HYP

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In a recent issue of the Gazette [2], Michael Hirschhorn demonstrated how many summation identities involving binomial coefficients can be proved purely algorithmically by recasting them in the language of hypergeometric series, and by then using standard look-up tables for such series. Although this method has been advocated by experts in the field of hypergeometric series for many years now, Hirschhorn rightly points out that it is still not uncommon to find authors stating newly discovered binomial coefficient identities that, when recast in terms of hypergeometric series, turn out to be hundreds of years old.

Unfortunately, Hirschhorn stopped short of pointing out that the algorithmic nature of the method makes it well-suited for computer implementation, and that such an implementation is in fact provided by Christian Krattenthaler's Mathematica package HYP [3]. Indeed, the following is a quote from Krattenthaler taken from one of his reviews for Mathematical Reviews. (For obvious reasons the names of the authors under review will remain undisclosed.)

> Binomial series identities are a very popular topic in the combinatorial community. Despite G. E. Andrews' paper [SIAM Rev. 16 (1974), 441–484], which emphasizes that binomial series should be considered within the framework of hypergeometric series, there is still reservation among some people to accept this fact. The authors certainly belong to this (hopefully shrinking) community. They frankly admit that they do not like hypergeometric series.
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> ... authors derive several transformation formulas. The reviewer checked for two of them that they are easily derived from well-known hypergeometric identities. Probably the same is true for the rest.
>
> The only excuse to avoid hypergeometric or basic hypergeometric series, namely that it is difficult for a non-expert to find identities that one might need, does not hold anymore. The reviewer designed Mathematica packages for manipulating hypergeometric and basic hypergeometric series which have built-in a large list of identities.

As an example we use HYP to tackle the first of the three identities considered by Hirschhorn,

$$\sum_k \binom{m}{2k}\binom{k}{n} = 2^{m-2n-1}\left\{\binom{m-n}{n} + \binom{m-n-1}{n-1}\right\}. \tag{1}$$

Of course, the point here is not to show that this identity holds; even Mathematica can do this evaluation and

In[1]:=Sum[Binomial[$m, 2k$] Binomial[k, n], {k, n,Infinity}]

leads to

$$\mathrm{Out}[1] = \frac{2^{m-2n-1} m\,\Gamma(m-n)}{\Gamma(m-2n+1)\Gamma(n+1)}$$

in accordance with the right side of (1). Rather, the aim of the exercise is to find out whether the summation (1) is already known. Hence we proceed with loading HYP

In[2]:=<<hyp.m

and then type

In[3]:=SUM[Binomial[$m, 2k$] Binomial[k, n], {k, n,Infinity}];

To now convert this into hypergeometric notation is easy and

In[4]:=%3 /. SUMH

yields the desired result

$$\text{Out}[4] = \binom{m}{2n} {}_2F_1\left(\begin{matrix} n - m/2, n - m/2 + 1/2 \\ n + 1/2 \end{matrix}; 1\right).$$

Here the ${}_2F_1$ stands for Gauss' hypergeometric series [2]. After this conversion of our sum we ask HYP to do the table look-up

In[5]:=%4 /. SListe

to which it answers with

Out[6]:={{S2101}}

Here S2101 refers to one of the many summations in HYP's database. To get a reference to the literature we type

In[7]:=? S2101

to be told that it is (III.4) in the appendix of Slater's book [4] which happens to be the Chu–Vandermonde sum, first found by Chu back in 1303 and clearly not a result one wishes to claim as a new discovery in 2002. To actually see the Chu–Vandermonde sum in action we type

In[8]:=%4 /. S2101

to find once again that

$$\text{Out}[8] = \frac{2^{m-2n-1} m\, \Gamma(m-n)}{\Gamma(m-2n+1)\Gamma(n+1)}.$$

The careful reader may object that in the above several implicit assumptions have been made by HYP concerning the parameters m and n. This is merely due our choice to keep all technicalities out it and not to display several message windows that appear during the calculations requesting information about m and n.

Apart from dealing with binomial and hypergeometric summations, HYP is an expert on hypergeometric transformations and contiguous relations and knows about Gosper's algorithm for doing indefinite summation and Zeilberger's creative telescoping algorithm. Moreover, its cousin HYPQ can handle q-binomials and basic hypergeometric series. For those preferring Maple over Mathematica there is a Maple version of HYP, called HYPERG developed by Bruno Gauthier [1].

References

1. B. Gauthier, http://www-igm.univ-mlv.fr/∼gauthier/ (document)
2. M. D. Hirschhorn, *Binomial coefficient identities and hypergeometric series*, Gazette **29** (2002), 203–208. (document)
3. C. Krattenthaler, http://euler.univ-lyon1.fr/home/kratt/ (document)
4. L. J. Slater, *Generalized Hypergeometric Functions*, (Cambridge University Press, Cambridge, 1966). (document)

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