac{(q^n;q)_n q^n}{(q,aq,q/a;q)_n} = \frac{1-a}{1+a}\bigg(1 - 2\sum_{n=1}^{\infty}\frac{(-1)^n a^n q^{\binom{n}{2}}}{(q,a,q/a;q)_\infty}\bigg) \tag{4.10}$$

which simplifies nicely for $a = 1$ to

$$1 + 2\sum_{n=1}^{\infty} \frac{(q^n;q)_n q^n}{(q;q)_n^3} = \frac{1}{(q;q)_\infty^3}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}. \tag{4.11}$$

Similar to the previous example we take $b = q$ and $c \to \infty$ in (3.4) and (3.6) to obtain

$$\alpha_n = q^{n^2}(1-q^{2n+1})/(1-q) \quad \text{and} \quad \beta_n = \frac{1}{(q;q)_n^2}. \tag{4.12}$$

Carrying out the usual calculations this yields the second-last theorem of this section.

Theorem 4.3. *For k a nonnegative integer and $N_j = n_j + n_{j+1} + \cdots + n_k$,*

$$\sum_{n=0}^{\infty} \frac{(q;q)_{2n} q^n}{(a;q)_{n+1}(q/a;q)_n} \sum_{n_1,\dots,n_k=0}^{\infty} \frac{q^{N_1^2+\cdots+N_k^2+N_1+\cdots+N_k}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}(q;q)_{n_k}^2}$$
$$= \sum_{n=0}^{\infty}(-1)^n a^n q^{(2k+1)\binom{n+1}{2}} + \sum_{r=1}^{\infty}\Big\{\sum_{n=0}^{\infty} - \sum_{n=-\infty}^{-1}\Big\} \frac{(-1)^{r+1}a^r q^{(2k+1)\binom{n+1}{2}+\binom{r-n}{2}}}{(q,a,q/a;q)_\infty}.$$

Again a dramatic simplification occurs for the smallest value of k. By

$$\sum_{r=1}^{\infty}\Big\{\sum_{n=0}^{\infty} - \sum_{n=-\infty}^{-1}\Big\}(-1)^{r+1}a^r q^{\binom{n+1}{2}+\binom{r-n}{2}} = \sum_{r=1}^{\infty}\sum_{n=0}^{r-1}(-1)^{r+1}a^r q^{\binom{n+1}{2}+\binom{r-n}{2}}$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{r+n}a^{r+n+1}q^{\binom{n+1}{2}+\binom{r+1}{2}} = a\bigg(\sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n+1}{2}}\bigg)^2,$$

the $k = 0$ instance of Theorem 4.3 becomes

$$\sum_{n=0}^{\infty} \frac{(q^{n+1};q)_n q^n}{(a;q)_{n+1}(q,q/a;q)_n} = \sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n+1}{2}} + a\frac{\bigg(\sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n+1}{2}}\bigg)^2}{(q,a,q/a;q)_\infty}. \tag{4.13}$$

Our next Bailey pair can be expressed most concisely as

$$\alpha_{2n}=\alpha_{-2n-1}=(-1)^n q^{n(3n-1)}(1-q^{4n+1})/(1-q) \quad\text{and}\quad \beta_n=\frac{1}{(q;q)_n(q;q^2)_n},$$

where n in α_{-2n-1} is assumed to be negative. This Bailey pair can be read off from the polynomial identity

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j-1)}(1-q^{4j+1}){\textstyle\left[{2n+1 \atop n-2j}\right]_q}=(1-q^{2n+1})(-q;q)_n \tag{4.14}$$

established in the appendix. The resulting theorem is similar to those of the introduction but not quite as beautiful since we cannot carry out the usual reduction of the sum over r. (One can still use the quasi-periodicity of the triple product on the right to write the sum over r as a finite sum over i and an infinite sum over n such that the new triple product is n-independent, but due to lack of symmetry, the resulting equation lacks the usual elegance.)

Theorem 4.4. *For k a nonnegative integer and $N_j=n_j+n_{j+1}+\cdots+n_k$,*

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(q;q)_{2n}q^n}{(a;q)_{n+1}(q/a;q)_n}\sum_{n_1,\dots,n_k=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}(q;q^2)_{n_{k-1}}}\\
&=\sum_{n=0}^{\infty}(-1)^n a^{2n}q^{n^2+2(k-1)(2n+1)n}\bigl(1-aq^{(2k-1)(2n+1)}\bigr)\\
&\qquad+\Bigl\{\sum_{r=1}^{\infty}a^r-\sum_{r=-\infty}^{-1}a^{-r}\Bigr\}(-1)^{r+1}q^{\binom{r}{2}}\frac{(q^{6k-2r-2},q^{2k+2r},q^{8k-2};q^{8k-2})_\infty}{(q,a,q/a;q)_\infty}.\end{aligned}$$

As we have come to expect, the $k=1$ case permits a simplification:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q;q)_n q^n}{(a;q)_{n+1}(q/a;q)_n}&=1+(1+a)\sum_{n=1}^{\infty}(-1)^n a^{2n-1}q^{n^2}\\
&\qquad+\frac{a-(1+a)\sum_{n=1}^{\infty}a^{3n-1}q^{n(3n-1)/2}(1-aq^n)}{(q;q^2)_\infty(a,q/a;q)_\infty}.\end{aligned}\tag{4.15}$$

This generalizes the not at all deep, but elegant

$$\sum_{n=0}^{\infty}\frac{q^n}{(-q;q)_n}=2-\frac{1}{(-q;q)_\infty}$$

obtained for $a=-1$.

It will by now be overly clear that the list of nice applications of Corollary 4.1 is sheer endless, and without too much effort one can obtain many more new Bailey pairs relative to q such that α_n has a desired factor $(1-q^{2n+1})/(1-q)$. Most obvious would of course be to use the dual Bailey pairs corresponding to, for example, (4.4) and (4.12) (note that (4.9) is self-dual), which are given by

$$\alpha_n=(-1)^n q^{(n-3)n/4}(1-q^{2n+1})/(1-q) \quad\text{and}\quad \beta_n=\frac{(-1)^n q^{n(n-2)/2}}{(q^2;q^2)_n(-q^{1/2};q)_n}$$

$$\alpha_n=q^{-n}(1-q^{2n+1})/(1-q) \quad\text{and}\quad \beta_n=\frac{q^{-n}}{(q;q)_n^2}.$$

The more adventurous reader might also further explore the possibility of transforming known Bailey pairs into new ones. This has been our main technique (see the appendix) for deriving new Bailey pairs. As long as one is willing to allow for Bailey pairs somewhat more complicated than those presented so far, the number of possible pairs appears to be limitless. As an example, we have found numerous pairs of the type $\beta_n = q^n/(q;q)_{2n}$ and

$$\begin{aligned}
\alpha_{3n} &= q^{2n(3n-1)}(1-q^{6n+1})/(1-q) \\
\alpha_{3n-1} &= -q^{2n(3n-2)+1}(1-q^{6n-1})/(1-q) \\
\alpha_{3n+1} &= -q^{2n(3n+1)}(1-q^{2n+1})(1-q^{6n+3})/(1-q).
\end{aligned}$$

Unlike our earlier Bailey pairs this follows from a polynomial identity that can be viewed as a linear combination of alternating sums over q-binomial coefficients,

$$\begin{aligned}
\text{(4.16)}\qquad \sum_{j=-\infty}^{\infty} q^{2j(3j-1)}\Bigl\{ {\textstyle\begin{bmatrix}2n+1\\ n-3j\end{bmatrix}_q - \begin{bmatrix}2n+1\\ n-3j+2\end{bmatrix}_q} \Bigr\} \\
- q\sum_{j=-\infty}^{\infty} q^{2j(3j+2)}\Bigl\{ {\textstyle\begin{bmatrix}2n+1\\ n-3j\end{bmatrix}_q - \begin{bmatrix}2n+1\\ n-3j-1\end{bmatrix}_q} \Bigr\} = q^n(1-q^{2n+1}).
\end{aligned}$$

Admittedly the identities obtained after iterating pairs such as the one above are too involved to be of great interest, but direct substitution in (4.3) often leads to results not much beyond those of Ramanujan. Our present example, for example, yields (after some tedious but elementary calculations)

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{q^{2n}}{(a;q)_{n+1}(q/a;q)_n} = 1 + a + (1+a^2)\sum_{n=1}^{\infty}(-1)^n a^{3n-2} q^{n(3n-1)/2}(1+aq^n) \\
+ \frac{a^2 + (1+a^2)\sum_{n=1}^{\infty}(-1)^n a^{2n} q^{\binom{n+1}{2}}}{(a,q/a;q)_\infty}.
\end{aligned}$$

This is so close to (1.1) that it is surprising Ramanujan missed it. A rather curious formula arises when we set $a=-1$,

$$\sum_{n=0}^{\infty}\frac{q^{2n}}{(-q;q)_n^2} - \frac{1+2\sum_{n=1}^{\infty}(-1)^n q^{\binom{n+1}{2}}}{(-q;q)_\infty^2} = 4\sum_{n=1}^{\infty} q^{n(3n-1)/2}(1-q^n).$$

We challenge the reader to explain why all nonzero coefficients on the right are ± 4.

5. Residual identities

In the following two sections we exploit Andrews' observation [8] that by calculating the residue around the pole $a = q^N$ in Ramanujan's partial theta function identities and by then invoking analyticity to replace q^N by a, new, nontrivial identities arise. Of course, instead of considering just Ramanujan's identities we will apply Andrews' trick to the more general theorems obtained in the previous section. As a first example however, we treat (3.7) in some detail. Let $a_0 = q^N$ with N a nonnegative integer and multiply both sides of (3.7) by $(a-a_0)$. The resulting identity is then of the form

$$\sum_{n=0}^{\infty} f_n(a) - \sum_{r=0}^{\infty} g_r(a) = \sum_{n=1}^{\infty} h_n(a),$$

with $\lim_{a\to a_0} f_n(a) = f_n(a_0)\chi(n\geq N)$ and $\lim_{a\to a_0} g_r(a)=0$. We thus infer that

$$\sum_{n=0}^{\infty} f_{n+N}(a_0) = \sum_{n=1}^{\infty} h_n(a_0).$$

By straightforward calculations this can be put in the form

$$\begin{aligned}(5.1)\quad \frac{(aq^2/b,aq^2/c,q^2/bc;q)_\infty}{(q^2/b,q^2/c,aq^2/bc;q)_\infty}&\sum_{n=0}^{\infty}\frac{(a^2q^{n+1},aq^2/bc;q)_n q^n}{(q,aq,aq^2/b,aq^2/c;q)_n}\\ &=\sum_{n=1}^{\infty}\frac{(-1)^n a^{n-1} q^{\binom{n+1}{2}}}{(q,q,aq;q)_\infty}\sum_{r=-\infty}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}(b,c;q)_r}{(q^2/b,q^2/c;q)_r}\Bigl(\frac{q^{n+3}}{bc}\Bigr)^r,\end{aligned}$$

where we have replaced a_0 by a. By standard analyticity arguments we may now assume a to be an indeterminate. When b and c tend to infinity the sum over r can be carried out by the triple product identity leading to [6, Eq. (8.1)]

$$(5.2)\qquad \sum_{n=0}^{\infty}\frac{(a^2q^{n+1};q)_n q^n}{(q,aq;q)_n}=\frac{1}{(q,aq;q)_\infty}\sum_{n=0}^{\infty} a^{3n}q^{n(3n+2)}(1-aq^{2n+1}).$$

Similarly, when $b=-q$ and $c\to\infty$ we obtain

$$(5.3)\qquad \sum_{n=0}^{\infty}\frac{(aq^{n+1};q)_n q^n}{(q;q)_n(aq^2;q^2)_n}=\frac{1}{(q;q)_\infty(aq^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n a^n q^{n(n+1)}.$$

The last two identities are in fact closely related to other results from the lost notebook. If we take $a=-1$ in (5.2) and then apply Heine's fundamental transformation [16, Eq. (III.1)]

$$(5.4)\qquad {}_2\phi_1(a,b;c;q,z)=\frac{(b,az;q)_\infty}{(c,z;q)_\infty}\,{}_2\phi_1(c/b,z;az;q,b),$$

with $b=-a=q^{1/2}$, $c=0$ and $z=q$ to transform the left side, we obtain after replacing q by q^2 [19, p. 37]

$$\sum_{n=0}^{\infty}\frac{q^n}{(-q;q^2)_{n+1}}=\sum_{n=0}^{\infty}(-1)^n q^{2n(3n+2)}(1+q^{4n+2}).$$

This is the second "marvelous" formula given by Andrews in his introduction to the lost notebook [5, Eq. (1.2)]. In [5] Andrews proofs this result by different means.

To transform (5.3) into another of Ramanujan's formulas is only slightly more involved. First we use

$$\frac{(aq^{n+1};q)_n}{(aq^2;q^2)_n}=\frac{(aq;q^2)_n}{(aq;q)_n}=\frac{(-aq;q)_n(aq;q^2)_n}{(a^2q^2;q^2)_n}$$

to write (5.3) as

$$\sum_{n=0}^{\infty}\frac{(-aq;q)_n(aq;q^2)_n q^n}{(q;q)_n(a^2q^2;q^2)_n}=\frac{1}{(q;q)_\infty(aq^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n a^n q^{n(n+1)}.$$

By the quadratic analogue of Heine's fundamental transformation [1, Thm. A_3]

$$\sum_{n=0}^{\infty}\frac{(a;q^2)_n(b;q)_{2n}z^n}{(q^2;q^2)_n(c;q)_{2n}}=\frac{(b;q)_\infty(az;q^2)_\infty}{(c;q)_\infty(z;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(c/b;q)_n(z;q^2)_n b^n}{(q;q)_n(az;q^2)_n},$$

with $a = z \to aq$, $b \to q$ and $c \to -aq^2$ this gives the result

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n (aq;q^2)_n (aq)^n}{(-aq;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n a^n q^{n(n+1)}.$$

For $a = 1$ this is a lost notebook identity [19, p. 13] proved by Andrews [6, Eq. (6.2)].

A further interesting specialization of (5.1) arises when we set $a = -1$. Then the double sum on the right can be simplified to a single sum resulting in a special case of Hall's ${}_3\phi_2$ transformation. First we keep a general and use [7, Lemma 2]

$$\sum_{n=0}^{\infty} \frac{(q/a;q)_{n+1}(a,b,c;q)_n}{(q;q)_{2n+1}} \Big(\frac{q^2}{bc}\Big)^n \\ = \frac{(q^2/b, q^2/c;q)_\infty}{(q,q^2/bc;q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} (b,c;q)_r}{(q^2/b,q^2/c;q)_r} \Big(\frac{aq^2}{bc}\Big)^r,$$

with $a = q^{n+1}$ to write (5.1) as

$$\frac{(aq^2/b, aq^2/c;q)_\infty}{(aq^2/bc;q)_\infty} \sum_{n=0}^{\infty} \frac{(a^2q^{n+1}, aq^2/bc;q)_n q^n}{(q,aq,aq^2/b,aq^2/c;q)_n} \\ = \frac{1}{(q,aq;q)_\infty} \sum_{r=0}^{\infty} (b,c;q)_r \Big(\frac{aq^2}{bc}\Big)^r \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{\binom{n+1}{2}} (q^{2r+2};q)_n}{(q;q)_n}.$$

Here we have changed the order of summation on the right and shifted $n \to n + r$. For $a = -1$ the sum over n on the right yields $1/(q;q^2)_{r+1}$ by the $b \to \infty$ limit of the Bailey–Daum sum [16, Eq. (II.9)]

$${}_2\phi_1(a,b;aq/b;q,-q/b) = \frac{(aq, aq^2/b^2;q^2)_\infty}{(q,q^2)_\infty (-q/b, aq/b;q)_\infty}.$$

Therefore

$$\frac{(q,-q,-q^2/b,-q^2/c;q)_\infty}{(-q^2/bc;q)_\infty} \sum_{n=0}^{\infty} \frac{(q;q^2)_n (-q^2/bc;q)_n q^n}{(q,-q^2/b,-q^2/c;q)_n} = \sum_{r=0}^{\infty} \frac{(b,c;q)_r}{(q;q^2)_{r+1}} \Big(\frac{-q^2}{bc}\Big)^r$$

which can be recognized as a specialization of the ${}_3\phi_2$ transformation [16, Eq. (III.10)].

6. Garrett–Ismail–Stanton-type identities

Before calculating more residual identities we review a recent development in the theory of Rogers–Ramanujan identities initiated by Garrett, Ismail and Stanton [15] and further exploited by Andrews, Knopfmacher and Paule [10], and Berkovich and Paule [12, 13]. As will be shown later, calculating the residues of Theorems 1.1–1.4 provides a surprising connection between partial theta function identities and Garrett–Ismail–Stanton-type identities.

The famous Rogers–Ramanujan identities are given by

$$\sum_{r=0}^{\infty} \frac{q^{r^2}}{(q;q)_r} = \frac{1}{(q,q^4;q^5)_\infty} \quad \text{and} \quad \sum_{r=0}^{\infty} \frac{q^{r(r+1)}}{(q;q)_r} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{6.1}$$

It is easy to see that the left-hand side of the first (second) Rogers–Ramanujan identity is the generating function of partitions of n with difference between parts at least 2 (and no part equal to 1). Schur exploited this fact in one of his proofs of the Rogers–Ramanujan identities [22] and introduced two sequences of polynomials $\{e_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ where e_n (d_n) is the generating function corresponding to the left-side of the first (second) Rogers–Ramanujan identity with the added condition on the partitions that their largest part does not exceed $n-1$. He then went on to show that both e_n and d_n satisfy the recurrence

$$x_{n+1} = x_n + q^n x_{n-1}. \tag{6.2}$$

Schur's main result was the following closed form expressions for e_n and d_n

$$e_n = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j-1)/2} \begin{bmatrix} n \\ \lfloor (n-5j+1)/2 \rfloor \end{bmatrix}_q \tag{6.3a}$$

$$d_n = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j-3)/2} \begin{bmatrix} n \\ \lfloor (n-5j+2)/2 \rfloor \end{bmatrix}_q . \tag{6.3b}$$

To see that this settles the Rogers–Ramanujan identities observe that by the triple product identity (1.2) e_n and d_n tend to the respective right-hand sides of (6.1) in the large n limit. Alternative representations for e_n and d_n, probably known to Schur, but first explicitly given by MacMahon [17, §286 and §289] are

$$e_n = \sum_{r=0}^{\infty} q^{r^2} \begin{bmatrix} n-r \\ r \end{bmatrix}_q \quad \text{and} \quad d_n = \sum_{r=0}^{\infty} q^{r(r+1)} \begin{bmatrix} n-r-1 \\ r \end{bmatrix}_q .$$

The two polynomial analogues of the Rogers–Ramanujan identities obtained by equating the different representations for e_n and d_n were first given by Andrews in [2]. After this introduction we now come to the beautiful discovery of Garrett, Ismail and Stanton who found that for m a nonnegative integer [15, Eq. (3.5)]

$$\sum_{r=0}^{\infty} \frac{q^{r(r+m)}}{(q;q)_r} = \frac{(-1)^m q^{-\binom{m}{2}} d_{m-1}}{(q,q^4;q^5)_\infty} - \frac{(-1)^m q^{-\binom{m}{2}} e_{m-1}}{(q^2,q^3;q^5)_\infty}. \tag{6.4}$$

Here $e_{-1} = d_0 = 0$ and $e_0 = d_{-1} = 1$ consistent with the recurrence (6.2). For $m=0$ and $m=1$ we of course just find the first and second Rogers–Ramanujan identity. A polynomial analogue of (6.4) was found by Andrews *et al.* [10, Prop. 1] in the course of proving (6.4) via an extended Engel expansion. Garrett *et al.* also found the inverse of (6.4) given by [15, Thm. 3.1]

$$\frac{(q^{3-2i}, q^{2i+2}, q^5; q^5)_\infty}{(q;q)_\infty} = \sum_{j=0}^{\lfloor i/2 \rfloor} (-1)^j q^{2j(j-i)+\binom{j+1}{2}} \begin{bmatrix} i-j \\ j \end{bmatrix}_q \sum_{r=0}^{\infty} \frac{q^{r(r+i-2j)}}{(q;q)_r}. \tag{6.5}$$

Next we need a major generalization of (6.4) due to Berkovich and Paule [12]. As a first step they use the recurrence (6.2) to obtain the following "negative m analogue" of (6.4) [12, Eq. (1.10)]

$$\sum_{r=0}^{\infty} \frac{q^{r(r-m)}}{(q;q)_r} = \frac{e_m(1/q)}{(q,q^4;q^5)_\infty} + \frac{d_m(1/q)}{(q^2,q^3;q^5)_\infty}$$

for m a nonnegative integer. Berkovich and Paule then proceed to generalize this to the Andrews–Gordon identities given by [3]

$$\sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}} = \frac{(q^i, q^{\kappa-i}, q^\kappa; q^\kappa)_\infty}{(q;q)_\infty}$$

for $i \in \{1,\dots,k\}$, $\kappa = 2k+1$ and N_j defined as usual. Before we can give their result we need to define generalizations of the polynomials e_n and d_n as follows

$$
\begin{aligned}
(6.6)\qquad X_{s,b}^{(p,p')}(n;q) &= X_{s,b}^{(p,p')}(n) \\
&= \sum_{j=-\infty}^{\infty} \Big\{ q^{j(pp'j+p'-ps)} {\begin{bmatrix} n \\ (n+s-b)/2-p'j \end{bmatrix}}_q - q^{(pj+1)(p'j+s)} {\begin{bmatrix} n \\ (n-s-b)/2-p'j \end{bmatrix}}_q \Big\},
\end{aligned}
$$

where p, p', s, b and n are integers such that $n+s+b$ is even. The following duality relation [21, Eq. (2.3) and (2.9)] will be needed later

$$
X_{s,1}^{(p,p')}(n;q) = q^{(n-s+1)(n+s-3)/4} X_{s,1}^{(p'-p,p')}(n;1/q). \tag{6.7}
$$

Comparing definition (6.6) with (6.3a) and (6.3b) shows that $e_n = X_{2,2+\sigma}^{(2,5)}(n)$ and $d_n = X_{1,3-\sigma}^{(2,5)}(n)$ where $\sigma \in \{0,1\}$ is fixed by the condition that $n+\sigma$ is even. We are now prepared to state the generalization of (6.4) as found by Berkovich and Paule [12, Eq. (3.21)]

$$
\begin{aligned}
(6.8)\qquad & \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_{i'}+\cdots+N_{k-1}-mN_1}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}} \\
&\qquad = \sum_{\substack{i=1 \\ i+i'+m \text{ even}}}^{\kappa-1} \frac{(q^i, q^{\kappa-i}, q^\kappa; q^\kappa)_\infty}{(q;q)_\infty} X_{i,i'}^{(2,\kappa)}(m;1/q)
\end{aligned}
$$

for $i' \in \{1,\dots,k\}$ and m a nonnegative integer. For $k=2$ and $i'=2$ this is (6.4) but even for $k=2$ and $i'=1$ this is new. A polynomial analogue of the $i'=k$ case of the above identity can be found in [13, Eq. (1.31)].

After this long introduction into Garrett–Ismail–Stanton-type generalizations of identities of the Rogers–Ramanujan-type we return to our partial theta function identities and calculate the corresponding residual identities. First we consider Theorem 1.1 which implies the identity.

Corollary 6.1. *For $k \geq 2$ and $\kappa = 2k+1$,*

$$
\begin{aligned}
(6.9)\qquad & \sum_{n=0}^{\infty} \frac{(a^2q^{n+1};q)_n q^n}{(q;q)_n} \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(aq;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}} \\
&\qquad = \sum_{i=1}^{\kappa-1} (-1)^{i+1} a^{i-1} q^{\binom{i}{2}} \frac{(q^i, q^{\kappa-i}, q^\kappa; q^\kappa)_\infty}{(q,q,aq;q)_\infty} \sum_{n=0}^{\infty} a^{\kappa n} q^{kn(\kappa n+2i)}.
\end{aligned}
$$

The identity corresponding to $k=1$ is given by (5.2).

The similarity between (6.8) and (6.9) is quite striking and in the following we will show how equating coefficients of a^n in the power series expansion of (6.9) leads to (6.8) for $i'=1$. As the obvious first step we use (1.6) and (3.8) to rewrite the above identity as

$$\sum_{n=0}^{\infty}\frac{(a^2q^{n+1};q)_nq^n}{(q;q)_n}\sum_{n_1,\dots,n_{k-1}=0}^{\infty}\frac{(aq^{n-N_1+1};q)_\infty q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}} = \sum_{r=1}^{\infty}(-1)^{r+1}a^{r-1}q^{\binom{r}{2}}\frac{(q^r,q^{\kappa-r},q^\kappa;q^\kappa)_\infty}{(q;q)_\infty^2}. \tag{6.10}$$

Now expanding the left side by (2.2) and (2.3) gives

$$\mathrm{LHS}(6.10) = \sum_{j,l,n,n_1,\dots,n_{k-1}=0}^{\infty}\frac{(-1)^{j+l}a^{j+2l}q^{\binom{j+1}{2}+\binom{l+1}{2}+n(j+l+1)-N_1j+\sum_{i=1}^{k-1}N_i(N_i+1)}}{(q;q)_j(q;q)_l(q;q)_{n-l}(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}}.$$

Shifting $n\to n+l$, $j\to j-2l$ and summing over n using (2.4) yields

$$\mathrm{LHS}(6.10) = \frac{1}{(q;q)_\infty}\sum_{j,l=0}^{\infty}(-1)^{j+l}a^jq^{\binom{j-l+1}{2}+l(l+1)}\begin{bmatrix}j-l\\ l\end{bmatrix}_q \times\sum_{n_1,\dots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}-(j-2l)N_1}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}}.$$

Equating coefficients of a^j with the right side of (6.10) thus gives

$$\sum_{l=0}^{\lfloor r/2\rfloor}(-1)^lq^{l(3l-2r+1)/2}\begin{bmatrix}r-l\\ l\end{bmatrix}_q\sum_{n_1,\dots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}-(r-2l)N_1}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}} = \frac{(q^{r+1},q^{\kappa-r-1},q^\kappa;q^\kappa)_\infty}{(q;q)_\infty} \tag{6.11}$$

for r a nonnegative integer. Comparing this result with (6.5) it becomes clear we should now invert. By the connection coefficient formula [15, Eq. (7.2)]

$$\frac{H_n(x|q)}{(q;q)_n}=\sum_{j=0}^{\lfloor n/2\rfloor}\frac{(1-q^{n-2j+1})q^j}{(q;q)_j(q;q)_{n-j+1}}\sum_{l=0}^{\lfloor n/2\rfloor-j}\frac{(-1)^lp^{\binom{l+1}{2}}(p;p)_{n-2j-l}}{(p;p)_l}\frac{H_{n-2j-2l}(x|p)}{(p;p)_{n-2j-2l}}$$

for the q-Hermite polynomials and by the q-Hermite orthogonality [16, Exercise 7.22] it follows that

$$\frac{h_n}{(q;q)_n}=\sum_{j=0}^{\lfloor n/2\rfloor}\frac{(1-q^{n-2j+1})q^j}{(q;q)_j(q;q)_{n-j+1}}\sum_{l=0}^{\lfloor n/2\rfloor-j}\frac{(-1)^lq^{\binom{l+1}{2}}(q;q)_{n-2j-l}}{(q;q)_l}\frac{h_{n-2j-2l}}{(q;q)_{n-2j-2l}}$$

for an arbitrary sequence $\{h_n\}_{n=0}^{\infty}$. Choosing $h_n=q^{n^2/4}f_n$ this implies the following inversion

$$g_r=\sum_{l=0}^{\lfloor r/2\rfloor}(-1)^lq^{l(3l-2r+1)/2}\begin{bmatrix}r-l\\ l\end{bmatrix}_q f_{r-2l} \tag{6.12a}$$

$$f_m=(q;q)_m\sum_{j=0}^{\lfloor m/2\rfloor}\frac{(1-q^{m-2j+1})q^{j(j-m+1)}}{(q;q)_j(q;q)_{m-j+1}}\,g_{m-2j}. \tag{6.12b}$$

This may also be derived without resorting to q-Hermite polynomials using the q-Dougall sum (3.5) with $bc = aq = q^{-n}$. Since (6.11) is of the form (6.12a) we may rewrite it using (6.12b) to find

$$
\begin{aligned}
(6.13)\qquad &\sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}-mN_1}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}} \\
&= (q;q)_m \sum_{j=0}^{\lfloor m/2 \rfloor} \frac{(1-q^{m-2j+1})q^{j(j-m+1)}}{(q;q)_j(q;q)_{m-j+1}} \frac{(q^{m-2j+1},q^{\kappa+2j-m-1},q^\kappa;q^\kappa)_\infty}{(q;q)_\infty} \\
&= (q;q)_m \sum_{\substack{j=1\\ m+j \text{ odd}}}^{m+1} \frac{(1-q^j)q^{(j-m-1)(j+m-3)/4}}{(q;q)_{(m-j+1)/2}(q;q)_{(m+j+1)/2}} \frac{(q^j,q^{\kappa-j},q^\kappa;q^\kappa)_\infty}{(q;q)_\infty}.
\end{aligned}
$$

The left side coincides with the Berkovich–Paule result (6.8) for $i' = 1$. To also show that the above right side agrees with the right side of (6.8) requires some manipulations. If we denote the summand on the right by S_j then a little calculation shows that $S_j = S_{-j}$. Since also $S_j = 0$ if $j \equiv 0 \pmod{\kappa}$ or $j > m+1$ we may therefore write

$$
\sum_{\substack{j=1\\ m+j \text{ odd}}}^{m+1} S_j = \sum_{\substack{i=1\\ m+i \text{ odd}}}^{\kappa-1} \sum_{r=-\infty}^{\infty} S_{2\kappa r+i}.
$$

Using this as well as (3.8) we arrive at

$$
\begin{aligned}
\mathrm{LHS}(6.13) = (q;q)_m &\sum_{\substack{i=1\\ m+i \text{ odd}}}^{\kappa-1} q^{(i-m-1)(i+m-3)/4} \frac{(q^i,q^{\kappa-i},q^\kappa;q^\kappa)_\infty}{(q;q)_\infty} \\
&\times \sum_{r=-\infty}^{\infty} \frac{(1-q^{2\kappa r+i})q^{((\kappa-2)(\kappa r+i)-\kappa)r}}{(q;q)_{(m-2\kappa r-i+1)/2}(q;q)_{(m+2\kappa r+i+1)/2}}.
\end{aligned}
$$

By the easily verified

$$
\frac{(1-q^j)(q;q)_m}{(q;q)_{(m-j+1)/2}(q;q)_{(m+j+1)/2}} = {m \brack (m-j+1)/2}_q - q^j {m \brack (m-j-1)/2}_q
$$

it thus follows that

$$
\begin{aligned}
\mathrm{LHS}(6.13) = &\sum_{\substack{i=1\\ m+i \text{ odd}}}^{\kappa-1} q^{(i-m-1)(i+m-3)/4} \frac{(q^i,q^{\kappa-i},q^\kappa;q^\kappa)_\infty}{(q;q)_\infty} \\
&\times \sum_{r=-\infty}^{\infty} q^{((\kappa-2)(\kappa r+i)-\kappa)r} \Big\{ {m \brack (m-2\kappa r-i+1)/2}_q - q^{2\kappa r+i} {m \brack (m-2\kappa r-i-1)/2}_q \Big\}.
\end{aligned}
$$

In the first term within the curly braces we replace $r \to -r$ and use ${m+n \brack m}_q = {m+n \brack n}_q$. Comparing the resulting expression with (6.6) we find that the second line of the above equation is $X_{i,1}^{(\kappa-2,\kappa)}(m)$. By the duality relation (6.7) we therefore find

$$
\mathrm{LHS}(6.13) = \sum_{\substack{i=1\\ m+i \text{ odd}}}^{\kappa-1} \frac{(q^i,q^{\kappa-i},q^\kappa;q^\kappa)_\infty}{(q;q)_\infty} X_{i,1}^{(2,\kappa)}(m;1/q)
$$

in agreement with the right side of (6.8) for $i'=1$.

Before we continue to treat the other theorems of sections 1 and 4 let us briefly comment on (6.11). Because of the ocurrence of the q-binomial coefficient it is clear that for negative values of r the left side trivially vanishes. The right side, on the other hand, is nonvanishing for any integer r as long as $r+1\not\equiv 0 \pmod{\kappa}$, and is in fact symmetric under the transformation $r\to\kappa-r-2$. To obtain an identity valid for all integers r one can use the symmetry of the right side (or more precisely, the quasi-periodicity under the transformation $r\to -r-2$) to prove that

$$\sum_{l=-\infty}^{\lfloor r/2\rfloor}(-1)^l q^{l(3l-2r+1)/2}\begin{bmatrix} r-l\\ r-2l\end{bmatrix}_q \sum_{n_1,\dots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}-(r-2l)N_1}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}} = \frac{(q^{r+1},q^{\kappa-r-1},q^{\kappa};q^{\kappa})_\infty}{(q;q)_\infty},$$

where r is now an arbitrary integer and where the q-binomial coefficient is redefined as $\left[\begin{smallmatrix} m+n\\ m\end{smallmatrix}\right]_q=(q^{n+1};q)_m/(q;q)_m$ for m a nonnegative integer and zero otherwise. Note that this implies that the lower bound in the sum over l may be optimized to $\min\{0,r+1\}$.

The other theorems on partial theta functions may be applied in a similar manner. Since each time the calculations only marginally differ, we will leave out the details and give the most important equations only. The residual equation corresponding to Theorem 1.2 can be stated as follows.

Corollary 6.2. *For $k\geq 2$ and $\kappa=2k$,*

$$\sum_{n=0}^{\infty}\frac{(a^2q^{n+1};q)_n q^n}{(q;q)_n}\sum_{n_1,\dots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(aq;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q^2;q^2)_{n_{k-1}}} = \sum_{i=1}^{\kappa-1}(-1)^{i+1}a^{i-1}q^{\binom{i}{2}}\frac{(q^i,q^{\kappa-i},q^{\kappa};q^{\kappa})_\infty}{(q,q,aq;q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{\kappa n}q^{(\kappa-1)(kn+i)n}.$$

The identity corresponding to $k=1$ is given by (5.3).

By equating coefficients of a^n one finds

$$\sum_{l=0}^{\lfloor r/2\rfloor}(-1)^l q^{l(3l-2r+1)/2}\begin{bmatrix} r-l\\ l\end{bmatrix}_q \sum_{n_1,\dots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}-(r-2l)N_1}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q^2;q^2)_{n_{k-1}}} = \frac{(q^{r+1},q^{\kappa-r-1},q^{\kappa};q^{\kappa})_\infty}{(q;q)_\infty}$$

for r a nonnegative integer. Inversion of this equation gives the $i'=1$ instance of

$$\sum_{n_1,\dots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_{i'}+\cdots+N_{k-1}-mN_1}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q^2;q^2)_{n_{k-1}}} = \sum_{\substack{i=1\\ m+i+i'\text{ odd}}}^{\kappa-1}\frac{(q^i,q^{\kappa-i},q^{\kappa};q^{\kappa})_\infty}{(q;q)_\infty}X_{i,i'}^{(2,\kappa)}(m;1/q),$$

where we remind the reader that $\kappa=2k$. In view of (6.8) it is not difficult to guess that the above is true for all $i'\in\{1,\dots,k\}$.

The Garrett–Ismail–Stanton-type identity associated to Theorem 1.3 takes a slightly different form. First we calculate the residual identity.

Corollary 6.3. *For $k \geq 2$ and $\kappa = 2k - 1/2$,*

$$\sum_{n=0}^{\infty} \frac{(a^2q^{n+1};q)_n q^n}{(q;q)_n} \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(aq;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}(-q^{1/2};q^{1/2})_{2n_{k-1}}}$$
$$= \sum_{i=1}^{2k-1} (-1)^{i+1} a^{i-1} q^{\binom{i}{2}} \frac{(q^i, q^{\kappa-i}, q^{\kappa}; q^{\kappa})_\infty}{(q,q,aq;q)_\infty}$$
$$\times \sum_{n=0}^{\infty} (-1)^n a^{2\kappa n} q^{2(\kappa-1)(\kappa n+i)n} \bigl(1 + a^{2\kappa-2i} q^{2(\kappa-1)(\kappa-i)(2n+1)}\bigr).$$

The equation corresponding to $k=1$ is

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^n}{(q,a^2;q)_n} = \frac{(a;q)_\infty}{(q,a^2;q)_\infty} \sum_{n=0}^{\infty} (-1)^n a^{3n} q^{(3n-1)n/2}(1+aq^n),$$

where we have replaced a by $a/q^{1/2}$. By (5.4) with $b \to 0$, $c \to a^2$ and $z \to q$ this can be transformed into

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{\binom{n}{2}}}{(a;q)_{n+1}} = \sum_{n=0}^{\infty} (-1)^n a^{3n} q^{(3n-1)n/2}(1+aq^n)$$

which for $a = q^{1/2}$ yields Rogers' false theta function identity [20, p. 333; Eq. (4)]

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(q;q^2)_{n+1}} = 1 + \sum_{n=1}^{\infty} (-1)^n q^{3n^2}(q^{2n} - q^{-2n}).$$

Returning to the more general case, we equate powers of a^n in Corollary 6.3 to obtain

$$\sum_{l=0}^{\lfloor r/2 \rfloor} (-1)^l q^{l(3l-2r+1)/2} \begin{bmatrix} r-l \\ l \end{bmatrix}_q \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}-(r-2l)N_1}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}(-q^{1/2};q^{1/2})_{2n_{k-1}}}$$
$$= \frac{(q^{r+1}, q^{\kappa-r-1}, q^{\kappa}; q^{\kappa})_\infty}{(q;q)_\infty}.$$

After inversion this gives the $i'=1$ instance of

$$\sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_{i'}+\cdots+N_{k-1}-mN_1}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}(-q^{1/2};q^{1/2})_{2n_{k-1}}}$$
$$= \sum_{\substack{i=1 \\ m+i+i' \text{ odd}}}^{2\kappa-1} \frac{(q^i, q^{\kappa-i}, q^{\kappa}; q^{\kappa})_\infty}{(q;q)_\infty} X_{i,i'}^{(4,2\kappa)}(m;1/q).$$

Again we conjecture this to hold for all $i' \in \{1,\dots,k\}$ and $\kappa = 2k-1/2$. Note the subtle difference with earlier cases in that the sum over i on the right exceeds κ. Since $X_{i,i'}^{(4,2\kappa)}(m;1/q)$ (for $i \in \{1,\dots 2\kappa-1\}$ and $i' \in \{1,\dots,k\}$) is a polynomial with only nonnegative coefficients [9] this implies that for $i > \kappa$ the summand on the right (as a power series in q) has nonpositive coefficients.

Finally we treat Theorems 1.4 and 4.1 together.

Corollary 6.4. *For $\sigma \in \{0,1\}$, $k \geq 2$ and $\kappa = 3k-\sigma-1$,*

$$\sum_{n=0}^{\infty} \frac{(a^2q^{n+1};q)_n q^n}{(q;q)_n} \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}+\sigma N_{k-1}(N_{k-1}-1)}}{(aq;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q;q)_{2n_{k-1}}}$$
$$= \sum_{i=1}^{\kappa-1}(-1)^{i+1}a^{i-1}q^{\binom{i}{2}} \frac{(q^i,q^{2\kappa-i},q^{2\kappa};q^{2\kappa})_\infty (q^{2\kappa-2i},q^{2\kappa+2i};q^{4\kappa})_\infty}{(q,q,aq;q)_\infty}$$
$$\times \Big[1-\sum_{n=1}^{\infty} a^{2\kappa n-2i} q^{(2\kappa-3)(\kappa n-i)n}\Big\{1-a^{2i}q^{2(2\kappa-3)in}\Big\}\Big].$$

The identity corresponding to $\sigma=0$ and $k=1$ turns out to be a special case of Heine's transformation (5.4) and has therefore been omitted. Equating the coefficients of a^n yields

$$\sum_{l=0}^{\lfloor r/2\rfloor}(-1)^l q^{l(3l-2r+1)/2}\begin{bmatrix} r-l \\ l\end{bmatrix}_q$$
$$\times \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}+\sigma(N_{k-1}^2-N_{k-1})-(r-2l)N_1}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q;q)_{2n_{k-1}}}$$
$$= \frac{(q^{r+1},q^{2\kappa-r-1},q^{2\kappa};q^{2\kappa})_\infty (q^{2\kappa-2r-2},q^{2\kappa+2r+2};q^{4\kappa})_\infty}{(q;q)_\infty},$$

provided r is a nonnegative integer. The inversion of this equation gives the $i'=1$ case of

$$\sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_{i'}+\cdots+N_{k-1}+\sigma N_{k-1}(N_{k-1}-1)-mN_1}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q;q)_{2n_{k-1}}}$$
$$= \sum_{\substack{i=1\\ m+i+i' \text{ odd}}}^{\kappa-1} \frac{(q^i,q^{2\kappa-i},q^{2\kappa};q^{2\kappa})_\infty (q^{2\kappa-2i},q^{2\kappa+2i},q^{4\kappa};q^{4\kappa})_\infty}{(q;q)_\infty} X_{i,i'}^{(3,\kappa)}(m;1/q).$$

Yet again we conjecture this to be true for all $i' \in \{1,\dots,k\}$.

Since the outcomes are less spectacular, we leave the calculation of the residual identities of the remaining theorems of section 4 to the reader. Instead we just list some of the simplest cases which we hope are of some interest.

Calculating the residue around $a=q^N$ in (3.9) results in

$$\sum_{n=0}^{\infty}\frac{(a^2q^{n+1};q)_n q^n}{(q,aq,aq^2;q)_n} = \frac{1+(1+1/a)\sum_{n=1}^{\infty}(-1)^n a^n q^{\binom{n+1}{2}}}{(1-q)(q,aq,aq^2;q)_\infty}$$

which for $a=1$ simplifies to

$$\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_n q^n}{(q,q,q^2;q)_n} = \frac{1+2\sum_{n=1}^{\infty}(-1)^n q^{\binom{n+1}{2}}}{(q;q)_\infty^3}$$

reminiscent of (4.11).

The residual identity corresponding to (4.10) is

$$\sum_{n=0}^{\infty}\frac{(a^2;q)_{2n} q^n}{(q,a,aq,a^2q;q)_n} = \frac{1}{(q,aq,aq;q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n+1}{2}}$$

which generalizes (4.11) obtained for $a = 1$. This last result becomes more interesting if we compare it with the analogous result obtained from (4.13):

$$\sum_{n=0}^{\infty} \frac{(a^2q^{n+1};q)_n q^n}{(q,aq,aq;q)_n} = \frac{1}{(q,aq,aq;q)_\infty} \biggl(\sum_{n=0}^{\infty} (-1)^n a^n q^{\binom{n+1}{2}} \biggr)^2.$$

Noting the similarity of the above two right-hand sides and using basic hypergeometric notation we infer that

$$\left({}_3\phi_2 \left[\begin{matrix} -a, aq^{1/2}, -aq^{1/2} \\ aq, a^2q \end{matrix} ; q, q \right] \right)^2 = \frac{1}{(q,aq,aq;q)_\infty} {}_3\phi_2 \left[\begin{matrix} -aq, aq^{1/2}, -aq^{1/2} \\ aq, a^2q \end{matrix} ; q, q \right].$$

Finally, calculating the residue around $a = q^N$ in (4.15) yields

$$\sum_{n=0}^{\infty} \frac{(-aq;q)_n q^n}{(q,a^2q;q)_n} = \frac{(-aq;q)_\infty}{(q,a^2q;q)_\infty} \Bigl[1 - (1+a) \sum_{n=1}^{\infty} a^{3n-2} q^{n(3n-1)/2} (1-aq^n) \Bigr].$$

By Heine's transformation (5.4) with $a \to -aq$, $b \to 0$, $c \to a^2q$ and $z \to q$ this becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{\binom{n+1}{2}}}{(-aq;q)_{n+1}} = 1 - (1+a) \sum_{n=1}^{\infty} a^{3n-2} q^{n(3n-1)/2} (1-aq^n).$$

For $a = 1$ this yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(-q^2;q^2)_{n+1}} = 1 + 2 \sum_{n=1}^{\infty} q^{3n^2} (q^n - q^{-n}). \tag{6.14}$$

Although this is a false theta function identity not in Rogers' paper it readily follows that (6.14) – (6.15) = (6.15) – 1 with [20, p. 333; Eq. (6)]

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(-q^2;q^2)_n} = 1 + \sum_{n=1}^{\infty} q^{3n^2} (q^n - q^{-n}). \tag{6.15}$$

7. Discussion

In the abstract we stated that many of Ramanujan's partial theta function identities can be generalized by the method developed in this paper. In the main text, however, we restricted ourselves to Ramanujan's identities (1.1) and (1.3)–(1.5), which all have a very similar structure dictated by Proposition 3.1. To conclude we will give one example of how simple modifications lead to generalizations of other partial theta function identities of the lost notebook. The identity we will generalize in our example is [19, p. 37]

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(aq,q/a;q^2)_{n+1}} \\ &\qquad = \sum_{n=0}^{\infty} (-1)^{n+1} a^{3n+1} q^{n(3n+2)} (1+aq^{2n+1}) + \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1} q^{n(n+1)}}{(aq,q/a;q^2)_\infty} \end{aligned}$$

which was first proved by Andrews [6, Eq. (3.9)].

First we take $b=q^2/a$ in (3.1) and divide both sides by $(1-q/a)$. This yields

$$\sum_{n=0}^{\infty}\frac{(q;q)_{2n+1}q^n}{(a,q/a;q)_{n+1}}\sum_{r=0}^{n}\frac{(-1)^r q^{\binom{r}{2}}f_r(1-q^{2r+2})}{(q;q)_{n-r}(q;q)_{n+r+2}}+\sum_{r=0}^{\infty}(a/q)^{r+1}f_r \tag{7.1}$$
$$=\frac{1}{(q,a,q/a;q)_\infty}\sum_{n=1}^{\infty}\Big\{\sum_{r=0}^{\infty}f_r+q^{-n}\sum_{r=-\infty}^{-1}f_{-r-1}\Big\}(-1)^{n+r}a^n q^{\binom{n+r+1}{2}}$$

provided all sums converge. Recalling definition (4.1) of a Bailey pair this can be restated as follows.

Corollary 7.1. *For (α,β) a Bailey pair relative to q^2 there holds*

$$\sum_{n=0}^{\infty}\frac{\beta_n(q^2;q)_{2n}q^n}{(a,q/a;q)_{n+1}}+(1-q^2)\sum_{n=0}^{\infty}\frac{\alpha_n(-1)^n(a/q)^{n+1}q^{-\binom{n}{2}}}{1-q^{2n+2}} \tag{7.2}$$
$$=\frac{(1-q^2)}{(q,a,q/a;q)_\infty}\sum_{r=1}^{\infty}(-1)^{r+1}(a/q)^r q^{\binom{r}{2}}\sum_{n=0}^{\infty}\alpha_n q^{(1-r)n}\frac{1-q^{r(2n+2)}}{1-q^{2n+2}}.$$

The Bailey pair required to turn this into (7.1) is

$$\alpha_n=(-1)^{\lfloor 4n/3\rfloor}q^{n(2n+1)/3}\frac{1-q^{2n+2}}{1-q^2}\chi(n\not\equiv 2 \pmod 3) \tag{7.3}$$
$$\beta_n=\frac{1}{(q^2;q)_{2n}}.$$

The proof of this pair comes down to the proof of the polynomial identity

$$\sum_{j=-\infty}^{\infty}\Big\{q^{j(6j+1)}{\begin{bmatrix}2n+2\\ n-3j\end{bmatrix}}_q-q^{(2j+1)(3j+1)}{\begin{bmatrix}2n+2\\ n-3j-1\end{bmatrix}}_q\Big\}=1-q^{2n+2} \tag{7.4}$$

established in the appendix. If we insert (7.3) into (7.2), replace a by $aq^{1/2}$ followed by $q\to q^2$ we find (7.1). If first we use the Bailey lemma to obtain the iterated Bailey pair

$$\alpha_n=(-1)^{\lfloor 4n/3\rfloor}q^{n(2n+1)/3+(k-1)n(n+2)}\frac{1-q^{2n+2}}{1-q^2}\chi(n\not\equiv 2 \pmod 3)$$
$$\beta_n=\sum_{n_1,\dots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-1}^2+2N_1+\cdots+2N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q^2;q)_{2n_{k-1}}}$$

relative to q^2 and insert this into (7.2) with a replaced by $aq^{1/2}$, we obtain our final theorem.

Theorem 7.1. *For $k \geq 1$, $\kappa = 3k-1$ and $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$,*

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(q^2;q)_{2n} q^{n+1/2}}{(aq^{1/2}, q^{1/2}/a;q)_{n+1}} &\sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+2N_1+\cdots+2N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q^2;q)_{2n_{k-1}}} \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} a^{3n+1} q^{(2\kappa-3)(3n+2)n/2}\big(1+aq^{(2\kappa-3)(2n+1)/2}\big) \\ &\quad + \sum_{i=1}^{\kappa-1} (-1)^{i+1} a^i q^{(i-1)^2/2} \frac{(q^{\kappa+i}, q^{\kappa-i}, q^{2\kappa}; q^{2\kappa})_\infty (q^{2i}, q^{4\kappa-2i}; q^{4\kappa})_\infty}{(q, aq^{1/2}, q^{1/2}/a;q)_\infty} \\ &\qquad \times \sum_{n=0}^{\infty} a^{2\kappa n} q^{(2\kappa-3)(\kappa n+i)n} \Big\{1 - a^{2\kappa-2i} q^{(2\kappa-3)(\kappa-i)(2n+1)}\Big\}.\end{aligned}$$

As a final comment we should note that it does not seem possible to prove and generalize all of Ramanujan's partial theta function formulas using (3.1), and it is an open problem whether one can modify our approach to extend an identity like [19, p. 39]

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{3n^2}}{(a;q^3)_{n+1}(q^3/a;q^3)_n} - q\sum_{n=1}^{\infty} \frac{q^{3n(n-1)}}{(aq, q^2/a;q^3)_n} + q/a \sum_{n=1}^{\infty} \frac{q^{3n(n-1)}}{(aq^2, q/a;q^3)_n} \\ = \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty (a, q/a;q)_\infty}\end{aligned}$$

which contains partial and complete theta functions of different moduli.

Appendix A. Proofs of polynomial identities

In this appendix we prove the various polynomial identities used in the main text for extracting Bailey pairs. All proofs are based on identities obtained by Rogers in his classic 1917 paper [20] on Rogers–Ramanujan-type identities, which we transform using the q-binomial recurrences

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \begin{bmatrix} m+n-1 \\ m \end{bmatrix}_q + q^{n-m} \begin{bmatrix} m+n-1 \\ m-1 \end{bmatrix}_q = \begin{bmatrix} m+n-1 \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} m+n-1 \\ m \end{bmatrix}_q.$$

For the proof of (4.5) we require

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j-1)/4} \begin{bmatrix} 2n \\ n-j \end{bmatrix}_q = (q^{1/2};q)_n \tag{A.1}$$

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{3j(j-1)/4} \begin{bmatrix} 2n \\ n-j \end{bmatrix}_q = q^n (q^{1/2};q)_n \tag{A.2}$$

equivalent to the Bailey pairs G(1) and G(3) in Slater's list [23]. Applying the first q-binomial recurrence to the left-hand side of (4.5) yields

$$\begin{aligned}\mathrm{LHS}(4.5) &= \mathrm{LHS}(\mathrm{A}.1) + q^{n+1} \sum_{j=-\infty}^{\infty} (-1)^j q^{3(j+1)j/4} \begin{bmatrix} 2n \\ n-j-1 \end{bmatrix}_q \\ &= (q^{1/2};q)_n - q^{n+1}\mathrm{LHS}(\mathrm{A}.2) = (1-q^{2n+1})(q^{1/2};q)_n,\end{aligned}$$

where the second equality follows after the variable change $j \to j-1$.

The ingredients needed for the proofs of (4.7), (4.16) and (7.4) are polynomial identities equivalent to the Bailey pairs A(1)–A(4) [23],

$$\sum_{j=-\infty}^{\infty}\Big\{q^{j(6j-1)}\begin{bmatrix}2n\\n-3j\end{bmatrix}_q - q^{(2j+1)(3j+1)}\begin{bmatrix}2n\\n-3j-1\end{bmatrix}_q\Big\}=1 \tag{A.3}$$

$$\sum_{j=-\infty}^{\infty}\Big\{q^{j(6j+1)}\begin{bmatrix}2n+1\\n-3j\end{bmatrix}_q - q^{(2j+1)(3j+1)}\begin{bmatrix}2n+1\\n-3j-1\end{bmatrix}_q\Big\}=1 \tag{A.4}$$

$$\sum_{j=-\infty}^{\infty} q^{j(6j+2)}\Big\{\begin{bmatrix}2n\\n-3j\end{bmatrix}_q - \begin{bmatrix}2n\\n-3j-1\end{bmatrix}_q\Big\}=q^n \tag{A.5}$$

$$\sum_{j=-\infty}^{\infty} q^{j(6j+4)}\Big\{\begin{bmatrix}2n+1\\n-3j\end{bmatrix}_q - \begin{bmatrix}2n+1\\n-3j-1\end{bmatrix}_q\Big\}=q^n. \tag{A.6}$$

To show (4.7) is true we take its left-hand side and apply the first (second) q-binomial recurrence to the first (second) term of the summand to find

$$\mathrm{LHS}(4.7)=\mathrm{LHS}(\mathrm{A}.3)-q^{n+1}\mathrm{LHS}(\mathrm{A}.5)=1-q^{2n+1}.$$

To establish (7.4) we take its left-hand side and apply the first q-binomial recurrences to both terms of the summand to find

$$\begin{aligned}\mathrm{LHS}(7.4)&=\mathrm{LHS}(\mathrm{A}.4)-q^{n+2}\sum_{j=-\infty}^{\infty}\Big\{q^{(j+1)(6j+2)}\begin{bmatrix}2n+1\\n-3j-2\end{bmatrix}_q-q^{j(6j+4)}\begin{bmatrix}2n+1\\n-3j-1\end{bmatrix}_q\Big\}\\&=1-q^{n+2}\mathrm{LHS}(\mathrm{A}.6)=1-q^{2n+2},\end{aligned}$$

where the second equality follows by the variable change $j\to -j-1$ in the first term of the sum over j and the symmetry $\begin{bmatrix}m+n\\m\end{bmatrix}_q=\begin{bmatrix}m+n\\n\end{bmatrix}_q$. The polynomial identity (4.16) requires some more work. First observe that the second line on the left of (4.16) is precisely $-q\mathrm{LHS}(\mathrm{A}.6)=-q^{n+1}$. Using the first (second) q-binomial recurrence on the first (second) term of the summand on the first line (and making some trivial variable changes) thus gives

$$\begin{aligned}\mathrm{LHS}(4.16)&=\mathrm{LHS}(\mathrm{A}.5)-q^{n+1}\\&\qquad+q^{n+1}\sum_{j=-\infty}^{\infty}\Big\{q^{j(6j+1)}\begin{bmatrix}2n\\n-3j-1\end{bmatrix}_q-q^{(2j+1)(3j+1)}\begin{bmatrix}2n\\n-3j-2\end{bmatrix}_q\Big\}.\end{aligned}$$

Next we expand the remaining sum over j using the first q-binomial recurrence. This leads to

$$\begin{aligned}\mathrm{LHS}(4.16)&=q^n-q^{n+1}+q^{n+1}\mathrm{LHS}(\mathrm{A}.4)|_{n\to n-1}\\&\qquad-q^{2n+2}\sum_{j=-\infty}^{\infty}\Big\{q^{(j+1)(6j+2)}\begin{bmatrix}2n-1\\n-3j-3\end{bmatrix}_q-q^{j(6j+4)}\begin{bmatrix}2n-1\\n-3j-2\end{bmatrix}_q\Big\}\\&=q^n-q^{2n+2}\mathrm{LHS}(\mathrm{A}.6)|_{n\to n-1}=q^n(1-q^{2n+1}).\end{aligned}$$

Finally we deal with (4.14) for which we need three polynomial identities equivalent to the Bailey pairs C(1), C(2) and C(4) [23],

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j-1)}\begin{bmatrix}2n\\n-2j\end{bmatrix}_q=(-q;q)_n \tag{A.7}$$

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j+1)}\left\{\begin{bmatrix}2n\\n-2j\end{bmatrix}_q-\begin{bmatrix}2n\\n-2j-1\end{bmatrix}_q\right\}=q^n(-q;q)_n \tag{A.8}$$

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{3j(j+1)}\begin{bmatrix}2n+1\\n-2j\end{bmatrix}_q=q^n(-q;q)_n. \tag{A.9}$$

First note that the second sum in (4.14) (corresponding to the term q^{4j+1} in $(1-q^{4j+1})$) is $-q\,\mathrm{LHS(A.9)}=-q^{n+1}(-q;q)_n$. We therefore need to show that the first sum, which will be denoted by S_1, equals $(1+q^{n+1}(1-q^n))(-q;q)_n$. Now, by the first q-binomial recurrence,

$$\begin{aligned}S_1&=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j-1)}\begin{bmatrix}2n\\n-2j\end{bmatrix}_q+q^{n+1}\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j+1)}\begin{bmatrix}2n\\n-2j-1\end{bmatrix}_q\\&=\mathrm{LHS(A.7)}+q^{n+1}(\mathrm{LHS(A.7)}-\mathrm{LHS(A.8)})=(1+q^{n+1}(1-q^n))(-q;q)_n.\end{aligned}$$

Here we note that in calculating LHS(A.7) − LHS(A.8) one should first replace $j\to -j$ in (A.7) and use $\begin{bmatrix}m+n\\m\end{bmatrix}_q=\begin{bmatrix}m+n\\n\end{bmatrix}_q$.

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