Algebraic construction of higher-rank dilute A models

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Received 27 April 1994; revised 12 September 1994; accepted 21 September 1994

Abstract

Starting from representations of the Birman–Wenzl–Murakami algebra underlying the critical $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ RSOS models of Jimbo, Miwa and Okado, we derive four series of solvable, critical RSOS models associated with the twisted affine Lie algebra $A_n^{(2)}$. Two of these are the critical limit of the $A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$ models obtained previously by Kuniba. The other two series, again one of the $A_{2n-1}^{(2)}$ and one of the $A_{2n}^{(2)}$ type, are new, and the latter generalizes the dilute A models to arbitrary rank n. For the two new series we present an elliptic extension which satisfies the Yang–Baxter equation, and show that for certain values of the parameters the higher-rank dilute A models break the $\mathbb{Z}_2$ symmetry of the underlying adjacency graph $\mathcal{G}$, where $\mathcal{G}$ is the level-l $C_n^{(1)}$ weight lattice.

1. Introduction

Since Baxter's celebrated solution of the eight-vertex model [1], the relevance of the star–triangle or Yang–Baxter equation (YBE) in solving two-dimensional lattice models has been widely acknowledged. Although many solutions to the YBE have already been found, a complete classification of all possible solutions seems still far from being established. Nevertheless, many of the known solutions nicely fit into the classification scheme of affine Kac–Moody algebras [2], and it is generally believed that a solvable model can be associated to each affine Lie algebra. For vertex models this was to a large extend established independently by Bazhanov and Jimbo [3] by quantization of the Belavin–Drinfeld solutions to the classical YBE [4].

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Dual to the vertex models there is another class of so-called restricted solid-on-solid (RSOS) models and, presumably, to each vertex model corresponds an RSOS counterpart. The first such restricted model, dual to the eight-vertex model, was found by Andrews, Baxter and Forrester [5]. This ABF model was subsequently generalized by Jimbo, Miwa and Okado (JMO), who constructed RSOS models based on the non-twisted affine Lie algebras $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ [6]. These results in turn, were extended by Kuniba, who gave similar results for the twisted affine Lie algebras $A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$ [7]. However, shortly after, a second series of RSOS models based on $A_2^{(2)}$ was found [8,9], which is quite distinct from Kuniba's $A_2^{(2)}$ model. This new series of so-called dilute A models has the interesting property that it provides an example of a solvable model in a magnetic field at $T = T_c$.

In this paper we explain the occurrence of two different series of critical $A_2^{(2)}$ RSOS models, and generalize this result to arbitrary rank n. We thus obtain higher-rank generalizations of the critical dilute A models. For these new models we also present an elliptic extension, and we point out that, like the dilute A models, the symmetry of the underlying adjacency graph is not obeyed.

The setup of this paper is as follows. We first recall several known facts about the critical $X_n^{(1)} = B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ JMO models. Apart from their definition given in Section 2, we point out in Section 3, following Deguchi et al. [10], that the JMO models yield representations of the Birman–Wenzl–Murakami (BWM) algebra. We than show in Subsection 4.1, owing to a recent observation by Grimm [11], that the $X_n^{(1)}$ BWM representations admit a dual baxterization, resulting in three series of models based on the twisted affine Lie algebra $A_n^{(2)}$. In Subsection 4.2 we extend the BWM algebra to allow for vacancies and, using the baxterization of this dilute BWM algebra as found in Ref. [11], we construct higher-rank dilute A models. In Section 5 we present explicit expressions for the face weights of the new twisted models in the off-critical elliptic case, and show that the dilute models break the level-l $C_n^{(1)}$ weight lattice symmetry of the underlying adjacency graph. Using the elliptic weights we also try to clarify the intimate relation between the two different series of $A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$ models.

Finally, in Section 6, we summarize and discuss our results and point out some interesting open problems.

2. The $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ face models

Let us start with reviewing some basic facts about the $B_n^{(1)}$ ($n \geqslant 2$), $C_n^{(1)}$ ($n \geqslant 1$) and $D_n^{(1)}$ ($n \geqslant 3$) face, or RSOS models of Jimbo, Miwa and Okado [6], of relevance to subsequent sections.

2.1. Local states and adjacency rules

The JMO models are solvable lattice models defined on the square lattice $\mathcal{L}$, in which the local states a, also referred to as heights, are the dominant integral weights of $X_n^{(1)}$ at some arbitrary but fixed level l. Let L be

$$L = t(l + g),\tag{2.1}$$

where g and t, the dual Coxeter number and the (long root)$^2/2$ of $X_n^{(1)}$, respectively, are listed in Table 1. Furthermore let $\Lambda_0, \ldots, \Lambda_n$ denote the fundamental weights of $X_n^{(1)}$. We then define the local states of the respective models as

$$
B_n^{(1)} \begin{cases} a = (L - a_1 - a_2 - 1)\Lambda_0 + \displaystyle\sum_{i=1}^{n-1} (a_i - a_{i+1} - 1)\,\Lambda_i + (2a_n - 1)\Lambda_n, \\[2mm] L > a_1 + a_2, \quad a_1 > a_2 > \ldots > a_n > 0, \quad \text{all } a_i \in \mathbb{Z}, \text{ or all } a_i \in \mathbb{Z} + \tfrac{1}{2}, \end{cases}
$$

$$
C_n^{(1)} \begin{cases} a = (L/2 - a_1 - 1)\Lambda_0 + \displaystyle\sum_{i=1}^{n-1} (a_i - a_{i+1} - 1)\,\Lambda_i + (a_n - 1)\Lambda_n, \\[2mm] L/2 > a_1 > a_2 > \ldots > a_n > 0, \quad a_i \in \mathbb{Z}, \end{cases}
$$

$$
D_n^{(1)} \begin{cases} a = (L - a_1 - a_2 - 1)\Lambda_0 + \displaystyle\sum_{i=1}^{n-1} (a_i - a_{i+1} - 1)\,\Lambda_i + (a_{n-1} + a_n - 1)\Lambda_n, \\[2mm] L > a_1 + a_2, \quad a_1 > a_2 > \ldots > a_n, \quad a_{n-1} + a_n > 0, \\[1mm] \text{all } a_i \in \mathbb{Z}, \text{ or all } a_i \in \mathbb{Z} + \tfrac{1}{2}. \end{cases}\tag{2.2}
$$

Apart from this definition of the states, we need to specify the allowed adjacency or admissibility condition for the states a and b of two neighbouring sites on $\mathcal{L}$. For this purpose we define $\mathcal{A}$ as the set of weights in the vector representation of the classical Lie algebra X_n, and we express the elements of $\mathcal{A}$ in terms of a set of orthonormal vectors ϵ_i, $1 \leqslant i \leqslant n$, $\langle \epsilon_i, \epsilon_j \rangle = \delta_{i,j}$,

$$
\mathcal{A} = \begin{cases} \{\pm\epsilon_1, \ldots, \pm\epsilon_n, 0\}, & B_n, \\ \{\pm\epsilon_1, \ldots, \pm\epsilon_n\}, & C_n, D_n. \end{cases}\tag{2.3}
$$

We can now write the classical part of the weights, denoted by $\bar{\Lambda}_i$, as follows:

$$
B_n^{(1)} \begin{cases} \bar{\Lambda}_i = \epsilon_1 + \ldots \epsilon_i, & 1 \leqslant i \leqslant n-1, \\[1mm] \quad\; = \tfrac{1}{2}(\epsilon_1 + \ldots \epsilon_n) & i = n, \end{cases}
$$

$$
C_n^{(1)} \quad \bar{\Lambda}_i = \epsilon_1 + \ldots + \epsilon_i, \quad 1 \leqslant i \leqslant n,
$$

$$
D_n^{(1)} \begin{cases} \bar{\Lambda}_i = \epsilon_1 + \ldots + \epsilon_i, & 1 \leqslant i \leqslant n-2, \\[1mm] \quad\; = \tfrac{1}{2}(\epsilon_1 + \ldots + \epsilon_{n-2} + \epsilon_{n-1} - \epsilon_n), & i = n-1, \\[1mm] \quad\; = \tfrac{1}{2}(\epsilon_1 + \ldots + \epsilon_{n-2} + \epsilon_{n-1} + \epsilon_n), & i = n. \end{cases}\tag{2.4}
$$

If we finally introduce one more symbol $\rho = \Lambda_0 + \ldots + \Lambda_n$, and set $\epsilon_{-i} = -\epsilon_i$, we get from (2.2) for the classical part of $a + \rho$: $\bar{a} + \bar{\rho} = \sum_{i=1}^{n} a_i \epsilon_i$, and hence $a_\mu = \langle a + \rho, \epsilon_\mu \rangle$, $-n \leqslant \mu \leqslant n$, $\mu \neq 0$.

With the above definitions, we can now formulate the adjacency rule for two neighbouring sites on $\mathcal{L}$ with heights a and b, respectively. If such a pair is admissible it will be denoted by $a \sim b$ and we have, $V(\bar{a})$ being an irreducible X_n module with highest weight $\bar{a}$,

$a \sim b$ iff, for any Dynkin diagram automorphism μ, the tensor

$$\text{module } V(\mu(\bar{a})) \otimes V(\bar{\Lambda}_1) \text{ includes } V(\mu(\bar{b})).\tag{2.5}$$

For $C_n^{(1)}$ and $D_n^{(1)}$ this simply means that $a \sim b$ iff $a - b \in \mathcal{A}$. For $B_n^{(1)}$ we have, in addition to this, to implement the rule that $a \nsim b$ if $a = b \wedge a_n = \frac{1}{2}$.

In our discussion of the higher-rank dilute A models in Sections 4–6, it will be convenient to also adopt a slightly different terminology. Instead of referring to the adjacency rule (2.5), we use the concept of an *adjacency graph* $\mathcal{G}$. Each allowed state or height corresponds to a node on $\mathcal{G}$, and $a \sim b$ on $\mathcal{L}$ if a and b on $\mathcal{G}$ are connected by a bond. For the JMO models at level l the adjacency graphs just correspond to the level-l $X_n^{(1)}$ *weight lattices*. Some weight lattices or adjacency graphs for the $C_n^{(1)}$ JMO model have been drawn in Fig. 3a. In Subsection 4.2 we will show that these same graphs also encode the adjacency rule for the generalized dilute A models.

2.2. Boltzmann weights

With the previous definitions and adjacency rules we can now list the non-zero Boltzmann weights of the $X_n^{(1)}$ JMO models. Defining the function

$$[u] = \sin\left(\frac{s\pi u}{L}\right), \qquad s \in \mathbb{Z},\tag{2.6}$$

with s and L coprime, the allowed local face configurations can be expressed as

$$W\begin{pmatrix} a & a+\epsilon_\mu \\ a+\epsilon_\mu & a+2\epsilon_\mu \end{pmatrix} = \frac{[\lambda-u][1-u]}{[\lambda][1]}, \qquad \mu \neq 0,$$

$$W\begin{pmatrix} a & a+\epsilon_\mu \\ a+\epsilon_\mu & a+\epsilon_\mu+\epsilon_\nu \end{pmatrix} = \frac{[a_\mu-a_\nu+u][\lambda-u]}{[a_\mu-a_\nu][\lambda]}, \qquad \mu \neq \pm\nu,$$

$$W\begin{pmatrix} a & a+\epsilon_\nu \\ a+\epsilon_\mu & a+\epsilon_\mu+\epsilon_\nu \end{pmatrix} = \left(\frac{[a_\mu-a_\nu+1][a_\mu-a_\nu-1]}{[a_\mu-a_\nu]^2}\right)^{1/2}\frac{[\lambda-u][u]}{[\lambda][1]},$$

$$\mu \neq \pm\nu,$$

$$W\begin{pmatrix} a & a+\epsilon_\nu \\ a+\epsilon_\mu & a \end{pmatrix} = \left(G_{a,\mu}G_{a,\nu}\right)^{1/2}\frac{[a_\mu+a_\nu-\lambda+1+u][u]}{[a_\mu+a_\nu+1][\lambda]}, \qquad \mu \neq \nu,$$

$$W\begin{pmatrix} a & a+\epsilon_\mu \\ a+\epsilon_\mu & a \end{pmatrix} = \frac{[2a_\mu+1+u][\lambda-u]}{[2a_\mu+1][\lambda]} + G_{a,\mu}\frac{[2a_\mu-\lambda+1+u][u]}{[2a_\mu+1][\lambda]},$$

$$\mu \neq 0,$$

$$= \frac{[2a_\mu-2\lambda+1+u][\lambda+u]}{[2a_\mu-2\lambda+1][\lambda]} - H_{a,\mu}\frac{[2a_\mu-\lambda+1+u][u]}{[2a_\mu-2\lambda+1][\lambda]},$$

$$\tag{2.7}$$

where

Table 1

$X_n^{(1)}$	g	t	λ	σ	$h(a)$
$B_n^{(1)}$	$2n-1$	1	$n-\frac{1}{2}$	1	$[a]$
$C_n^{(1)}$	$n+1$	2	$n+1$	-1	$[2a]$
$D_n^{(1)}$	$2n-2$	1	$n-1$	1	1

$$G_{a,\mu} = \sigma \frac{h(a_\mu + 1)}{h(a_\mu)} \prod_{\nu \neq 0, \pm \mu} \frac{[a_\mu - a_\nu + 1]}{[a_\mu - a_\nu]}, \qquad \mu \neq 0,$$

$$G_{a,0} = 1,$$

$$H_{a,\mu} = \sum_{\nu \neq \mu} G_{a,\nu} \frac{[a_\mu + a_\nu - 2\lambda + 1]}{[a_\mu + a_\nu + 1]}, \tag{2.8}$$

and $a_0 = -\frac{1}{2}$. As follows from Eq. (2.3), $\mu, \nu = 0, \pm 1, \ldots, \pm n$ for $B_n^{(1)}$ and $\pm 1, \ldots, \pm n$ for $C_n^{(1)}$ and $D_n^{(1)}$. The *crossing* parameter $\lambda = \frac{1}{2}tg$, the sign factor σ and the function h are given in Table 1.

In Ref. [6] it was shown that the above Boltzmann weights satisfy the Yang–Baxter equation [12]

$$\sum_g W\left(\begin{array}{cc} f & g \\ a & b \end{array} \middle| u\right) W\left(\begin{array}{cc} e & d \\ f & g \end{array} \middle| u+v\right) W\left(\begin{array}{cc} d & c \\ g & b \end{array} \middle| v\right)$$

$$= \sum_g W\left(\begin{array}{cc} e & g \\ f & a \end{array} \middle| v\right) W\left(\begin{array}{cc} g & c \\ a & b \end{array} \middle| u+v\right) W\left(\begin{array}{cc} e & d \\ g & c \end{array} \middle| u\right). \tag{2.9}$$

Some useful properties of the weights are the standard initial condition, crossing symmetry and inversion relation

$$W\left(\begin{array}{cc} d & c \\ a & b \end{array} \middle| 0\right) = \delta_{a,c}, \tag{2.10}$$

$$W\left(\begin{array}{cc} d & c \\ a & b \end{array} \middle| \lambda - u\right) = \left(\frac{G_a G_c}{G_b G_d}\right)^{1/2} W\left(\begin{array}{cc} c & b \\ d & a \end{array} \middle| u\right), \tag{2.11}$$

$$\sum_g W\left(\begin{array}{cc} d & g \\ a & b \end{array} \middle| u\right) W\left(\begin{array}{cc} d & c \\ g & b \end{array} \middle| -u\right) = \rho(u)\rho(-u)\delta_{a,c}, \tag{2.12}$$

with the function ρ given by

$$\rho(u) = \frac{[\lambda - u][1 - u]}{[\lambda][1]}. \tag{2.13}$$

3. The BWM algebra

In this section we follow Deguchi et al. [10] to show that the JMO models defined in the previous section in fact provide realizations of the well-known Birman–Wenzl–Murakami algebra [13].

3.1. Braid–monoid algebra

To establish this, we first recast the JMO models into operator form by defining the *face* operators X_i,

$$X_i(u)_{\{a\},\{b\}} = W\left(\begin{array}{cc} a_{i-1} & b_i \\ a_i & a_{i+1} \end{array} \middle| u \right) \prod_{j \neq i} \delta_{a_j, b_j}, \qquad i = 1, \ldots, N. \tag{3.1}$$

The entries $\{a\} = a_0, \ldots, a_{N+1}$ and $\{b\}$ are elements of the $N + 1$ step *path space*, that is $a_i \in \mathcal{G}$ and $a_i \sim a_{i+1}$, for all i.

With this definition, the YBE (2.9) can be rewritten as

$$X_{i+1}(u) X_i(u + v) X_{i+1}(v) = X_i(v) X_{i+1}(u + v) X_i(u). \tag{3.2}$$

Together with the obvious relation

$$X_i(u) X_j(v) = X_j(v) X_i(u) \qquad \text{for } |i - j| \geqslant 2, \tag{3.3}$$

the operators X_i form a so-called Yang–Baxter algebra [12].

If we now define the braid and inverse braid operators b_i and b_i^{-1}, as well as the Temperley–Lieb (TL) operators or monoids e_i, by

$$b_i^{\pm 1} = q^{\mp 1} \lim_{u \to \pm i \infty} \frac{X_i(u)}{\rho(u)}, \qquad e_i = X_i(\lambda), \tag{3.4}$$

we can recast the $X_n^{(1)}$ models into the following simple from:

$$X_i(u) = I - \frac{[u]}{2 i [\lambda][1]} \left(q^{u-\lambda} b_i - q^{-u+\lambda} b_i^{-1} \right). \tag{3.5}$$

Here $I = X_i(0)$ is the identity operator, and

$$q = e^{-s i \pi / L}, \tag{3.6}$$

with L as in (2.1).

The braids and the TL operators, which will be given explicitly in Eq. (3.15), satisfy the following *braid–monoid* algebra [14]:

$$e_i^2 = \sqrt{Q} e_i,$$

$$e_i e_{i \pm 1} e_i = e_i,$$

$$e_i e_j = e_j e_i \qquad \text{for } |i - j| \geqslant 2, \tag{3.7}$$
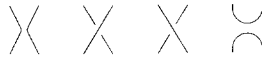
$$b_i b_i^{-1} = b_i^{-1} b_i = I,$$

Fig. 1. From left to right: graphical representation of the local identity $\mathcal{I}$, the braid b, the inverse braid b^{-1} and the TL operator e.

$$b_i\, b_{i+1}\, b_i = b_{i+1}\, b_i\, b_{i+1},$$

$$b_i\, b_j = b_j\, b_i \qquad \text{for } |i - j| \geqslant 2, \tag{3.8}$$

$$b_i\, e_i = e_i\, b_i = \omega\, e_i,$$

$$b_{i\pm1}\, b_i\, e_{i\pm1} = e_i\, b_{i\pm1}\, b_i = e_i\, e_{i\pm1},$$

$$b_i\, e_j = e_j\, b_i \qquad \text{for } |i - j| \geqslant 2, \tag{3.9}$$

$$(b_i - q^{-1}I)(b_i + qI)(b_i - \omega I) = 0, \tag{3.10}$$

$$e_i = I + \frac{b_i - b_i^{-1}}{q - q^{-1}} = \frac{\omega^{-1}}{q - q^{-1}}(b_i - q^{-1}I)(b_i + qI), \tag{3.11}$$

where

$$\sqrt{Q} = 1 + \frac{\omega - \omega^{-1}}{q - q^{-1}} = \frac{[2\lambda][1 + \sigma\lambda]}{[\lambda][1]},$$

$$\omega = \sigma\, q^{2\lambda + \sigma}. \tag{3.12}$$

We remark that the above braid–monoid algebra, where the braids satisfy the cubic reduction relation (3.10) and the TL operators can be expressed as quadratics in the braids, was first introduced by Birman and Wenzl, and independently by Murakami, and is known as the BWM algebra [13].

In our construction of the higher-rank dilute A models, presented in Subsection 4.2, it will be convenient to represent the braid and TL operators graphically. In order to do so, we view these operators as acting on strands of $N + 1$ strings, where $b_i^{\pm1}$ and e_i act non-trivially on the ith and $(i + 1)$th position only. Removing the trivial part, we introduce local operators $X(u)$, $\mathcal{I} = X(0)$, $b^{\pm1}$ and e, shown in Fig. 1, acting in a two-step path space, or, equivalently, acting on strands of two strings. We can thus write the local equivalent of Eq. (3.5) pictorially as

$$X(u) = \Big) \Big(- \frac{[u]}{2\,i\,[\lambda][1]} \left(q^{u-\lambda} \Big)\!\!\Big(- q^{-u+\lambda} \Big)\!\!\Big(\right). \tag{3.13}$$

3.2. Representations of the BWM algebra

To give explicit expressions for the non-zero elements of e_i and $b_i^{\pm1}$ arising from the JMO models, we write, following the notation of (3.1),

$$(e_i)_{\{a\},\{b\}} = e \begin{pmatrix} a_{i-1} & b_i \\ a_i & a_{i+1} \end{pmatrix} \prod_{j \neq i} \delta_{a_j, b_j},$$

$$(b_i^{\pm 1})_{\{a\},\{b\}} = b^{\pm 1} \begin{pmatrix} a_{i-1} & b_i \\ a_i & a_{i+1} \end{pmatrix} \prod_{j \neq i} \delta_{a_j, b_j}. \tag{3.14}$$

With the help of these definitions, the representations of the BWM algebra following from the JMO models read

$$e \begin{pmatrix} a & a + \epsilon_\nu \\ a + \epsilon_\mu & a \end{pmatrix} = (G_{a,\mu} G_{a,\nu})^{1/2},$$

$$b^{\pm 1} \begin{pmatrix} a & a + \epsilon_\mu \\ a + \epsilon_\mu & a + 2\epsilon_\mu \end{pmatrix} = q^{\mp 1}, \qquad \mu \neq 0,$$

$$b^{\pm 1} \begin{pmatrix} a & a + \epsilon_\mu \\ a + \epsilon_\mu & a + \epsilon_\mu + \epsilon_\nu \end{pmatrix} = -q^{\pm(a_\mu - a_\nu)} \frac{[1]}{[a_\mu - a_\nu]}, \qquad \mu \neq \pm\nu,$$

$$b^{\pm 1} \begin{pmatrix} a & a + \epsilon_\nu \\ a + \epsilon_\mu & a + \epsilon_\mu + \epsilon_\nu \end{pmatrix} = - \left(\frac{[a_\mu - a_\nu + 1][a_\mu - a_\nu - 1]}{[a_\mu - a_\nu]^2} \right)^{1/2}, \qquad \mu \neq \pm\nu,$$

$$b^{\pm 1} \begin{pmatrix} a & a + \epsilon_\nu \\ a + \epsilon_\mu & a \end{pmatrix} = (G_{a,\mu} G_{a,\nu})^{1/2} q^{\pm(a_\mu + a_\nu + 1)} \frac{[1]}{[a_\mu + a_\nu + 1]}, \qquad \mu \neq \nu,$$

$$b^{\pm 1} \begin{pmatrix} a & a + \epsilon_\mu \\ a + \epsilon_\mu & a \end{pmatrix} = (G_{a,\mu} - 1) q^{\pm(2a_\mu + 1)} \frac{[1]}{[2a_\mu + 1]}, \qquad \mu \neq 0,$$

$$= (1 - H_{a,\mu}) q^{\pm(2a_\mu + 1)} \frac{[1]}{[2a_\mu - 2\lambda + 1]}, \tag{3.15}$$

with $G_{a,\mu}$ and $H_{a,\mu}$ as in (2.8).

4. Construction of $A_n^{(2)}$ models

Using the representations of the BWM algebra as listed in Eq. (3.15), we construct four different series of $A_n^{(2)}$ RSOS models.

4.1. Dual baxterization

The first three are in fact obtained straightforwardly, following an observation made in Ref. [11] that to each representation of the BWM algebra correspond two distinct operators $X_i(u)$ satisfying the Yang–Baxter algebra. That is, apart from the *baxterization* (3.5) or, equivalently, (3.13), with λ related to the rank n of $X_n^{(1)}$ as in Table 1, we can make a second choice for λ as a function of n such that (3.5) satisfies the YBE. This alternative choice for the crossing parameter λ, together with the twisted affine Lie

Table 2

$X_n^{(1)}$	$X_m^{(2)}$	λ
$B_n^{(1)}$	$A_{2n}^{(2)}$	$n + \frac{1}{2} - \frac{1}{2}L$
$C_n^{(1)}$	$A_{2n-1}^{(2)}$	$n - \frac{1}{2}L$
$D_n^{(1)}$	$A_{2n-1}^{(2)}$	$n - \frac{1}{2}L$

algebra associated with the resulting model, is given in Table 2. In this table we also list the untwisted model from which these "new" models arise.

The proof of this *dual* baxterization is very simple. If we replace q^σ by $-q^{-\sigma}$, keeping ω as a function of n fixed, all BWM relations (3.7)–(3.11) remain unchanged, and hence the form (3.5) still baxterizes the $X_n^{(1)}$ representations. The condition that ω is invariant under the above transformation is met by replacing λ by $\lambda - \sigma + L/2$ in Table 1, resulting in the new relation between λ and n of Table 2.

As a function of λ (not as a function n) the values of $\sqrt{Q}$ and ω have of course changed, and, by replacing λ in (3.12) by $\lambda - \sigma + L/2$, now read

$$\sqrt{Q} = \frac{[2\lambda][1 - \sigma\lambda]}{[\lambda][1]},$$

$$\omega = -\sigma q^{2\lambda - \sigma}, \tag{4.1}$$

where we note that the relation between $\sqrt{Q}$ and ω as given in (3.12) remains unaltered.

The $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$ models arising from the $B_n^{(1)}$ and $C_n^{(1)}$ algebras, respectively, are precisely the critical cases of the models obtained previously in Ref. [7]. In order to make the correspondence with the parametrization in Ref. [7] exact, one has to make the transformation $\lambda \rightarrow -\lambda - L/2$ and $u \rightarrow -u$ in expression (3.5). The $A_{2n-1}^{(2)}$ model obtained via $D_n^{(1)}$ is new, though it was already suggested in Ref. [7] that such a model could be found. In Section 5 we give explicit expressions for the Boltzmann weights for this model in the more general, but still solvable, elliptic case.

4.2. Construction of higher-rank dilute A models

4.2.1. Dilute BWM algebra

To obtain a second series of $A_{2n}^{(2)}$ models, we have to generalize the BWM algebra to allow for vacancies, i.e., the absence of strings. This idea of so-called *dilution* was first applied in Refs. [8,9] to the TL algebra, yielding a dilute version of this algebra. It was shown in these same papers that this *dilute TL* algebra, like the ordinary TL algebra, admits a baxterization. The resulting models admit a classification in terms of the classical and affine, simply laced Lie algebras, and were, for obvious reasons, termed dilute A–D–E models. (For a somewhat more algebraic setting of dilute algebras, see also Ref. [15].) Recently, also a dilute version of the BWM algebra has been found which can be baxterized [11]. It is in fact precisely this same *dilute BWM* algebra that underlies our construction of higher-rank dilute A models.

Fig. 2. Graphical representation of the local operators in the dilute BWM algebra.

The idea of dilution amounts to including operators acting on vacant states. That is, instead of the usual braids and monoids shown in Fig. 1, we allow for the full set of operators $\mathcal{O}_1, \ldots, \mathcal{O}_{11}$ depicted in Fig. 2. The first four operators act simply as projectors on the various local subspaces. Also the "shift" operators 6 and 7 act trivially, their product being a projector of type 3 or 4 depending on their relative ordering. The only non-trivial new objects are the TL like operators 10 and 11, which, for example, obey the relations

$$(4.2)$$

For the complete set of defining relations of the dilute BWM algebra we refer the reader to the original paper [15]. Clearly however, the operators $\mathcal{O}_1$, $\mathcal{O}_7$–$\mathcal{O}_9$, generate a BWM subalgebra.

4.2.2. Baxterization of the dilute BWM algebra

As was shown in Ref. [11], the dilute BWM algebra admits a baxterization as follows:

$$(4.3)$$

In the case of our interest, we take the BWM subalgebra to be generated by $C_n^{(1)}$. Hence the expression (3.15) for the braids $b_i^{\pm 1}$ and the TL operator e_i still holds, though the relation between the crossing parameter λ and the rank n of $C_n^{(1)}$ as given in Table 1 changes to [2]

$$\lambda = n + \tfrac{1}{2}.$$

$$(4.4)$$

---

[2] Some care has to be taken in interpreting the very last line in (3.15), as it contains the variable λ explicitly, as well as implicitly via the definition of $H_{a,\mu}$. Only after first replacing λ by $n + 1$ before transforming λ to $\lambda = n + \tfrac{1}{2}$, the expression remains valid. We do note however that, since for $C_n^{(1)}$ $\mu \neq 0$, we can always use the alternative form independent of λ.

Consequently, the face operators generated by the BWM subalgebra do not yield a solution to the YBE. Due to the different relation between λ and n, the "constants" $\sqrt{Q}$ and ω now read

$$\sqrt{Q} = \frac{[2\lambda+1][\frac{1}{2}-\lambda]}{[\lambda+\frac{1}{2}][1]},$$

$$\omega = -q^{2\lambda}, \tag{4.5}$$

where the relation (3.12) between $\sqrt{Q}$ and ω is, again, still valid.

4.2.3. $A_{2n}^{(2)}$ representation of the dilute BWM algebra

As already mentioned before, the model obtained by diluting the $C_n^{(1)}$ BWM representations is naturally associated with the twisted affine Lie algebra $A_{2n}^{(2)}$. The generators of the "$A_{2n}^{(2)}$ dilute BWM algebra" underlying the RSOS model are, apart from $(\mathcal{O}_7)_i = b_i$, $(\mathcal{O}_8)_i = b_i^{-1}$ and $(\mathcal{O}_9)_i = e_i$ listed in (3.15),

$$\mathcal{O}_1 \begin{pmatrix} a & a+\epsilon_\mu \\ a+\epsilon_\mu & a+\epsilon_\mu+\epsilon_\nu \end{pmatrix} = 1,$$

$$\mathcal{O}_2 \begin{pmatrix} a & a+\epsilon_\mu \\ a+\epsilon_\mu & a+\epsilon_\mu \end{pmatrix} = \mathcal{O}_3 \begin{pmatrix} a & a \\ a & a+\epsilon_\mu \end{pmatrix} = 1,$$

$$\mathcal{O}_4 \begin{pmatrix} a & a \\ a & a \end{pmatrix} = 1,$$

$$\mathcal{O}_5 \begin{pmatrix} a & a \\ a+\epsilon_\mu & a+\epsilon_\mu \end{pmatrix} = \mathcal{O}_6 \begin{pmatrix} a & a+\epsilon_\mu \\ a & a+\epsilon_\mu \end{pmatrix} = 1,$$

$$\mathcal{O}_{10} \begin{pmatrix} a & a \\ a+\epsilon_\mu & a \end{pmatrix} = \mathcal{O}_{11} \begin{pmatrix} a & a+\epsilon_\mu \\ a & a \end{pmatrix} = (G_{a,\mu})^{1/2}, \tag{4.6}$$

with $\mu = \pm 1, \ldots, \pm n$. Here we have employed notation similar to that of Eq. (3.14).

4.2.4. Higher-rank dilute A models

We note that for $n = 1$ we can, using Eq. (3.15) and $G_{a,\mu} = -[2a_\mu + 2]/[2a_\mu]$, eliminate the braids in favour of the TL operators, to rewrite (4.3) as

$$X(u) = \frac{[\lambda-u][1-u]}{[\lambda][1]} \,\bigg)\bigg(\, - \frac{[\lambda-1-u][u]}{[\lambda][1]} \,\overset{\cup}{\underset{\cap}{}}$$

$$+ \frac{[u][\lambda-u]}{[\lambda][1]} \left(\diagup + \diagdown \right) + \frac{[\lambda-u]}{[\lambda]} \left(\diagdown + \diagup \right)$$

$$+ \frac{[u]}{[\lambda]} \left(\overset{\cdots}{\cap} + \overset{\cup}{\cdots} \right) + \frac{[\lambda][1]+[u][\lambda-u]}{[\lambda][1]} \,\diagup\!\!\!\diagdown. \tag{4.7}$$

This is exactly expression (2) combined with (7) of Ref. [9] for the dilute A models. Hence we propose the name *higher-rank dilute A models* for the rank $n > 1$ case.

5. Elliptic solutions to the YBE

The procedure outlined in the previous section to find $A_n^{(2)}$ RSOS models only applies at criticality. It is however possible to extend all the $A_n^{(2)}$ models to yield elliptic solutions of the YBE. For the models based on the $B_n^{(1)}$ and $C_n^{(1)}$ BWM representations, the elliptic solution has already been found in Ref. [7]. We therefore restrict our attention to the $A_{2n-1}^{(2)}$ models based on $D_n^{(1)}$, and to the higher-rank dilute A models.

Before we present the solutions, we need the following elliptic theta functions, with argument u and arbitrary but fixed nome $p = \exp(i\pi\tau)$, $\text{Im}(\tau) > 0$:

$$\vartheta_1(u,p) = \sin u \prod_{n=1}^{\infty} \left(1 - 2p^{2n}\cos 2u + p^{4n}\right)\left(1 - p^{2n}\right),$$

$$\vartheta_4(u,p) = \prod_{n=1}^{\infty} \left(1 - 2p^{2n-1}\cos 2u + p^{4n-2}\right)\left(1 - p^{2n}\right). \tag{5.1}$$

5.1. Elliptic $A_{2n-1}^{(2)}$ models

The elliptic extension of the $A_{2n-1}^{(2)}$ models arising from the $D_n^{(1)}$ algebra is particularly simple. In fact, the Boltzmann weights are again just given by Eqs. (2.7) and (2.8), where we now interpret the function $[\cdot]$ as a theta function:

$$[u] = \vartheta_1\left(\frac{s\pi u}{L}, p\right), \tag{5.2}$$

which reduces to (2.6) in the critical $p \to 0$ limit. The function h which occurs in the definition of $G_{a,\mu}$, assumes the off-critical value

$$h(a) = \vartheta_4\left(\frac{2s\pi a}{L}, p^2\right). \tag{5.3}$$

To show that the YBE holds for the elliptic face weights, we refer to the proof for the $X_n^{(1)}$ models [6]. Due to the similarity between the weights of the JMO models and the $A_{2n-1}^{(2)}$ model, only some trivial modifications have to be carried out to extend their proof to the above model.

Although the $A_{2n-1}^{(2)}$ model is defined for values of the rank $n \geqslant 3$ only (see Section 2), it is interesting to note that for $n = 1$ the Boltzmann weights simplify to that of the CSOS model [16].

5.2. Elliptic $A_{2n}^{(2)}$ models

The elliptic Boltzmann weights for the higher-rank dilute A models are somewhat more involved. Apart from the function $[u]$ in (5.2), we need a second theta function

$$[u]_4 = \vartheta_4\left(\frac{s\pi u}{L}, p\right). \tag{5.4}$$

With these two functions we get the following non-zero face weights:

$$W\begin{pmatrix} a & a+\epsilon_\mu \\ a+\epsilon_\mu & a+2\epsilon_\mu \end{pmatrix} = \frac{[\lambda-u][1-u]}{[\lambda][1]},$$

$$W\begin{pmatrix} a & a+\epsilon_\mu \\ a+\epsilon_\mu & a+\epsilon_\mu+\epsilon_\nu \end{pmatrix} = \frac{[a_\mu-a_\nu+u][\lambda-u]}{[a_\mu-a_\nu][\lambda]}, \qquad \mu \neq \pm\nu,$$

$$W\begin{pmatrix} a & a+\epsilon_\nu \\ a+\epsilon_\mu & a+\epsilon_\mu+\epsilon_\nu \end{pmatrix} = \left(\frac{[a_\mu-a_\nu+1][a_\mu-a_\nu-1]}{[a_\mu-a_\nu]^2}\right)^{1/2} \frac{[\lambda-u][u]}{[\lambda][1]},$$

$$\mu \neq \pm\nu,$$

$$W\begin{pmatrix} a & a+\epsilon_\nu \\ a+\epsilon_\mu & a \end{pmatrix} = \left(G_{a,\mu}G_{a,\nu}\right)^{1/2}\frac{[a_\mu+a_\nu-\lambda+1+u][u]}{[a_\mu+a_\nu+1][\lambda]}, \qquad \mu \neq \nu,$$

$$W\begin{pmatrix} a & a+\epsilon_\mu \\ a+\epsilon_\mu & a \end{pmatrix} = \frac{[2a_\mu+1+u][\lambda-u]}{[2a_\mu+1][\lambda]} + G_{a,\mu}\frac{[2a_\mu-\lambda+1+u][u]}{[2a_\mu+1][\lambda]},$$

$$= \frac{[2a_\mu-2\lambda+1+u][\lambda+u]}{[2a_\mu-2\lambda+1][\lambda]} - H_{a,\mu}\frac{[2a_\mu-\lambda+1+u][u]}{[2a_\mu-2\lambda+1][\lambda]},$$

$$W\begin{pmatrix} a & a+\epsilon_\mu \\ a & a+\epsilon_\mu \end{pmatrix} = W\begin{pmatrix} a & a \\ a+\epsilon_\mu & a+\epsilon_\mu \end{pmatrix} = \left(\frac{[a_\mu+\frac{3}{2}]_4[a_\mu-\frac{1}{2}]_4}{[a_\mu+\frac{1}{2}]_4^2}\right)^{1/2}\frac{[\lambda-u][u]}{[\lambda][1]},$$

$$W\begin{pmatrix} a & a+\epsilon_\mu \\ a & a \end{pmatrix} = W\begin{pmatrix} a & a \\ a+\epsilon_\mu & a \end{pmatrix} = \left(G_{a,\mu}\right)^{1/2}\frac{[a_\mu-\lambda+\frac{1}{2}+u]_4[u]}{[a_\mu+\frac{1}{2}]_4[\lambda]},$$

$$W\begin{pmatrix} a & a \\ a & a+\epsilon_\mu \end{pmatrix} = W\begin{pmatrix} a+\epsilon_\mu & a \\ a & a \end{pmatrix} = \frac{[a_\mu+\frac{1}{2}-u]_4[\lambda-u]}{[a_\mu+\frac{1}{2}]_4[\lambda]},$$

$$W\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \frac{[2\lambda-u][\lambda+u]}{[2\lambda][\lambda]} - H_{a,0}\frac{[\lambda-u][u]}{[2\lambda][\lambda]}, \tag{5.5}$$

with $\mu = \pm 1, \ldots, \pm n$. The function $G_{a,\mu}$ as given in (2.8) is still correct, but the function h therein generalizes to

$$h(a) = \frac{[2a]}{[a]_4}. \tag{5.6}$$

The functions $H_{a,\mu}$ and $H_{a,0}$ read

$$H_{a,\mu} = \sum_{\nu\neq\mu} G_{a,\nu}\frac{[a_\mu+a_\nu-2\lambda+1]}{[a_\mu+a_\nu+1]} + \frac{[a_\mu-2\lambda+\frac{1}{2}]_4}{[a_\mu+\frac{1}{2}]_4},$$

$$H_{a,0} = \sum_{\mu} G_{a,\mu} \frac{[a_{\mu} - 2\lambda + \frac{1}{2}]_4}{[a_{\mu} + \frac{1}{2}]_4}, \tag{5.7}$$

where here, as well as in Eq. (5.5), $\mu, \nu = \pm 1, \ldots, \pm n$. In order to avoid confusion, we remark that the above relation between $H_{a,\mu}$ and $G_{a,\mu}$ does not reproduce the relation (2.8) in the $p \to 0$ limit. This is a consequence of the different relation between the rank n and the crossing parameter λ for the $C_n^{(1)}$ model and the $A_{2n}^{(2)}$ model, see Table 1 and Eq. (4.4), and also the footnote on page 472.

The full proof that the weights (5.5) satisfy the YBE is rather involved, though the approach followed in Ref. [6] still does apply. A simpler method of proof is provided by noting that the above model can be obtained by restricting the unrestricted $A_n^{(2)}$ SOS model of Ref. [7], after first performing an imaginary transformation. To show that after restriction the YBE is still satisfied, it suffices to inspect equations where some of the external spins in the YBE take their values at the boundary of the level-l $C_n^{(1)}$ weight lattice. From the zeros of the function h,

$$h(a) = 0 \qquad \text{if } a = 0, L/2, \tag{5.8}$$

and the particular form of the function $G_{a,\mu}$ in (2.8), we readily find that the "boundary" YB equations are indeed satisfied for $a_i \in \mathbb{Z}$, $L/2 > a_1 > a_2 > \ldots > a_n > 0$.

It is known that the off-critical $n = 1$ dilute A models break the $\mathbb{Z}_2$ symmetry of the underlying adjacency graph for odd values of the parameter s. This led [17] for example to the identification of the dilute A_3 model ($n = 1$, $l = 2$, $s = 5$) and Zamolodchikov's E_8 S-matrix of the critical Ising model in a field [18].

Here we extend this broken $\mathbb{Z}_2$ observation to the general-rank dilute A models. The adjacency graphs for the rank-n dilute models correspond to the level-l $C^{(1)}$ weight lattices, some examples of which are shown in Fig. 3a[3]. First of all we note that the $C_n^{(1)}$ weight lattice at level l is invariant under the following transformation:

$$a_i \to L/2 - a_{n-i+1} \equiv a_i' \qquad \text{for all } i = 1, \ldots, n, \tag{5.9}$$

where we recall that $L/2 = l + n + 1$. Denoting the corresponding height as a',

$$a' = (L/2 - a_1 - 1)\Lambda_n + \sum_{i=1}^{n-1} (a_i - a_{i+1} - 1)\,\Lambda_{n-i} + (a_n - 1)\Lambda_0, \tag{5.10}$$

we find that

$$W\begin{pmatrix} d' & c' \\ a' & b' \end{pmatrix} \neq W\begin{pmatrix} d & c \\ a & b \end{pmatrix} \qquad \text{for } s \text{ odd and } p \neq 0. \tag{5.11}$$

---

[3] Note that by construction the dilute models are based on the same graphs as their "undiluted" counterparts. This implies that although two neighbouring sites on $\mathcal{L}$ can have the same height, the adjacency graphs $\mathcal{G}$ do not have tadpoles. This meets our earlier convention |9,19| that only if two neighbouring sites on $\mathcal{L}$ have different heights, they must be adjacent on $\mathcal{G}$.
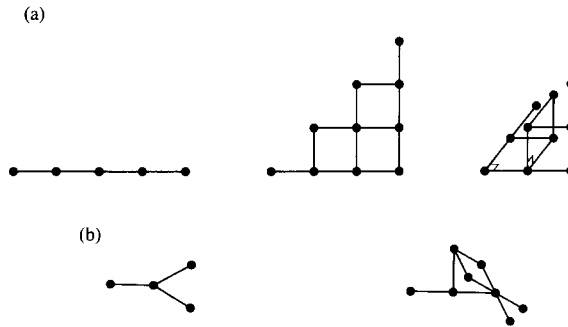
(a)

(b)

Fig. 3. (a) Some simple adjacency graphs for the rank-n, level-l dilute A-model. Note that the $(n,l) = (3,2)$ and the $(n,l) = (2,3)$ graphs are in fact identical. This level–rank duality is true more generally for the pairs (n,l) and (l,n). (b) The D type graphs obtained by modding out the $\mathbb{Z}_2$ symmetry of graphs in (a).

5.3. Unrestricted $A_n^{(2)}$ models

Our approach to obtain the four series of (critical) $A_n^{(2)}$ RSOS models has been to utilize representations of the (dilute) BWM algebra based on the non-twisted affine Lie algebra $X_n^{(1)}$, X=B,C,D.

Here we wish to briefly mention an alternative method [5–7] to obtain the (elliptic) $A_n^{(2)}$ RSOS models. This serves the purpose of explaining the close relation between the two series of $A_{2n-1}^{(2)}$ models as well as between the two series of $A_{2n}^{(2)}$ models.

As already mentioned in the previous section, the $A_n^{(2)}$ RSOS models can be viewed as restricted versions of some $A_n^{(2)}$ SOS model. This is similar to the ABF model being the restriction of the eight-vertex SOS model [5].

In the case of $A_n^{(2)}$, the unrestricted SOS model was found in Ref. [7]. In such an SOS model the states or heights are not restricted to a finite adjacency graph (for the models in this paper the level-l X_n weight lattices), but range over an infinite set.

For the elliptic models of Subsections 5.1 and 5.2, this is established as follows. We replace Eq. (2.2) by

$$a \in \sum_{i=0}^{n} \mathbb{C}\, \Lambda_i, \qquad (5.12)$$

with Λ_i the fundamental weights of $D_n^{(1)}$ and $C_n^{(1)}$, respectively. Furthermore, instead of the adjacency rule (2.5) we have that $a \sim b$ if $a - b \in \mathcal{A}$. Finally, we let L, which now appears only in the definitions (5.2) and (5.4) of $[\cdot]$ and $[\cdot]_4$, be an arbitrary complex variable, $L \in \mathbb{C}$, $L \neq 0$.

Of course, since only $a - b \in \mathcal{A}$, we effectively have that $a = w_0 + \sum_{i=1}^{n} \mathbb{Z}\, \epsilon_i$, with w_0 a fixed element of $\sum_{i=0}^{n} \mathbb{C}\, \Lambda_i$. It is this freedom in the choice of "origin" w_0 which allows us to make different types of restrictions.

Let us first show how this works in the $A_{2n-1}^{(2)}$ case. If we choose L as in (2.1) with X=D and we restrict the local states to be the level-l dominant integral weights of $D_n^{(1)}$, i.e., we have (2.2), then the elliptic Boltzmann weights (2.7) with (2.8) and (5.3)

satisfy the YBE provided we impose the adjacency rule (2.5). This gives us back the $A_{2n-1}^{(2)}$ RSOS model based on $D_n^{(1)}$ as presented in Subsection 5.1. The other way to restrict is that of Ref. [7]. First we carry out the imaginary transformation

$$a_i \rightarrow \frac{\tau L}{2s} + a_i \qquad (5.13)$$

in the weights (2.7). It is the freedom in w_0 which allows us to perform this translation. To actually obtain a model based on C_n, we now "reinterpret" the Λ_i's in (5.12) to be the fundamental weights of $C_n^{(1)}$. Since formally $\mathcal{A}$ is the same for C_n and D_n, see (2.3), the SOS model still satisfies the YBE. Now we again choose L according to (2.1) but with X=C, and restrict the states to the dominant integral weights of $C_n^{(1)}$. Provided (2.5) holds, this yields the $A_{2n-1}^{(2)}$ RSOS model based on $C_n^{(1)}$. It is a simple matter to check that in the critical limit the thus obtained Boltzmann weights are, up to a gauge transformation, those of Subsection 4.1 with X=C.

In an almost similar manner we can restrict the $A_{2n}^{(2)}$ SOS model in two ways. One, based on $C_n^{(1)}$ leads to the rank-n dilute A models and the other, based on $B_n^{(1)}$ leads to the $A_{2n}^{(2)}$ RSOS models of Ref. [7]. The only slight complication is that the set of weight vectors $\mathcal{A}$ for B_n and C_n are not equal (equal in a formal sense). However, the extra 0 for B_n is compensated by the fact that for the dilute models we have the rule that only if two neighbouring sites on $\mathcal{L}$ have different height they must be adjacent on $\mathcal{G}$. As a remark we add to this that though it might be tempting to reformulate the dilute models such that the adjacency graphs $\mathcal{G}$ explicitly encode the fact that neighbouring sites on $\mathcal{L}$ can take equal values, we believe this to be unnatural. This reformulation would, following the notation of Fig. 4, result in the same graphs as those of Fig. 3, but with all nodes replaced by open circles. The problem with this approach however is that dilution of models based on graphs with tadpoles becomes ill-defined, or requires additional spin variables on the edges of $\mathcal{L}$. See for e.g., the dilution of the $B_n^{(1)}$ JMO models as carried out in Ref. [20].

Despite the intimate relation between the several series of $A_n^{(2)}$ models, we wish to stress here that as far as the RSOS versions are concerned the models have quite different properties. The most apparent one is perhaps the quite distinct incidence or adjacency rule depending on the underlying untwisted algebra. For example, the graphs of Fig. 3a for the dilute A models have to be compared with those of Fig. 4 for the $A_{2n}^{(2)}$ models of Ref. [7]. But also the difference in BWM structure (ordinary or dilute), in critical behaviour [7,21,17], and broken $\mathbb{Z}_2$ symmetry are worth mentioning.

To end this our discussion on restricted versus unrestricted SOS model, we remark that the two possible ways to restrict the $A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$ SOS models have a simple algebraic origin. That is, both $A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$ admit two realizations in terms of the classical Lie algebras X_n, X=B,C,D. In particular, $A_{2n-1}^{(2)}$ can be based on the root systems of either C_n or D_n, and $A_{2n}^{(2)}$ can be based on the root systems of both B_n and C_n. Moreover, the imaginary transformation (5.13) precisely corresponds to the interchange of the two associated (finite) root subsystems of $A_n^{(2)}$.
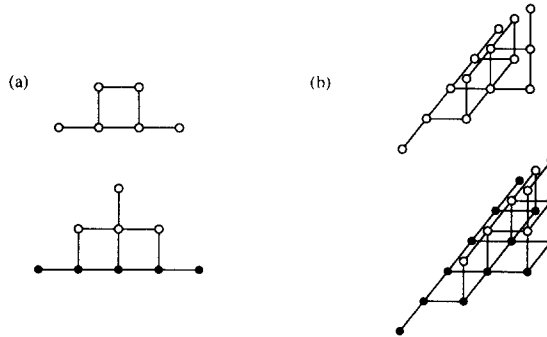
Fig. 4. Some adjacency graphs for the $A_{2n}^{(2)}$ model based on $B_n^{(1)}$. The rank and level read (a) $n = 2$, $l = 4$ and (b) $n = 3$, $l = 5$. Nodes admissible to themselves ($a \sim a$) are represented by an open circle.

6. Summary and discussion

In this paper we have derived four series of solvable RSOS models associated with the twisted affine Lie algebra $A_n^{(2)}$. All four series did arise as baxterizations of known $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ representations of the BWM algebra. The $A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$ models obtained from the $C_n^{(1)}$ and $B_n^{(1)}$ algebras, respectively, are the RSOS models found previously in Ref. [7]. The $A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$ models arising from the $D_n^{(1)}$ and, again, the $C_n^{(1)}$ algebra, are new.

To obtain the second series of $A_{2n}^{(2)}$ models, we have generalized, following recent work by Grimm [11], the BWM algebra to allow for vacancies. We have furthermore shown that the second $A_{2n}^{(2)}$ model reduces to the dilute A-model [8,9] in the $n = 1$ case.

Finally, for the two new series of RSOS models we presented, without proof, an elliptic, off-critical extension that satisfies the YBE.

From the results presented in this paper, various questions naturally arise. First of all, it is known that the critical dilute A models are members of a larger class of dilute A–D–E lattice models. These models give a complete realization [21] of the $SU(2)$ modular-invariant partition functions as classified by Cappelli et al. [22]. An interesting question is whether not also the higher-rank dilute A models belong to some larger family of models, and, if so, to what modular invariants they correspond. Clearly, from the form of the higher-rank models as given in Eq. (4.3), such a larger class is not apparent. However, by eliminating the inverse braid operator in favour of the TL operator, a form very close to the dilute TL algebra emerges. Indeed, for $n = 1$ we exactly recover this algebra, see Eq. (4.7). Starting from this generalized dilute TL algebra, one might try to extend the method of Ref. [19] to find higher-rank dilute D and E models. Of course, some of these (D type models) readily follow from the fact that the dilute A models at criticality always obey the $\mathbb{Z}_2$ symmetry of the $C_n^{(1)}$ weight lattice, and hence from modding out this symmetry. For the graphs in Fig. 3a, the resulting D graphs are shown in Fig. 3b. A property of the higher-rank dilute A models

supporting the possibility of finding D and E type models is the fact that the function $G_{a,\mu}$ in (2.8) closely relates to eigenvectors S of the level-l $C_n^{(1)}$ weight lattice[4],

$$G_{a,\mu} = \frac{S(a + \epsilon_\mu)}{S(a)},$$

$$S(a) = (-)^{a_1 + \ldots + a_n} \prod_{i=1}^{n} h(a_i) \prod_{1 \leqslant i < j \leqslant n} [a_i + a_j][a_i - a_j]. \tag{6.1}$$

Here S satisfies the eigenvalue equation

$$\sum_{a \sim b} S(a) = \sqrt{Q} S(b), \tag{6.2}$$

where the eigenvalue $\sqrt{Q}$ is given by (4.5). Like for the known $n = 1$ case, other models might occur for adjacency graphs that fit this same eigenvalue equation, or in other words that intertwine the $C_n^{(1)}$ weight lattice.

Another question concerns the dilution of BWM representations other than those yielding the $A_{2n}^{(2)}$ models. As was shown in Ref. [11], the R-matrix of the $D_{n+1}^{(2)}$ vertex model [3] can be found by diluting $B_n^{(1)}$ BWM representations. This can actually easily be carried out at the level of the RSOS models as well. This and the dilution of the $D_n^{(1)}$ JMO model, resulting in a new series of models related to $B_n^{(1)}$, will be the issue of a separate publication [20].

A final intriguing point is the problem of generalizing the Hecke algebra to a dilute Hecke algebra. If such an algebra could be defined, it would open the possibility to generalize the rank-one dilute A models in a completely different fashion as described in this paper. This would be similar to the two distinct ways of generalizing the ABF model [5]. That is, the critical ABF model can either be viewed as a TL model, naturally leading to a $A_n^{(1)}$ generalization based on the Hecke algebra [6], or we can view it as a model based on the BWM algebra, leading to the $C_n^{(1)}$ JMO models of Eq. (2.7). In a naive attempt to generalize the Hecke algebra to allow for vacancies, simply following the same procedure as applied in diluting the TL or BWM algebra, we however encounter the problem that by combining the defining relations between the generators, we can always reduce the algebra to the dilute TL case. Whether this rather unwanted feature can be repaired by some more elaborate dilution process remains an open question.

Acknowledgement

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[4] We remark that, unlike the $n = 1$ case, we can for $n > 1$ in general not choose the parameter s in (2.6) such that $\sqrt{Q}$ is the largest eigenvalue of G. Consequently S is generally not of the Perron–Frobenius type.

Note added in proof

The author was informed by A. Kuniba that the idea of dual baxterization as described in Subsection. 4.1 has been applied to vertex models in Ref. [23].

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