THE GENERALIZED BORWEIN CONJECTURE. II. REFINED q-TRINOMIAL COEFFICIENTS

S. OLE WARNAAR

Abstract. Transformation formulas for four-parameter refinements of the q-trinomial coefficients are proven. The iterative nature of these transformations allows for the easy derivation of several infinite series of q-trinomial identities, and can be applied to prove many instances of Bressoud's generalized Borwein conjecture.

1. Introduction

This is the second in a series of papers addressing Bressoud's generalized Borwein conjecture. Defining the Gaussian polynomial or q-binomial coefficient as

$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \begin{cases} \displaystyle\prod_{k=1}^{m} \frac{1-q^{n+k}}{1-q^k} & m,n \in \mathbb{Z}_+ \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

(with $\mathbb{Z}_+ = \{0,1,2,\dots\}$) Bressoud [11] considered the polynomials

$$G(N,M;\alpha,\beta,K) = \sum_{j=-\infty}^{\infty} (-1)^j q^{Kj((\alpha+\beta)j+\alpha-\beta)/2} \begin{bmatrix} M+N \\ N-Kj \end{bmatrix}.$$

Writing $P \geq 0$ if P is a polynomial with nonnegative coefficients, he then conjectured the truth of the following statement concerning G.

Conjecture 1.1. *Let K be a positive integer and $N, M, \alpha K, \beta K$ be nonnegative integers such that $1 \leq \alpha + \beta \leq 2K - 1$ (strict inequalities when $K = 2$) and $\beta - K \leq N - M \leq K - \alpha$. Then $G(N,M;\alpha,\beta,K) \geq 0$.*

This generalizes an earlier conjecture of P. Borwein [5] stating that the coefficients of the polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$, defined by

$$\prod_{k=1}^{n} (1-q^{3k-2})(1-q^{3k-1}) = A_n(q^3) - qB_n(q^3) - q^2 C_n(q^3)$$

are all nonnegative. By the q-binomial theorem it readily follows that [5]

$$\begin{aligned} A_n(q) &= G(n,n;4/3,5/3,3) \\ B_n(q) &= G(n+1,n-1;2/3,7/3,3) \\ C_n(q) &= G(n+1,n-1;1/3,8/3,3). \end{aligned}$$

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For a more comprehensive introduction to the above conjectures we refer to our first paper in this series [31] and to the original publications by Andrews [5] and Bressoud [11].

Several special cases of the generalized Borwein conjecture have already been settled in the literature. When α and β are integers $G(N,M;\alpha,\beta,K;q)$ has a combinatorial interpretation as the generating function of partitions that fit in a rectangle of dimensions $M \times N$ and satisfy certain restrictions on their hook-differences [7]. For later reference and comparison we formalize the $M=N$ case of this in a theorem.

Theorem 1.1. *$G(M,M;\alpha,\beta,K) \geq 0$ for $\alpha,\beta,K \in \mathbb{Z}$ such that $0 \leq \alpha,\beta \leq K$.*

When at least one of α and β is fractional, no combinatorial interpretation of $G(N,M;\alpha,\beta,K;q)$ is known, except for a few very simple cases. $G(M,M;1/2,1,2)$, for example, is the generating function of partitions with largest part at most M and no parts below its Durfee square. Despite this lack of a combinatorial interpretation, Ismail, Kim and Stanton [19, Thm. 5] have proven Conjecture 1.1 to hold for $\alpha+\beta=K$ with $\alpha=(K-N+M\pm 1)/2$ and $M+N$ even. Again we put the $M=N$ case of this in a theorem. Because of the symmetry

$$G(M,M;\alpha,\beta,K) = G(M,M;\beta,\alpha,K) \tag{1.2}$$

we may without loss of generality assume $\alpha=(K-1)/2$.

Theorem 1.2. *For K a positive integer $G(M,M;(K-1)/2,(K+1)/2,K) \geq 0$.*

When K is odd this is of course contained in Theorem 1.1. Finally we quote a result obtained in our first paper by use of the Burge transform [31, Cor. 3.2].

Theorem 1.3. *$G(M,M;b,b+1/a,a) \geq 0$ for a,b coprime integers such that $0 < b < a$.*

Similar results were obtained for $N \neq M$ with both α and β noninteger [31, Cor. 5.1]. It is quite clear, however, that proving $G(M,M;\alpha,\beta,K) \geq 0$ when both α and β are fractional and not $\alpha=(K-1)/2$ and $\beta=(K+1)/2$ — $A_n(q)$ of the original Borwein conjecture falls in this class — is rather more difficult. In this paper new transformation formulas will be applied to make some progress in this direction. To state our results we define

$$d_{\bar{a}} = d = a_0 + a_1 + \cdots + a_n \tag{1.3a}$$

$$\mu_{\bar{a}} = \mu = \frac{2}{3}(4^{d-1}-1) - \sum_{i=1}^{n} 4^{a_i+\cdots+a_n-2}, \tag{1.3b}$$

where $\bar{a}=(a_0,a_1,\dots,a_n) \in \mathbb{Z}_+^{n+1}$. Note that $\mu \in \mathbb{Z}_+$ provided $a_n \geq 2-\delta_{n,0}$. With these definitions our main results are the following three theorems, which generalize Theorems 1.1–1.3.

Theorem 1.4. *For $n \geq 0$, let $\bar{a}=(a_0,\dots,a_n) \in \mathbb{Z}_+^{n+1}$ such that $a_0 \geq 0$, $a_1,\dots,a_{n-1} \geq 1$ and $a_n \geq 2-\delta_{n,0}$. Then*

$$G(M,M;(\alpha+\mu K)/2^{d-1},(\beta+\mu K)/2^{d-1},2^{d-1}K) \geq 0,$$

for $\alpha,\beta,K \in \mathbb{Z}$ such that $0 \leq \alpha,\beta \leq K$, and d and μ given by (1.3).

For $\bar{a}=(1)$ there holds $d=1$ and $\mu=0$ so that we recover Theorem 1.1.

Theorem 1.5. *With the same conditions as in Theorem 1.4 there holds*

$$G(M,M;((2\mu+1)K-1)/2^d,((2\mu+1)K+1)/2^d,2^{d-1}K)\geq 0,$$

for K a positive integer and d and μ given by (1.3).

For $\bar{a}=(1)$ this reduces to Theorem 1.2.

Theorem 1.6. *With the same conditions as in Theorem 1.4 there holds*

$$G(M,M;b/2^{d-1},(b+1/a)/2^{d-1},2^{d-1}a)\geq 0,$$

for a,b coprime integers such that $\mu a<b<(\mu+1)a$, and d and μ given by (1.3).

For $\bar{a}=(1)$ this reduces to Theorem 1.3. A slight reformulation of Theorem 1.6 will be given in Theorem 6.2 of section 6.

Outline of the paper. The first part of this paper deals with the theory of q-trinomial coefficients. In the next section we review the basics of q-trinomial coefficients and then extend the theory to refined q-trinomial coefficients. Our main results are Theorems 2.1 and 2.2, which are two elegant transformation formulas for refined q-trinomial coefficients that can be viewed as trinomial analogues of the Burge transform. The technical section 3 contains proofs of some of our main claims concerning refined q-trinomials.

The second part of the paper contains applications of the transformation formulas of section 2, with sections 4 and 5 devoted to proving q-trinomial identities and Rogers–Ramanujan-type identities, and sections 6 and 7 devoted to the generalized Borwein conjecture. In the appendix some simple summation formulae needed in the main text are established.

2. Refined q-Trinomial coefficients

We employ the following standard notations for the q-shifted factorial: $(a;q)_n=(a)_n=\prod_{j=1}^{n}(1-aq^{j-1})$ for $n\geq 0$, $(a;q)_n=(a)_n=1/(aq^n;q)_{-n}$ for $n\in\mathbb{Z}$ (so that $1/(q)_{-n}=0$ for $n>0$) and $(a_1,\dots,a_k;q)_n=(a_1,\dots,a_k)_n=(a_1)_n\cdots(a_k)_n$. Whenever series are nonterminating it is tacitly assumed that $|q|<1$.

In analogy with the definition of binomial coefficients, the trinomial coefficients $\binom{L}{a}_2$ are defined by the expansion

$$(1+x+x^2)^L=\sum_{a=-L}^{L}\binom{L}{a}_2 x^{a+L}. \tag{2.1}$$

Double application of the binomial expansion shows that

$$\binom{L}{a}_2=\sum_{k=0}^{L}\binom{L}{k}\binom{L-k}{k+a}. \tag{2.2}$$

The analogy with binomials breaks down when it comes to defining q-analogues. The binomial expansion is readily generalized to the q-case by [2, Eq. (3.3.6)]

$$(x)_L=\sum_{a=0}^{L}(-x)^a q^{\binom{a}{2}}\begin{bmatrix}L\\a\end{bmatrix}, \tag{2.3}$$

but no q-analogue of (2.2) seems possible that yields a q-version of (2.1). Despite this complication, Andrews and Baxter [6] successfully defined useful q-trinomial coefficients. Here we need just two of the simplest q-analogues of (2.2) given by [6, Eq. (2.7); $B=A$]

$$\begin{bmatrix} L \\ a \end{bmatrix}_{2;q} = \begin{bmatrix} L \\ a \end{bmatrix}_2 = \sum_{k=0}^{L} q^{k(k+a)} \begin{bmatrix} L \\ k \end{bmatrix} \begin{bmatrix} L-k \\ k+a \end{bmatrix} \tag{2.4}$$

and [6, Eq. (2.8)]

$$T(L,a;q) = T(L,a) = q^{\frac{1}{2}(L^2-a^2)} \begin{bmatrix} L \\ a \end{bmatrix}_{2;q^{-1}}. \tag{2.5}$$

An explicit expression for $T(L,a)$ needed later is given by [6, Eq. (2.60)]

$$T(L,a) = \sum_{\substack{n=0 \\ n+a+L \text{ even}}}^{L} q^{\frac{1}{2}n^2} \begin{bmatrix} L-n \\ \frac{1}{2}(L-a-n) \end{bmatrix} \begin{bmatrix} L \\ n \end{bmatrix}. \tag{2.6}$$

It is easy to see from (2.4) and (2.6) that the q-trinomial coefficients obey the symmetry $\left[\begin{smallmatrix} L \\ a \end{smallmatrix}\right]_2 = \left[\begin{smallmatrix} L \\ -a \end{smallmatrix}\right]_2$ and $T(L,a) = T(L,-a)$. Almost as easy to establish are the large L limits. By a limit of the q-Gauss sum [16, Eq. (II.8)],

$$\lim_{L\to\infty} \begin{bmatrix} L \\ a \end{bmatrix}_2 = \sum_{k=0}^{\infty} \frac{q^{k(k+a)}}{(q)_k (q)_{k+a}} = \frac{1}{(q)_\infty}, \tag{2.7}$$

and by Euler's q-exponential sum [16, Eq. (II.2)],

$$\lim_{\substack{L\to\infty \\ L+a+\sigma \text{ even}}} T(L,a) = \sum_{\substack{n=0 \\ n+\sigma \text{ even}}}^{\infty} \frac{q^{\frac{1}{2}n^2}}{(q)_n} = \frac{(-q^{1/2})_\infty + (-1)^\sigma (q^{1/2})_\infty}{2(q)_\infty}.$$

To conclude our brief review of q-trinomial coefficients we mention that in identities one often encounters the same linear combination of two such coefficients. For this reason it is helpful to define [4]

$$U(L,a) = T(L,a) + T(L,a+1), \tag{2.8}$$

which has a limiting behaviour somewhat simpler to that of $T(L,a)$,

$$\lim_{L\to\infty} U(L,a) = \frac{(-q^{1/2})_\infty}{(q)_\infty}. \tag{2.9}$$

In the following we go well-beyond q-trinomial coefficients, and introduce two polynomials $\mathcal{S}$ and $\mathcal{T}$ that can be viewed as four-parameter extensions of $\left[\begin{smallmatrix} L \\ a \end{smallmatrix}\right]_2$ and $T(L,a)$, respectively. Assuming that L, M, a, b are all integers we define

$$\mathcal{S}(L,M,a,b;q) = \mathcal{S}(L,M,a,b) = \sum_{k=0}^{L} q^{k(k+a)} \left[\begin{smallmatrix} L+M-a-2k \\ M \end{smallmatrix}\right] \left[\begin{smallmatrix} M-a+b \\ k \end{smallmatrix}\right] \left[\begin{smallmatrix} M+a-b \\ k+a \end{smallmatrix}\right] \tag{2.10}$$

and

$$\begin{aligned} \mathcal{T}(L,M,a,b;q) &= \mathcal{T}(L,M,a,b) \\ &= \sum_{\substack{n=0 \\ n+a+L \text{ even}}}^{L} q^{\frac{1}{2}n^2} \left[\begin{smallmatrix} M \\ n \end{smallmatrix}\right] \left[\begin{smallmatrix} M+b+(L-a-n)/2 \\ M+b \end{smallmatrix}\right] \left[\begin{smallmatrix} M-b+(L+a-n)/2 \\ M-b \end{smallmatrix}\right]. \end{aligned} \tag{2.11}$$

Comparison with (2.4) and (2.6) shows that

$$(q)_L \lim_{M\to\infty} \mathcal{S}(L,M,a,b) = \begin{bmatrix} L \\ a \end{bmatrix}_2 \tag{2.12}$$

and

$$(q)_L \lim_{M\to\infty} \mathcal{T}(L,M,a,b) = T(L,a). \tag{2.13}$$

Before we list the most important properties of $\mathcal{S}$ and $\mathcal{T}$ let us remark that the polynomial $\mathcal{T}$ was recently introduced in [30]. Following the terminology of [30] we will call $\mathcal{T}(L,M,a,b)$ a refined q-trinomial coefficient (for reasons that will become clear shortly, and *not* because of (2.13)).

The first three properties of $\mathcal{S}$ and $\mathcal{T}$ listed below follow directly from the definitions. With $\mathcal{Q}$ to mean either $\mathcal{S}$ or $\mathcal{T}$ we have the range of support

$$\mathcal{S}(L,M,a,b) \neq 0 \ \text{ iff } \ |a| \le L,\ |b| \le M \text{ and } |a-b| \le M \tag{2.14a}$$

$$\mathcal{T}(L,M,a,b) \neq 0 \ \text{ iff } \ |a| \le L,\ |b| \le M \text{ and } \frac{1}{2}(a+L) \in \mathbb{Z} \text{ if } M=0, \tag{2.14b}$$

the symmetry

$$\mathcal{Q}(L,M,a,b) = \mathcal{Q}(L,M,-a,-b) \tag{2.15}$$

and the duality

$$\mathcal{Q}(L,M,a,b;1/q) = q^{ab-LM}\mathcal{Q}(L,M,a,b;q). \tag{2.16}$$

Whereas (2.14) and (2.15) are the obvious generalizations of analogous properties of q-trinomial coefficients, (2.16) is in clear contrast with (2.5). The next result, to be compared with (2.7), will be important when we address the generalized Borwein conjecture;

$$(q)_M \lim_{L\to\infty} \mathcal{S}(L,M,a,b) = \sum_{k=0}^{\infty} q^{k(k+a)} \begin{bmatrix} M-a+b \\ k \end{bmatrix} \begin{bmatrix} M+a-b \\ k+a \end{bmatrix} = \begin{bmatrix} 2M \\ M-b \end{bmatrix}. \tag{2.17}$$

Here the second equality follows from the q-Chu–Vandermonde sum [16, Eq. (II.7)]

$$\sum_{k=0}^{n} \frac{(a,q^{-n})_k}{(q,c)_k}\Big(\frac{cq^n}{a}\Big)^k = \frac{(c/a)_n}{(c)_n}, \tag{2.18}$$

with $a \to q^{-(M-a+b)}$, $n \to M-b$ and $c \to q^{a+1}$.

We now come to the main results of this section.

Theorem 2.1. *For $L,M,a,b \in \mathbb{Z}$ such that $ab \ge 0$*

$$\sum_{i=0}^{M} q^{\frac{1}{2}i^2} \begin{bmatrix} L+M-i \\ L \end{bmatrix} \mathcal{T}(L-i,i,a,b) = q^{\frac{1}{2}b^2}\mathcal{T}(L,M,a+b,b). \tag{2.19}$$

This transformation was announced in [30, Thm. 3.1]. Our next transform has not appeared before.

Theorem 2.2. *For $L,M,a,b \in \mathbb{Z}$ such that $ab \ge 0$, and such that $|a| \le M$ if $|b| \le M$ and $|a+b| \le L$,*

$$\sum_{i=0}^{M} q^{\frac{1}{2}i^2} \begin{bmatrix} L+M-i \\ L \end{bmatrix} \mathcal{T}(i,L-i,b,a) = q^{\frac{1}{2}b^2}\mathcal{S}(L,M,a+b,b). \tag{2.20}$$

A discussion of the conditions imposed on the parameters (which are *not* sharp) precedes the proofs given in sections 3.1 and 3.2.

Theorems 2.1 and 2.2 justify calling $\mathcal{T}(L,M,a,b)$ a refined q-trinomial coefficient because they imply

$$\sum_{i=0}^{L} q^{\frac{1}{2}(i^2-b^2)}\mathcal{T}(L-i,i,a-b,b)=T(L,a)$$

and

$$\sum_{i=0}^{L} q^{\frac{1}{2}(i^2-b^2)}\mathcal{T}(i,L-i,b,a-b)=\begin{bmatrix}L\\a\end{bmatrix}_2.$$

Here it is assumed in both formulae that $0\le b\le a$ or $a\le b\le 0$. The first equation follows by taking the large M limit in (2.19) using (2.13). The second equation follows from the first by application of (2.5) and (2.16) or from (2.20) by taking M to infinity and using (2.12).

Given the above two theorems, an obvious question is whether there also exist transformations from $\mathcal{S}$ to $\mathcal{T}$ or from $\mathcal{S}$ to $\mathcal{S}$. The only result we found in this direction is the following not-so-useful summation.

Lemma 2.1. *For $L,M,a,b\in\mathbb{Z}$ such that $L\le M$ and $|a-b|\le L$ if $|a|\le L$, $|b|\le M$ and $|a-b|\le\max\{L,M\}$,*

$$\sum_{i=0}^{M-L} q^{Li}\begin{bmatrix}M-L\\i\end{bmatrix}\mathcal{S}(L-i,M-i,a,b)=\mathcal{S}(M,L,b,a).$$

Since we will not use this transformation we omit its proof. We note however that for $a=b=0$ it coincides with Theorem 2.3 below (with $a=b=0$ and L and M interchanged), and the proof for more general a and b is a simple modification of the proof of that theorem as given in section 3.3.

Before we can state our next two results we first need to define

$$\mathcal{B}(L,M,a,b;q)=\mathcal{B}(L,M,a,b)=\begin{bmatrix}M+b+L-a\\M+b\end{bmatrix}\begin{bmatrix}M-b+L+a\\M-b\end{bmatrix}, \tag{2.21}$$

for $L,M,a,b\in\mathbb{Z}/2$ such that $L+a$ and $M+b$ are integers. Note that $\mathcal{B}(L,M,a,b)$ is nonzero for $|b|\le M$ and $|a|\le L$ only.

Lemma 2.2. *For $L,M,a,b\in\mathbb{Z}$*

$$\sum_{\substack{n=0\\ n+L+a\text{ even}}}^{M} q^{\frac{1}{2}n^2}\begin{bmatrix}M\\n\end{bmatrix}\mathcal{B}((L-n)/2,M,a/2,b)=\mathcal{T}(L,M,a,b). \tag{2.22}$$

Proof. Substituting the definitions of $\mathcal{T}$ and $\mathcal{B}$ gives the desired result. □

Theorem 2.3. *For $L,M,a,b\in\mathbb{Z}$ such that $M\le L$ and $|a-b|\le M$ if $|a|\le M/2$, $|a+b|\le L$ and $|a-b|\le\max\{L,M\}$,*

$$\sum_{i=0}^{L-M}\sum_{k=0}^{L} q^{Mi+k^2}\begin{bmatrix}L-M\\i\end{bmatrix}\begin{bmatrix}L+M-2i-2k\\L-i\end{bmatrix}\mathcal{B}(k,L-i-k,a,b)=q^{a^2}\mathcal{S}(L,M,a+b,2a). \tag{2.23}$$

The conditions imposed on the above summation formula are sharp. Their origin will be discussed in the proof given in section 3.3.

Next we derive two corollaries of Theorems 2.2 and 2.3. First, taking (2.20), inserting the definition of $\mathcal{T}$ and letting L tend to infinity using (2.17) yields

$$\sum_{i=0}^{M} \sum_{\substack{n=0 \\ n+b+i \text{ even}}}^{i} q^{\frac{1}{2}(i^2+n^2)} \begin{bmatrix} M \\ i \end{bmatrix} \begin{bmatrix} i-n \\ \frac{1}{2}(i-b-n) \end{bmatrix} \begin{bmatrix} i \\ n \end{bmatrix} = q^{\frac{1}{2}b^2} \begin{bmatrix} 2M \\ M-b \end{bmatrix}.$$

Replacing b by $2a$ and n by $i-2k$ gives rise to the following result.

Corollary 2.1. *For* $M, a \in \mathbb{Z}$

$$\sum_{k=0}^{\infty} C_{M,k}(q) \begin{bmatrix} 2k \\ k-a \end{bmatrix} = q^{2a^2} \begin{bmatrix} 2M \\ M-2a \end{bmatrix},$$

where

$$C_{M,k}(q) = \sum_{i=0}^{M} q^{(i-k)^2+k^2} \begin{bmatrix} M \\ i \end{bmatrix} \begin{bmatrix} i \\ 2k \end{bmatrix} \geq 0.$$

Taking (2.23), inserting the definition of $\mathcal{B}$ and sending L to infinity using (2.17) yields a very similar result.

Corollary 2.2. *For* $M, a \in \mathbb{Z}$

$$\sum_{k=0}^{\infty} \bar{C}_{M,k}(q) \begin{bmatrix} 2k \\ k-a \end{bmatrix} = q^{a^2} \begin{bmatrix} 2M \\ M-2a \end{bmatrix},$$

where

$$\bar{C}_{M,k}(q) = \sum_{i=0}^{M} q^{M(M-i)+k^2} \begin{bmatrix} M \\ i \end{bmatrix} \begin{bmatrix} i \\ 2k \end{bmatrix} \geq 0.$$

This can also be obtained from Corollary 2.1 by the substitution $q \mapsto 1/q$.

We conclude our discussion of refined q-trinomial coefficients by introducing the refined version of the polynomial U of equation (2.8) and another polynomial frequently needed;

$$\mathcal{U}(L,M,a,b) = \mathcal{T}(L,M,a,b) + \mathcal{T}(L,M,a+1,b) \tag{2.24}$$

$$\mathcal{V}(L,M,a,b) = \mathcal{S}(L,M,a,b) + q^{b+1/2}\mathcal{S}(L,M,a+1,b+1). \tag{2.25}$$

The following limits of $\mathcal{U}$ and $\mathcal{V}$ will be useful later

$$\lim_{L\to\infty} \mathcal{U}(L,M,a,b) = \frac{(-q^{1/2})_M}{(q)_{2M}} \begin{bmatrix} 2M \\ M-b \end{bmatrix} \tag{2.26}$$

and

$$\lim_{L,M\to\infty} \mathcal{U}(L,M,a,b) = \frac{(-q^{1/2})_\infty}{(q)_\infty^2} \tag{2.27}$$

$$\lim_{L,M\to\infty} \mathcal{V}(L,M,a,b) = \frac{1+q^{b+1/2}}{(q)_\infty^2}. \tag{2.28}$$

The first limit follows from the definitions of $\mathcal{U}$ and $\mathcal{T}$ and the q-binomial theorem (2.3) with $x = -q^{1/2}$. The second limit is obvious from the first, and the last limit follows from (2.7) and (2.12).

We finally compare some of our results for refined q-trinomial coefficients with known results for the polynomial $\mathcal{B}$ of equation (2.21). First we note that $\mathcal{B}$ obeys

$$\begin{aligned} \mathcal{B}(L,M,-a,-b)&=\mathcal{B}(L,M,a,b)\\ \mathcal{B}(L,M,a,b)&=\mathcal{B}(M,L,b,a)\\ \mathcal{B}(L,M,a,b;1/q)&=q^{2ab-2LM}\mathcal{B}(L,M,a,b;q). \end{aligned} \tag{2.29}$$

The first and last of these relations are similar to (2.15) and (2.16) satisfied by $\mathcal{S}$ and $\mathcal{T}$. Surprisingly, the analogy goes much further, and the following two theorems are clear analogues of Theorems 2.1 and 2.2.

Theorem 2.4. *For $L,M,a,b\in\mathbb{Z}$ such that $|a-b|\le L$ if $|b|\le M$ and $|a+b|\le L$,*

$$\sum_{i=0}^{M} q^{i^2}\begin{bmatrix}2L+M-i\\2L\end{bmatrix}\mathcal{B}(L-i,i,a,b)=q^{b^2}\mathcal{B}(L,M,a+b,b).$$

Theorem 2.5. *With the same conditions as above*

$$\sum_{i=0}^{M} q^{i^2}\begin{bmatrix}2L+M-i\\2L\end{bmatrix}\mathcal{B}(i,L-i,b,a)=q^{b^2}\mathcal{B}(L,M,a+b,b).$$

These two theorems are known as the Burge transform, see e.g., [8, 13, 15, 24, 31]. A proof follows from the q-Pfaff–Saalschütz sum (A.1). Unlike (2.19) and (2.20), which are independent transformations, the above transformations imply one another thanks to the symmetry (2.29). Another more important difference between the Burge transform and the transformations for $\mathcal{T}$ and $\mathcal{S}$ is that the two Burge transformations can be iterated to yield a binary tree of transformations [13, 15, 31], while the transformations for $\mathcal{T}$ and $\mathcal{S}$ only give rise to an infinite double chain.

Later in the paper we also need the following extension of Theorem 2.4 involving the polynomial

$$\mathcal{B}_{r,s}(L,M,a,b)=\begin{bmatrix}M+b+s+L-a+r\\M+b+s\end{bmatrix}\begin{bmatrix}M-b+L+a\\M-b\end{bmatrix}. \tag{2.30}$$

Theorem 2.6 ([13, 24])**.** *For $L,M,a,b,r,s\in\mathbb{Z}$ such that $-L\le a-b-s\le L+r$ if $-M-s\le b\le M$ and $-L\le a+b\le L+r$,*

$$\sum_{i=b}^{M} q^{i(i+s)}\begin{bmatrix}2L+M+r-i\\2L+r\end{bmatrix}\mathcal{B}_{r+s,s}(L-i-s,i,a,b)=q^{b(b+s)}\mathcal{B}_{r,s}(L,M,a+b,b).$$

3. Proof of Theorems 2.1, 2.2 and 2.3

3.1. **Proof of Theorem 2.1.** Before we commence with the proof a few comments are in order. From equation (2.14b) it follows that the summand on the left is zero if not $|b|\le i\le\min\{L-|a|,M\}$. This implies that the left-hand side vanishes trivially if $|b|>M$ or $|a|+|b|>L$. By the same equation (2.14b) the right-hand side vanishes if $|b|>M$ or $|a+b|>L$. One might thus hope that a sufficient condition for Theorem 2.1 to hold nontrivially would be $|a|+|b|\le L$ and $|b|\le M$. However this appears not to be the case and $|a|+|b|\le L$ needs to be replaced by (i) $|a|+|b|\le L$ with $ab\ge 0$, or (ii) $|a|+2|b|\le L$ with $ab\le 0$. Since in all interesting applications of the theorem it turns out that a and b have the same signature, we have omitted the cases where a and b have opposite sign in the statement of the theorem and in the proof given below. However, in [24, Thm. 1.2] a very general transformation formula is proven which for $N=2$, $M\to M-b$, $\ell\to 2b$, $L_1\to (L-a+2b)/2$ and $L_2\to (L+a-2b)$ coincides with Theorem 2.1. The condition $L_1,L_2\geq 0$ as given in [24] establishes the validity of (2.19) for $|a-2b|\leq L$.

Proof of Theorem 2.1. Substituting the definition of $\mathcal{T}$ in (2.19) we are to prove

$$\begin{aligned}(3.1)\qquad \sum_{i,n} q^{\frac12(i^2+n^2)} \begin{bmatrix} L+M-i\\ L\end{bmatrix}\begin{bmatrix} i\\ n\end{bmatrix}\begin{bmatrix} i+b+(L-i-a-n)/2\\ i+b\end{bmatrix}\begin{bmatrix} i-b+(L-i+a-n)/2\\ i-b\end{bmatrix}\\ = q^{\frac12 b^2}\sum_n q^{\frac12 n^2}\begin{bmatrix} M\\ n\end{bmatrix}\begin{bmatrix} M+b+(L-a-b-n)/2\\ M+b\end{bmatrix}\begin{bmatrix} M-b+(L+a+b-n)/2\\ M-b\end{bmatrix},\end{aligned}$$

with a,b in the ranges specified by the theorem, and where we assume that $L+i+n+b$ is even on the left and $L+n+a+b$ is even on the right.

Since both sides of (3.1) are symmetric under simultaneous negation of a and b we may without loss of generality assume that $a,b\geq 0$ in the following. In view of the previous discussion we may also assume that $b\leq M$ and $L\geq 0$. As a first step we use the symmetry

$$(3.2)\qquad \begin{bmatrix} m+n\\ m\end{bmatrix}=\begin{bmatrix} m+n\\ n\end{bmatrix}$$

to rewrite (3.1) in the less-symmetric form

$$\begin{aligned}(3.3)\qquad \sum_{i,n} q^{\frac12(i^2+n^2)} \begin{bmatrix} L+M-i\\ L\end{bmatrix}\begin{bmatrix} i\\ n\end{bmatrix}\begin{bmatrix} i+b+(L-i-a-n)/2\\ (L-i-a-n)/2\end{bmatrix}\begin{bmatrix} i-b+(L-i+a-n)/2\\ i-b\end{bmatrix}\\ = q^{\frac12 b^2}\sum_n q^{\frac12 n^2}\begin{bmatrix} M\\ n\end{bmatrix}\begin{bmatrix} M+b+(L-a-b-n)/2\\ (L-a-b-n)/2\end{bmatrix}\begin{bmatrix} M-b+(L+a+b-n)/2\\ (L+a+b-n)/2\end{bmatrix}.\end{aligned}$$

According to the definition (1.1) of the q-binomial coefficient, $\begin{bmatrix} m+n\\ m\end{bmatrix}$ is zero if $m<0$. We will now show that in the case of equation (3.3) this condition together with $a,b\geq 0$, $b\leq M$ and $L\geq 0$ implies that all of the top-entries of the various q-binomials are nonnegative. First consider the left side. The summand vanishes if not both $L-i-a-n$ and $i-b$ are nonnegative. This gives the following inequalities for the top-entries of the four q-binomials: $L+M-i\geq L+M-b\geq 0$, $i\geq b\geq 0$, $i+b+(L-i-a-n)/2\geq 2b\geq 0$ and $i-b+(L-i+a-n)/2\geq a\geq 0$. For the right side of (3.3) it is equally simple. Since $L-a-b-n\geq 0$ one has $M\geq 0$, $M+b+(L-a-b-n)/2\geq M+b\geq 0$ and $M-b+(L+a+b-n)/2\geq M+a\geq 0$.

Now recall the following modified definition of the q-binomial coefficient:

$$(3.4)\qquad \begin{bmatrix} m+n\\ m\end{bmatrix}=\begin{cases}\dfrac{(q^{n+1})_m}{(q)_m} & \text{for } m\in\mathbb{Z}_+,\ n\in\mathbb{Z}\\ 0 & \text{otherwise.}\end{cases}$$

The only difference between (1.1) and (3.4) is that in the latter $\begin{bmatrix} m+n\\ m\end{bmatrix}$ is nonzero for $m\geq 0$ and $m+n<0$. Since we have just argued that in equation (3.3) the top-entries of the q-binomials cannot be negative by the conditions on the lower entries we may in the remainder of our proof of (3.3) assume definition (3.4).

After these preliminaries we shall transform the left side of (3.3) into the right side. First we make the simultaneous changes $i\mapsto i+n$ and $n\mapsto i$ to get

$$(3.5)\qquad \mathrm{LHS}(3.3)=\sum_{i,n} q^{\frac12 n^2+i(i+n)}\begin{bmatrix} L+M-n-i\\ L\end{bmatrix}\begin{bmatrix} i+n\\ i\end{bmatrix}\begin{bmatrix} (L-a+n)/2+b\\ (L-a-n)/2-i\end{bmatrix}\begin{bmatrix} (L+a+n)/2-b\\ i+n-b\end{bmatrix}.$$

To proceed we need Sears' ${}_4\phi_3$ transformation [16, Eq. (III.15)]

$$\sum_{k=0}^{n} \frac{(a,b,c,q^{-n})_k\, q^k}{(q,d,e,f)_k} = a^n\, \frac{(e/a,f/a)_n}{(e,f)_n} \sum_{k=0}^{n} \frac{(a,d/b,d/c,q^{-n})_k\, q^k}{(q,d,aq^{1-n}/e,aq^{1-n}/f)_k}, \tag{3.6}$$

true for $def = abcdq^{1-n}$. Making the substitutions $n \mapsto d-e$, $a \mapsto q^{-c}$, $b \mapsto q^{a+1}$, $c \mapsto q^{g-f}$, $d \mapsto q^{g+1}$, $e \mapsto q^{-b}$ and $f \mapsto q^{e+1}$ results in the following transformation for sums of products of q-binomial coefficients:

$$\begin{aligned}\sum_{i=0}^{d-e} q^{i(i-c+e+g)} \begin{bmatrix} a-i \\ a-b\end{bmatrix} \begin{bmatrix} i+c \\ i\end{bmatrix} \begin{bmatrix} d \\ d-e-i\end{bmatrix} \begin{bmatrix} f \\ i+g\end{bmatrix} \\ = \sum_{i=0}^{d-e} q^{i(i-c+e+g)} \begin{bmatrix} a-d+e \\ i-c+e+f\end{bmatrix} \begin{bmatrix} c-g \\ i\end{bmatrix} \begin{bmatrix} b+d-i \\ d-e-i\end{bmatrix} \begin{bmatrix} i+f \\ i+g\end{bmatrix},\end{aligned} \tag{3.7}$$

for $a,b,c,d,e,f,g \in \mathbb{Z}$ such that $a+c=b+d+f$. Returning to (3.5), we utilize (3.7) to transform the sum over i. As a result

$$\mathrm{LHS}(3.3) = \sum_{i,n} q^{\frac{1}{2}n^2+i(i+n)} \begin{bmatrix} M+(L+a-n)/2 \\ i+(L+a+n)/2\end{bmatrix} \begin{bmatrix} b \\ i\end{bmatrix} \begin{bmatrix} M+b+(L-a-n)/2-i \\ (L-a-n)/2-i\end{bmatrix} \begin{bmatrix} i-b+(L+a+n)/2 \\ i+n-b\end{bmatrix}.$$

By the simultaneous changes $i \mapsto n-i$ and $n \to 2i-n+b$ this yields

$$\begin{aligned}\mathrm{LHS}(3.3) = q^{\frac{1}{2}b^2} \sum_n q^{\frac{1}{2}n^2} \begin{bmatrix} M+b+(L-a-b-n)/2 \\ (L-a-b-n)/2\end{bmatrix} \\ \times \sum_i q^{i(i-n+b)} \begin{bmatrix} M+(L+a-b+n)/2-i \\ (L+a+b+n)/2\end{bmatrix} \begin{bmatrix} b \\ n-i\end{bmatrix} \begin{bmatrix} (L+a-b+n)/2 \\ i\end{bmatrix}.\end{aligned}$$

Now set $c=g=0$ in (3.7) and eliminate b. This yields the well-known polynomial version of the q-Pfaff–Saalschütz theorem [2, 14, 18]

$$\sum_{i=0}^{d-e} q^{i(i+e)} \begin{bmatrix} a-i \\ d+f\end{bmatrix} \begin{bmatrix} d \\ d-e-i\end{bmatrix} \begin{bmatrix} f \\ i\end{bmatrix} = \begin{bmatrix} a-d+e \\ e+f\end{bmatrix} \begin{bmatrix} a-f \\ d-e\end{bmatrix}, \tag{3.8}$$

which, because of definition (3.4), is valid for all $a,d,e,f \in \mathbb{Z}$. Using (3.8) to carry out the sum over i results in the right-hand side of (3.3). □

3.2. **Proof of Theorem 2.2.** The first part of the discussion at the start of section 3.1 also applies here. That is, the content of Theorem 2.2 is nontrivial for $|b| \le M$ and $|a|+|b| \le L$ only. It is also true again that the bounds are not sharp, since for some a and b of opposite sign, such that $|a|+|b| \le L$, the theorem holds as well. The extra condition (compared to Theorem 2.1) $|a| \le M$ when both $|b| \le M$ and $|a|+|b| \le L$ does appear to be sharp.

Proof. Without loss of generality we may assume $a,b \ge 0$, $b \le M$ and $L \ge 0$. Of course we also have $a \le M$ as a necessary condition. Substituting the definitions of $\mathcal{T}$ and $\mathcal{S}$ in the above identity and using (3.2) to asymmetrize, we are to prove

$$\begin{aligned}\sum_{i,n} q^{\frac{1}{2}(i^2+n^2)} \begin{bmatrix} L+M-i \\ L\end{bmatrix} \begin{bmatrix} L-i \\ n\end{bmatrix} \begin{bmatrix} L-i+a+(i-b-n)/2 \\ (i-b-n)/2\end{bmatrix} \begin{bmatrix} L-i-a+(i+b-n)/2 \\ L-i-a\end{bmatrix} \\ = q^{\frac{1}{2}b^2} \sum_k q^{k(k+a+b)} \begin{bmatrix} L+M-a-b-2k \\ L-a-b-2k\end{bmatrix} \begin{bmatrix} M-a \\ k\end{bmatrix} \begin{bmatrix} M+a \\ k+a+b\end{bmatrix},\end{aligned} \tag{3.9}$$

where the parity rule that $i+n+b$ on the left must be even is implicit.

As in the proof of Theorem 2.1 we will now show that the top-entries of all seven q-binomial coefficients are nonnegative by the conditions on the parameters and by the condition that the lower entries are nonnegative. This allows us to again assume the modified definition (3.4) of the q-binomials in our proof.

First consider the left-hand side. Because $(i-b-n)/2 \geq 0$ and $L-i-a \geq 0$ we get $L+M-i \geq M+a \geq 0$, $L-i \geq a \geq 0$, $L-i+a+(i-b-n)/2 \geq 2a \geq 0$ and $L-i-a+(i+b-n)/2 \geq b$. On the right-hand side, since $L-a-b-2k \geq 0$, we get $L+M-a-b-2k \geq M$ for the top-entry of the first q-binomial. It is the above-discussed extra condition $a \leq M$ that ensures that also the second q-binomial on the right has a nonnegative top-entry. Since clearly also $M+a$ is nonnegative we are indeed in a position to assume (3.4) in the remainder of the proof.

We begin with the simultaneous changes $i \mapsto i+k+b$ and $n \mapsto k-i$ to find

$$\mathrm{LHS}(3.9) = \sum_{i,k} q^{\frac{1}{2}b^2+k(k+b)+i(i+b)} \begin{bmatrix} L+M-b-k-i \\ L \end{bmatrix} \begin{bmatrix} L-b-k-i \\ k-i \end{bmatrix} \begin{bmatrix} L+a-b-k \\ i \end{bmatrix} \begin{bmatrix} L-a-k \\ L-a-b-k-i \end{bmatrix}.$$

Next we apply the following transformation formula similar to (3.7), which can again be viewed as a corollary of the Sears transform (3.6):

$$\sum_{i=0}^{d} q^{i(i+f-g)} \begin{bmatrix} a-i \\ b \end{bmatrix} \begin{bmatrix} c-i \\ d-i \end{bmatrix} \begin{bmatrix} e \\ i \end{bmatrix} \begin{bmatrix} f \\ g-i \end{bmatrix} = \sum_{i=0}^{d} q^{i(i+c-d-g)} \begin{bmatrix} a-i \\ b-i \end{bmatrix} \begin{bmatrix} c-g \\ d-i \end{bmatrix} \begin{bmatrix} a-e \\ i \end{bmatrix} \begin{bmatrix} d+f-i \\ g-i \end{bmatrix}, \tag{3.10}$$

for $a,b,c,d,e,f,g \in \mathbb{Z}$ such that $b+c=d+e+f$. The proof of this requires the substitutions $n \mapsto d$, $a \mapsto q^{-g}$, $b \mapsto q^{-b}$, $c \mapsto q^{e-a}$, $d \mapsto q^{-a}$, $e \mapsto q^{-d-f}$ and $f \mapsto q^{c-g-d+1}$ in (3.6) and some simplifications. As a consequence of (3.10)

$$\mathrm{LHS}(3.9) = \sum_{i,k} q^{\frac{1}{2}b^2+k(k+b)+i(i+a-k)} \begin{bmatrix} L+M-b-k-i \\ L-i \end{bmatrix} \begin{bmatrix} a \\ k-i \end{bmatrix} \begin{bmatrix} M-a \\ i \end{bmatrix} \begin{bmatrix} L-a-i \\ L-a-b-k-i \end{bmatrix}.$$

Shifting $k \to k+i$ and then renaming i as k and k as i yields

$$\mathrm{LHS}(3.9) = q^{\frac{1}{2}b^2} \sum_{k} q^{k(k+a+b)} \begin{bmatrix} M-a \\ k \end{bmatrix} \sum_{i} q^{i(i+b+k)} \begin{bmatrix} L+M-b-2k-i \\ L-k \end{bmatrix} \begin{bmatrix} a \\ i \end{bmatrix} \begin{bmatrix} L-a-k \\ L-a-b-2k-i \end{bmatrix}.$$

Finally, summing over i using (3.8) leads to the right side of (3.9). □

3.3. **Proof of Theorem 2.3.** As with the Theorems 2.1 and 2.2, we first discuss the conditions imposed on the parameters.

On the left the summand vanishes unless $0 \leq i \leq \min\{L-M, L\}$, $i+2k \leq M$, $|b| \leq L-i-k$ and $|a| \leq k$. Hence the left-hand side is nonzero if and only if $L \geq M$, $|a| \leq M/2$ and $|a|+|b| \leq L$. By (2.14a) the right-hand side is nonzero if and only if $|a+b| \leq L$, $|a| \leq M/2$ and $|a-b| \leq M$. Because $|a|+|b| \leq L$ is equivalent to $|a+b| \leq L$ and $|a-b| \leq L$, both sides trivially vanish if any of the following three conditions is violated: $|a| \leq M/2$, $|a+b| \leq L$ and $|a-b| \leq \max\{L, M\}$. If these conditions are however satisfied, the mismatch between right and left side needs to be repaired by imposing that $M \leq L$ and $|a-b| \leq M$. (Note that $|a-b| \leq M$ implies $|a-b| \leq L$ when $M \leq L$.)

Proof of Theorem 2.3. After inserting the definitions (2.10) and (2.21) of $\mathcal{S}$ and $\mathcal{B}$, we use the symmetry (3.2) of the q-binomial coefficients to get

$$\begin{aligned}(3.11)\qquad \sum_{i,k} q^{Mi+k^2} \begin{bmatrix} L-M \\ L-M-i \end{bmatrix} \begin{bmatrix} L+M-2i-2k \\ M-i-2k \end{bmatrix} \begin{bmatrix} L-i-a+b \\ k-a \end{bmatrix} \begin{bmatrix} L-i+a-b \\ L-i-k-b \end{bmatrix} \\ = q^{a^2} \sum_k q^{k(k+a+b)} \begin{bmatrix} L+M-a-b-2k \\ L-a-b-2k \end{bmatrix} \begin{bmatrix} M+a-b \\ k \end{bmatrix} \begin{bmatrix} M-a+b \\ M-2a-k \end{bmatrix}.\end{aligned}$$

In view of the above discussion we may without loss of generality assume that $M \le L$ and $|a-b| \le M$. This, together with the condition that the lower entries of all seven q-binomial coefficients are nonnegative, implies that all the top-entries are nonnegative. Specifically, on the left we have $L-M \ge 0$, $L+M-2i-2k = (M-i-2k)+(L-M-i)+M \ge 0$, $L-i-a+b = (L-M-i)+(M-a+b) \ge 0$, $L-i+a-b = (L-M-i)+(M+a-b) \ge 0$. Similarly, on the right we have $L+M-a-b-2k = (L-a-b-2k)+M \ge 0$, $M-a+b \ge 0$, $M-a+b \ge 0$. Consequently we can again assume definition (3.4) in the proof of (3.11).

By the simultaneous changes $i \mapsto i+2k-M$ and $k \mapsto M-i-k$

$$\begin{aligned}\mathrm{LHS}(3.11) = \sum_{i,k} q^{k^2+i(i+2k-M)} \\ \times \begin{bmatrix} L-M \\ L-i-2k \end{bmatrix} \begin{bmatrix} L+M-2k \\ i \end{bmatrix} \begin{bmatrix} L+M-i-2k-a+b \\ M-i-k-a \end{bmatrix} \begin{bmatrix} L+M-i-2k+a-b \\ L-k-b \end{bmatrix}.\end{aligned}$$

Transforming the sum over i by (3.10) leads to

$$\mathrm{LHS}(3.11) = \sum_{i,k} q^{k^2+i(i+k+b)} \begin{bmatrix} L+M-i-2k+a-b \\ L-i-k-b \end{bmatrix} \begin{bmatrix} M-a+b \\ M-i-k-a \end{bmatrix} \begin{bmatrix} a-b \\ i \end{bmatrix} \begin{bmatrix} L-i-k-a \\ L-i-2k \end{bmatrix},$$

which, by the simultaneous change $k \mapsto i+a$ and $i \mapsto k-i$, becomes

$$\begin{aligned}\mathrm{LHS}(3.11) = q^{a^2} \sum_k q^{k(k+a+b)} \begin{bmatrix} M-a+b \\ M-k-2a \end{bmatrix} \\ \times \sum_i q^{i(i-k+a-b)} \begin{bmatrix} L+M-i-k-a-b \\ L-k-a-b \end{bmatrix} \begin{bmatrix} a-b \\ k-i \end{bmatrix} \begin{bmatrix} L-k-2a \\ L-i-k-2a \end{bmatrix}.\end{aligned}$$

An important difference between the definitions (1.1) and (3.4) of the q-binomial coefficients is that only the former satisfies the symmetry (3.2). However, the modified q-binomials do satisfy this symmetry provided $m+n \ge 0$. Now the above summand vanishes if $M-k-2a < 0$. Hence in the sum over i we may assume that $M-k-2a \ge 0$. This implies that $L-k-2a = (L-M)+(M-k-2a) \ge 0$, so that we may replace $\begin{bmatrix} L-k-2a \\ L-i-k-2a \end{bmatrix}$ by $\begin{bmatrix} L-k-2a \\ i \end{bmatrix}$. Then the sum over i can be performed by (3.8) resulting in the right-hand side of (3.11). $\square$

4. Two simple examples

4.1. **First example.** In our first application of the transformations (2.19) and (2.20) we start with the simplest possible identity for refined q-trinomials.

Lemma 4.1. *For $L, M \in \mathbb{Z}_+$*

$$(4.1)\qquad \sum_{j=-\infty}^{\infty} q^{j(j+1)} \{\mathcal{T}(L,M,2j,j) - \mathcal{T}(L,M,2j+2,j)\} = \delta_{L,0}\delta_{M,0}.$$

Proof. We begin with [15, Lem. 3.1]

$$\sum_{j=-\infty}^{\infty}\sum_{\tau=0}^{1}(-1)^{\tau}q^{j(j+1)}\mathcal{B}(L,M,j+\tau,j)=\delta_{L,0}\delta_{M,0}.$$

By (2.22) this implies

$$\sum_{j=-\infty}^{\infty}\sum_{\tau=0}^{1}(-1)^{\tau}q^{j(j+1)}\mathcal{T}(L,M,2j+2\tau,j)=\sum_{\substack{n=0\\ n+L\text{ even}}}^{M}q^{\frac12 n^2}\begin{bmatrix}M\\n\end{bmatrix}\delta_{L,n}\delta_{M,0}=\delta_{L,0}\delta_{M,0}.\qquad\square$$

A single application of the transformation of Theorem 2.1 yields

$$\sum_{j=-\infty}^{\infty}q^{\frac12 j(3j+2)}\{\mathcal{T}(L,M,3j,j)-\mathcal{T}(L,M,3j+2,j)\}=\delta_{L,0}.$$

Further iterating this, using (2.19) and a simple induction argument, shows that

$$\sum_{j=-\infty}^{\infty}q^{\frac12 j((k+3)j+2)}\{\mathcal{T}(L,M,(k+3)j,j)-\mathcal{T}(L,M,(k+3)j+2,j)\}$$
$$=\sum_{r_1,\dots,r_{k-1}\ge 0}q^{\frac12(r_1^2+\cdots+r_k^2)}\prod_{i=0}^{k-1}\begin{bmatrix}L-r_{i+1}-\sum_{j=1}^{i-1}r_j\\ r_i-r_{i+1}\end{bmatrix},$$

where $r_0:=M$, $r_k:=L-r_1-\cdots-r_{k-1}$ and $k\ge 1$. Here and in the rest of the paper we adopt the convention that $-\sum_{j=m}^{n-1}r_j=\sum_{j=n}^{m-1}r_j$ for $n<m$, so that

$$\begin{bmatrix}L-r_1-\sum_{j=1}^{-1}r_j\\ r_0-r_1\end{bmatrix}=\begin{bmatrix}L+r_0-r_1\\ r_0-r_1\end{bmatrix}=\begin{bmatrix}L+M-r_1\\ M-r_1\end{bmatrix}.$$

We cannot turn the above polynomial identity into a nontrivial q-series result because the large L limit gives zero on either side. What we can do is apply the transformation of Theorem 2.2. Note in particular that the condition $|a|\le M$ if $|b|\le M$ and $|a+b|\le L$ does not pose a problem since $|(k+3)j|\ge |j|$ and $|(k+3)j+2|\ge|j|$ for $k\ge 0$. By (2.20) we thus find

$$\begin{aligned}(4.2)\qquad \sum_{j=-\infty}^{\infty}\Big\{&q^{\frac12 j((k+3)(k+4)j+2)}\mathcal{S}(L,M,(k+4)j,(k+3)j)\\ &-q^{\frac12((k+3)j+2)((k+4)j+2)}\mathcal{S}(L,M,(k+4)j+2,(k+3)j+2)\Big\}\\ &=\sum_{r_1,\dots,r_k\ge 0}q^{\frac12(r_1^2+\cdots+r_{k+1}^2)}\begin{bmatrix}L+M-r_1\\ L\end{bmatrix}\begin{bmatrix}L-r_2\\ r_1\end{bmatrix}\prod_{i=2}^{k}\begin{bmatrix}r_1-r_{i+1}-\sum_{j=2}^{i-1}r_j\\ r_i-r_{i+1}\end{bmatrix},\end{aligned}$$

where $r_{k+1}=r_1-r_2-\cdots-r_k$. For $k=0$ the correct expression on the right side is $\begin{bmatrix}L+M\\ L\end{bmatrix}$. It is also worth separately stating the $k=1$ case, namely

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac12 j(5j+1)}\mathcal{S}(L,M,\lfloor(5j+1)/2\rfloor,2j)=\sum_{n=0}^{M}q^{n^2}\begin{bmatrix}L+M-n\\ L\end{bmatrix}\begin{bmatrix}L-n\\ n\end{bmatrix}.$$

This result is the first of four doubly-bounded analogues of the first Rogers–Ramanujan identity that will be obtained in this paper. Using (2.17) to take the limit when L

tends to infinity we find

$$G(M,M;1,3/2,2)=\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{2}j(5j+1)}\begin{bmatrix}2M\\M-2j\end{bmatrix}=\sum_{n=0}^{M}q^{n^2}\begin{bmatrix}M\\n\end{bmatrix}, \tag{4.3}$$

a result originally due to Bressoud [10, Eq. (9)].

If instead of L we let M become large and use (2.12) we obtain a result of Andrews [3, Eq. (1.11)]

$$\sum_{j=-\infty}^{\infty}\left\{q^{j(10j+1)}\begin{bmatrix}L\\5j\end{bmatrix}_2-q^{(2j+1)(5j+2)}\begin{bmatrix}L\\5j+2\end{bmatrix}_2\right\}=\sum_{n=0}^{\infty}q^{n^2}\begin{bmatrix}L-n\\n\end{bmatrix}.$$

For arbitrary k equation (4.2) is a generalization of q-trinomial identities of [9, Eq. (9.4)] and [29, Prop. 4.5]. Making the simultaneous replacements $r_i\to r_{i-1}-r_i$ $(i=2,\dots,k)$ in (4.2) and letting L and M tend to infinity gives identities for Virasoro characters of [29, Cor. IV.1].

Before we come to our next example let us point out that the above discussion can be repeated for the second Rogers–Ramanujan identity. In particular one can show by a generalization of (2.20) that

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{2}j(5j+3)}\mathcal{S}(L,M,\lfloor 5j/2\rfloor+1,2j+1)=\sum_{n=0}^{M-1}q^{n(n+1)}\begin{bmatrix}L+M-n-1\\L\end{bmatrix}\begin{bmatrix}L-n-1\\n\end{bmatrix}.$$

4.2. **Second example.** Our next example uses a slightly more complicated-to-prove identity as starting point.

Lemma 4.2. *For $L,M\in\mathbb{Z}_+$*

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{2}j(j+1)}\mathcal{U}(L,M,j,j)=\delta_{M,0}. \tag{4.4}$$

Proof. Let $\sigma\in\{0,1\}$. Then for $L-\sigma/2\in\mathbb{Z}_+$ there holds

$$\sum_{j=-\infty}^{\infty}\sum_{\tau=0}^{1}(-1)^\tau q^{j(2j+1)}\mathcal{B}(L,M,j+\sigma/2,2j+\tau)=\delta_{M,0}. \tag{4.5}$$

For $\sigma=0$ this is [15, Cor. 3.2]. For $\sigma=1$ we replace L by $L+1/2$ so that we are to show that

$$\sum_{j=-\infty}^{\infty}\sum_{\tau=0}^{1}(-1)^\tau q^{j(2j+1)}\begin{bmatrix}L+M+j+\tau\\M+2j+\tau\end{bmatrix}\begin{bmatrix}L+M-j-\tau+1\\M-2j-\tau\end{bmatrix}=\delta_{M,0}.$$

By the q-binomial recurrence

$$\begin{bmatrix}m+n\\m\end{bmatrix}=\begin{bmatrix}m+n-1\\m\end{bmatrix}+q^n\begin{bmatrix}m+n-1\\m-1\end{bmatrix} \tag{4.6}$$

the left side can be expanded as

$$\sum_{j=-\infty}^{\infty}\sum_{\tau=0}^{1}(-1)^\tau q^{j(2j+1)}\begin{bmatrix}L+M+j+\tau\\M+2j+\tau\end{bmatrix}\left\{\begin{bmatrix}L+M-j-\tau\\M-2j-\tau\end{bmatrix}+q^{L+j+1}\begin{bmatrix}L+M-j-\tau\\M-2j-\tau-1\end{bmatrix}\right\}.$$

The first term of the summand gives $\delta_{M,0}$ according to (4.5) with $\sigma=0$. The second term should thus vanish, which readily follows by the variable changes $j\mapsto -j-1$ and $\tau\to 1-\tau$.

If we now apply the transformation (2.22) to (4.5) we obtain

$$\sum_{j=-\infty}^{\infty}\sum_{\tau=0}^{1}(-1)^{\tau}q^{j(2j+1)}\mathcal{T}(L,M,2j+\sigma,2j+\tau)$$
$$=\sum_{\substack{n=0\\ n+\sigma+L \text{ even}}}^{M} q^{\frac{1}{2}n^2}\begin{bmatrix}M\\ n\end{bmatrix}\delta_{M,0}=\delta_{M,0}\chi(L+\sigma \text{ even}),$$

where $\chi(\text{true})=1$ and $\chi(\text{false})=0$. Summing over σ, replacing $j\to(j-\tau)/2$ and using the definition (2.24) of $\mathcal{U}$ yields (4.4). □

By application of (2.19) the identity (4.4) transforms into

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{2}j(2j+1)}\mathcal{U}(L,M,2j,j)=\begin{bmatrix}L+M\\ L\end{bmatrix}.$$

Letting L tend to infinity using (2.26) yields a q-binomial identity equivalent to item H(2) in Slater's list of Bailey pairs [27]. Next, by (2.19) and induction

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{2}j((k+1)j+1)}\mathcal{U}(L,M,(k+1)j,j)$$
$$=\sum_{r_1,\dots,r_{k-1}\geq 0} q^{\frac{1}{2}(r_1^2+\cdots+r_{k-1}^2)}\prod_{i=0}^{k-1}\begin{bmatrix}L-r_{i+1}-\sum_{j=1}^{i-1}r_j\\ r_i-r_{i+1}\end{bmatrix}, \tag{4.7}$$

for $k\geq 2$ and $r_0:=M$, $r_k:=0$. For $k=2$ and $L\to\infty$ this gives a q-binomial identity of Rogers [23] given as item B(1) in Slater's list. For $k=2,3$ and $M\to\infty$ this gives two q-trinomial identities of Andrews [4, Thm. 5.1], [3, Eq. (4.4)]. Before considering the identities arising when both L and M become large, we will show that one can easily derive a variation of (4.7) using the following lemma.

Lemma 4.3. *For $L,M\in\mathbb{Z}_+$*

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{3j(j+1)/2}\mathcal{U}(L,M,3j+1,j)=q^M\sum_{r=0}^{M}q^{\frac{1}{2}r^2}\begin{bmatrix}L+M-r-1\\ M-r\end{bmatrix}\begin{bmatrix}L-1\\ r\end{bmatrix}. \tag{4.8}$$

For $L\to\infty$ this yields the Bailey pair B(2).

Proof. Fix M and denote the left side of (4.7) for $k=2$ by f_L and the left side of (4.8) by g_L. Using

$$\begin{bmatrix}m+n\\ m\end{bmatrix}=\begin{bmatrix}m+n-1\\ m-1\end{bmatrix}+q^m\begin{bmatrix}m+n-1\\ m\end{bmatrix} \tag{4.9}$$

it then follows that

$$\begin{aligned}
f_{L-1} &= \sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{2}j(3j+1)}\sum_{\tau=0}^{1}\sum_{\substack{n=0\\ n+L+j+\tau\text{ odd}}}^{M} q^{\frac{1}{2}n^2}\begin{bmatrix}M\\n\end{bmatrix}\\
&\qquad\times\begin{bmatrix}M+(L-j-n-\tau-1)/2\\M+j\end{bmatrix}\begin{bmatrix}M+(L+j-n+\tau-1)/2\\M-j\end{bmatrix}\\
&= q^{-M}\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{2}j(3j+3)}\sum_{\tau=0}^{1}\sum_{\substack{n=0\\ n+L+j+\tau\text{ odd}}}^{M} q^{\frac{1}{2}n^2}\begin{bmatrix}M\\n\end{bmatrix}\\
&\qquad\times\begin{bmatrix}M+(L-j-n-\tau-1)/2\\M+j\end{bmatrix}\left\{\begin{bmatrix}M+(L+j-n+\tau+1)/2\\M-j\end{bmatrix}-\begin{bmatrix}M+(L+j-n+\tau-1)/2\\M-j-1\end{bmatrix}\right\}.
\end{aligned}$$

The first term in the summand on the right yields g_L and the second term vanishes by the substitutions $j\mapsto -j-1$ and $\tau\mapsto 1-\tau$. Hence $g_L=q^M f_{L-1}$. Since also the right sides of (4.7) for $k=2$ and (4.8) satisfy this equation we are done. □

We leave it to the reader to apply the transformation (2.19) to (4.8) to obtain the variation of (4.7) alluded to, but remark that a single application of (2.19) results in a generalization of the q-trinomial identity [3, Eq. (4.10)].

Next we take the large L and M limit in (4.7). By (2.27) and the triple product identity [16, Eq. (II.28)]

$$\sum_{k=-\infty}^{\infty}(-z)^k q^{\binom{k}{2}}=(z,q/z,q)_\infty \tag{4.10}$$

this yields

$$\sum_{m_1,\dots,m_{k-1}\geq 0}\frac{q^{\frac{1}{2}(M_1^2+\cdots+M_{k-1}^2)}}{(q)_{m_1}\cdots(q)_{m_{k-1}}}=\frac{(-q^{1/2})_\infty}{(q)_\infty}(q^{k/2},q^{k/2+1},q^{k+1};q^{k+1})_\infty,$$

with $M_i=m_i+\cdots+m_{k-1}$. To connect this with more familiar q-series results we make use of Lemma A.1 to reduce the number of summation variables on the left. First, for odd values of k we use the expression for $f_{2k-1}(0)$ with $a=1$ given by the lemma to find the following theorem.

Theorem 4.1. *For $k\geq 2$*

$$\begin{aligned}
\sum_{m_1,\dots,m_{2k-2}\geq 0}\frac{q^{\frac{1}{2}(M_1^2+\cdots+M_{2k-2}^2)}}{(q)_{m_1}\cdots(q)_{m_{2k-2}}}&=\sum_{n_1,\dots,n_{k-1}\geq 0}\frac{q^{N_1^2+\cdots+N_{k-1}^2}(-q^{1/2-N_1})_{N_1}}{(q)_{n_1}\cdots(q)_{n_{k-1}}}\\
&=\prod_{\substack{j=1\\ j\not\equiv 2 \pmod 4\\ j\not\equiv 0,\pm(2k-1) \pmod{4k}}}^{\infty}\frac{1}{(1-q^{j/2})}.
\end{aligned}$$

The last two expressions of this theorem constitute Andrews' generalization of the Göllnitz–Gordon identities [2, Eq. (7.4.4)]. The equality of these with the first expression was conjectured by Melzer [20] and recently proven by Bressoud, Ismail and Stanton [12, Thm. 5.1; $i=k,a=1$] using different techniques.

Next, when k is even we use the expression for $f_{2k}(0)$ with $a=1$.

Theorem 4.2. *For $k \geq 1$*

$$\sum_{m_1,\dots,m_{2k-1}\geq 0} \frac{q^{\frac{1}{2}(M_1^2+\cdots+M_{2k-1}^2)}}{(q)_{m_1}\cdots(q)_{m_{2k-1}}} = (-q^{1/2})_\infty \sum_{n_1,\dots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2}}{(q)_{n_1}\cdots(q)_{n_{k-1}}}$$
$$= (-q^{1/2})_\infty \prod_{\substack{j=1\\ j\not\equiv 0,\pm k \pmod{2k+1}}}^{\infty} \frac{1}{(1-q^j)}.$$

The equality between the second and third expression is the well-known (first) Andrews–Gordon identity [1].

Returning to (4.7) we apply the transformation (2.20), use definition (2.25) and replace k by $k-1$ to find

$$(4.11)\qquad \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{1}{2}j(k(k+1)j+1)} \mathcal{V}(L,M,(k+1)j,kj)$$
$$= \sum_{r_1,\dots,r_{k-1}\geq 0} q^{\frac{1}{2}(r_1^2+\cdots+r_{k-1}^2)} \begin{bmatrix} L+M-r_1 \\ L \end{bmatrix} \begin{bmatrix} L-r_2 \\ r_1 \end{bmatrix} \prod_{i=2}^{k-1} \begin{bmatrix} r_1-r_{i+1}-\sum_{j=2}^{i-1} r_j \\ r_i-r_{i+1} \end{bmatrix},$$

with $k \geq 2$ and $r_k := 0$. Although the right sides of (4.7) and (4.11) coincide for $k=2$, the large L limit of (4.11) for $k=2$ does not reproduce the Bailey pair B(1), but yields the pair I(3) due to Slater [27]. The large M limit of this same identity yields [4, Cor. 5.2] of Andrews. If for general k we apply (2.28) and collect even and odd powers of $q^{1/2}$ we obtain the Virasoro-character identity

$$\sum_{\substack{r_1,\dots,r_{k-1}\geq 0\\ \sigma+\sum_i r_i \text{ even}}} \frac{q^{\frac{1}{2}(r_1^2+\cdots+r_{k-1}^2-\sigma)}}{(q)_{r_1}} \prod_{i=2}^{k-1} \begin{bmatrix} r_1-r_{i+1}-\sum_{j=2}^{i-1} r_j \\ r_i-r_{i+1} \end{bmatrix} = \begin{cases} \chi^{(k,2k+2)}_{(k-1)/2,k+2\sigma}(q) & k \text{ odd} \\ \chi^{(k+1,2k)}_{k/2,k+2\sigma-1}(q) & k \text{ even,} \end{cases}$$

where $\sigma \in \{0,1\}$ and

$$\chi^{(p,p')}_{r,s}(q) = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} \Bigl\{ q^{j(pp'j+p'r-ps)} - q^{(pj+s)(p'j+r)} \Bigr\}.$$

This generalizes the identities (83) and (86) in Slater's list [28] of identities of the Rogers–Ramanujan type.

5. Two not-so-simple examples

In the two examples of the previous section the initial identities (4.1) and (4.4) were both straightforward consequences of known q-binomial identities. In this section we will give two further applications of Theorems 2.1 and 2.2 that show that not all irreducible identities for refined q-trinomials (irreducible in the sense that they do not follow by application of (2.19)) are trivial. Apart from leading to more examples of identities of the Rogers–Ramanujan type, this will result in the following remarkable pair of Virasoro-character identities:

$$\frac{1}{(q;q^2)_\infty}\Big\{\chi^{(2k-1,2k+1)}_{k,k+1}(q^2)+q^{1/2}\chi^{(2k-1,2k+1)}_{k,k}(q^2)\Big\}$$
$$=\sum_{r_1,\dots,r_{k-1}\geq 0}\frac{q^{\frac{1}{2}(r_1^2+\cdots+r_{k-1}^2)}}{(q)_{r_1}}\Big(\prod_{i=2}^{k-2}\begin{bmatrix} r_1-r_{i+1}-\sum_{j=2}^{i-1}r_j \\ r_i-r_{i+1}\end{bmatrix}\Big)\begin{bmatrix} r_1+r_{k-1}-\sum_{j=2}^{k-2}r_j \\ 2r_{k-1}\end{bmatrix}_{q^{1/2}}$$

and

$$\frac{1}{(-q^{1/2};q^{1/2})_\infty}\Big\{\chi^{(2k-1,2k+1)}_{k,k+1}(q^{1/2})+q^{1/2}\chi^{(2k-1,2k+1)}_{k,k-1}(q^{1/2})\Big\}$$
$$=\sum_{r_1,\dots,r_{k-1}\geq 0}\frac{q^{\frac{1}{2}(r_1^2+\cdots+r_{k-1}^2)}}{(q)_{r_1}}\Big(\prod_{i=2}^{k-2}\begin{bmatrix} r_1-r_{i+1}-\sum_{j=2}^{i-1}r_j \\ r_i-r_{i+1}\end{bmatrix}\Big)\begin{bmatrix} \lfloor(r_1+r_{k-1}-\sum_{j=2}^{k-2}r_j)/2\rfloor \\ r_{k-1}\end{bmatrix}_{q^2},$$

for $k\geq 3$ and $\lfloor x\rfloor$ the integer part of x. The strange similarity between these two formulas and how the roles of $q^{1/2}$ and q^2 are interchanged in going from one to the other is in our opinion quite amazing.

5.1. **A generalization of Bailey pair C(5).** In our next example we take the following identity as starting point.

Lemma 5.1. *For $L,M\in\mathbb{Z}_+$*

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j(j+1)}\mathcal{U}(L,M,j,2j)=q^{\binom{M}{2}}\begin{bmatrix} L+M+1 \\ 2M\end{bmatrix}_{q^{1/2}}. \tag{5.1}$$

Taking the large L limit using (2.26) gives

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j(j+1)}\begin{bmatrix} 2M \\ M-2j\end{bmatrix}=q^{\binom{M}{2}}(-q)_M.$$

This identity, which is equivalent to the Bailey pair C(5) of Rogers [23, 27], can be obtained as a specialization of the nonterminating q-Dougall sum [16, Eq. (II.20)]. This foreshadows that (5.1) will not be as easy to prove as our previous examples.

Proof. By (1.1) and (2.14b) the lemma is trivially true for $M=0$, and in the following we may assume $M\geq 1$. We now multiply both sides by z^L and sum L over the nonnegative integers. On the right this sum can be carried out thanks to [2, Eq. (3.3.7)]

$$\sum_{k=0}^{\infty}z^k\begin{bmatrix} n+k \\ k\end{bmatrix}=\frac{1}{(z)_{n+1}}. \tag{5.2}$$

On the left we insert the definition (2.24) of $\mathcal{U}$ to obtain two terms. In the second of these (corresponding to a sum over $\mathcal{T}(L,M,j+1,2j)$) we change $j\mapsto -j-1$ and use the symmetry (2.15). Substituting the definition (2.11) of $\mathcal{T}$ and shifting $L\to 2L+n+j$ then gives

$$\sum_{j=-\infty}^{\infty}\sum_{L=0}^{\infty}\sum_{\tau=0}^{1}\sum_{n=0}^{M}(-1)^{j+\tau}z^{2L+j+n}q^{j(j+1)+\frac{1}{2}n^2}\begin{bmatrix} M \\ n\end{bmatrix}\begin{bmatrix} L+M+2j+2\tau \\ L\end{bmatrix}\begin{bmatrix} L+M-j-2\tau \\ L+j\end{bmatrix}$$
$$=\frac{z^{M-1}q^{\binom{M}{2}}}{(z;q^{1/2})_{2M+1}}.$$

By (2.3) the sum over n yields $(-zq^{1/2})_M$. Dividing by this term and using that $(-zq^{1/2})_M(z;q^{1/2})_{2M+1}=(z^2)_{2M+1}/(-z)_{M+1}$, we are left to prove that

$$\sum_{j=-\infty}^{\infty}\sum_{L=0}^{\infty}\sum_{\tau=0}^{1}(-1)^{j+\tau}z^{2L+j}q^{j(j+1)}\begin{bmatrix}L+M+2j+2\tau\\L\end{bmatrix}\begin{bmatrix}L+M-j-2\tau\\L+j\end{bmatrix}=\frac{z^{M-1}q^{\binom{M}{2}}(-z)_{M+1}}{(z^2)_{2M+1}}.$$

We expand the right using (2.3) and (5.2) and equate coefficients of z^a. Renaming L as j and M as L this yields

$$\sum_{j=0}^{a}\sum_{\tau=0}^{1}(-1)^{a+\tau}q^{(2j-a)(2j-a-1)}\begin{bmatrix}L-3j+2a+2\tau\\j\end{bmatrix}\begin{bmatrix}L+3j-a-2\tau\\a-j\end{bmatrix}$$
$$=q^{\binom{L}{2}}\sum_{i=0}^{\infty}q^{\binom{2i+L-a}{2}}\begin{bmatrix}2L+i\\i\end{bmatrix}\begin{bmatrix}L+1\\2i+2L-a\end{bmatrix},$$

for $a\geq 0$ and $L\geq 1$. Next we write $a=2M+\sigma$ for $\sigma\in\{0,1\}$, and make the changes $j\mapsto j+M+\sigma$ on the left and $i\to M-i$ on the right. Recalling (2.30) this gives

$$\sum_{j=-\infty}^{\infty}\sum_{\tau=0}^{1}(-1)^{\tau}q^{2j(2j+1)}\mathcal{B}_{0,\sigma}(L,M,4j+2\tau,j)\tag{5.3}$$
$$=q^{\binom{L}{2}}\sum_{i=0}^{M}q^{\binom{L-2i-\sigma}{2}}\begin{bmatrix}2L+M-i\\2L\end{bmatrix}\begin{bmatrix}L+1\\2L-2i-\sigma\end{bmatrix},$$

for $M\geq 0$ and $L\geq 1$. To prove this identity we recall the following result due to Gessel and Krattenthaler [17, Thm. 12; $r=2$, $a=M$, $c=L$, $m=n=1=\epsilon=1$] [17, Thm. 13; $r=2$, $a=M+1$, $c=L+1$, $m=\epsilon=1$, $n=0$]

$$\sum_{j=-\infty}^{\infty}\sum_{\tau=0}^{1}(-1)^{\tau}q^{j(3j+2-\sigma)}\mathcal{B}_{\sigma,\sigma}(L,M,3j+2\tau,j)=q^{L(L+\sigma-1)}\begin{bmatrix}L+M+\sigma+1\\2L+\sigma\end{bmatrix},$$

for $L,M\geq 0$. Applying Theorem 2.6 with $r=0$ and $s=\sigma$ yields (5.3). □

If we now apply Theorem 2.1 to Lemma 5.1 and simplify the resulting right-side by (A.2) we obtain

$$\sum_{j=-\infty}^{\infty}(-1)^{j}q^{j(3j+1)}\mathcal{U}(L,M,3j,2j)=\begin{bmatrix}L+2M\\L\end{bmatrix}_{q^{1/2}},\tag{5.4}$$

which in the large L limit corresponds to the Bailey pair C(1) [23, 27]. We note that (5.4) can be used to also obtain a generalization of the Bailey pair C(2), or, more precisely, of a linear combination of C(1) and C(2).

Lemma 5.2. *For $L,M\in\mathbb{Z}_+$*

$$\sum_{j=-\infty}^{\infty}(-1)^{j}q^{j(3j+1)}\mathcal{U}(L,M,3j,2j+1)=\begin{bmatrix}L+2M-1\\L\end{bmatrix}_{q^{1/2}}.\tag{5.5}$$

Proof. We will be rather brief, omitting some details. Fixing M, let f_L and g_L denote the left side of (5.4) and (5.5), respectively. Then

$$g_L = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j+1)} \times \sum_{\tau=0}^{1} \sum_{\substack{n=0 \\ n+L+j+\tau \text{ even}}}^{M} q^{\frac{1}{2}n^2} \begin{bmatrix} M \\ n \end{bmatrix} \begin{bmatrix} M+(L+j-n-\tau+2)/2 \\ M+2j+1 \end{bmatrix} \begin{bmatrix} M+(L-j-n+\tau-2)/2 \\ M-2j-1 \end{bmatrix}.$$

Applying (4.9) to the second q-binomial yields two triple-sums. One of these vanishes as it changes sign under the substitutions $j \mapsto -j-1$ and $\tau \mapsto 1-\tau$. Hence

$$g_L = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j+1)} \times \sum_{\tau=0}^{1} \sum_{\substack{n=0 \\ n+L+j+\tau \text{ even}}}^{M} q^{\frac{1}{2}n^2} \begin{bmatrix} M \\ n \end{bmatrix} \begin{bmatrix} M+(L+j-n-\tau)/2 \\ M+2j \end{bmatrix} \begin{bmatrix} M+(L-j-n+\tau-2)/2 \\ M-2j-1 \end{bmatrix}.$$

Again by (4.9) we can subtract this in a straightforward manner from f_L. If in what then results we make the changes $j \mapsto -j$ and $\tau \to 1-\tau$ we find $q^M f_{L-1}$. Hence $g_L = f_L - q^M f_{L-1}$. If we now replace f_L in this equation by the right side of (5.4) and use the recurrence (4.6) we find that g_L equals the right side of (5.5). $\square$

Returning to (5.4) it follows by (2.19) and induction that for $k \geq 1$

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j((2k+1)j+1)} \mathcal{U}(L, M, (2k+1)j, 2j) = \sum_{r_1,\dots,r_{k-1} \geq 0} q^{\frac{1}{2}\sum_{i=1}^{k-1} r_i^2} \begin{bmatrix} L+r_{k-1}-\sum_{j=1}^{k-2} r_j \\ 2r_{k-1} \end{bmatrix}_{q^{1/2}} \prod_{i=0}^{k-2} \begin{bmatrix} L-r_{i+1}-\sum_{j=1}^{i-1} r_j \\ r_i - r_{i+1} \end{bmatrix}, \tag{5.6}$$

with $r_0 := M$. Letting M tend to infinity we obtain a chain of q-trinomial identities, the simplest of which is equivalent to [4, Thm. 6.1; (6.10)] by Andrews. (The other identity in Andrews' theorem follows by a single iteration of (5.5) and again taking the large M limit.) When also L tends to infinity we arrive at

$$\sum_{n_1,\dots,n_{k-1} \geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2}}{(q^2;q^2)_{n_1} \cdots (q^2;q^2)_{n_{k-1}} (q;q^2)_{n_{k-1}}} = \prod_{\substack{j=1 \\ j \not\equiv 2 \pmod 4 \\ j \not\equiv 0, \pm 4k \pmod{8k+4}}}^{\infty} \frac{1}{(1-q^j)},$$

with $N_i = n_i + \cdots + n_{k-1}$. Carrying out the same calculations starting with Lemma 5.2 leads to the analogous result

$$\sum_{n_1,\dots,n_{k-1} \geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+2N_1+\cdots+2N_{k-1}}}{(q^2;q^2)_{n_1} \cdots (q^2;q^2)_{n_{k-1}} (q;q^2)_{n_{k-1}+1}} = \prod_{\substack{j=1 \\ j \not\equiv 2 \pmod 4 \\ j \not\equiv 0, \pm 4 \pmod{8k+4}}}^{\infty} \frac{1}{(1-q^j)}.$$

For $k=2$ the above two identities are Rogers' [23, p. 330, Eq. (3)] listed as items (79) and (96) in Slater's list [28] of Rogers–Ramanujan identities.

When k is odd we can apply Lemma A.1 to obtain the following equivalent pair of Rogers–Ramanujan identities

$$\sum_{n_1,\dots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2}}{(q)_{n_1}\cdots(q)_{n_{k-1}}(q;q^2)_{n_{k-1}}} = \prod_{\substack{j=1\\ j\not\equiv 0,\pm 4k \pmod{8k-2}}}^{\infty} \frac{1}{(1-q^j)}$$

and

$$\sum_{n_1,\dots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+2N_1+\cdots+2N_{k-1}}}{(q)_{n_1}\cdots(q)_{n_{k-1}}(q;q^2)_{n_{k-1}+1}} = \prod_{\substack{j=1\\ j\not\equiv 0,\pm 2 \pmod{8k-2}}}^{\infty} \frac{1}{(1-q^j)}.$$

For $k=2$ these are [23, p. 329, Eqs. (1.1) and (1.3)] of Rogers, corresponding to items (61) and (59) in Slater's list. We leave it to the reader to carry out the corresponding rewritings when k is even.

So far, the results of this section are nothing unusual; typical examples of identities of the Rogers–Ramanujan type have been derived. More exciting q-series results arise if we apply (2.20) to (5.6). Replacing k by $k-1$ this yields

$$\begin{aligned}(5.7)\qquad &\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{2}j((4k^2-1)j+2)}\mathcal{V}(L,M,(2k+1)j,(2k-1)j)\\ &=\sum q^{\frac{1}{2}\sum_{i=1}^{k-1}r_i^2}\begin{bmatrix}L+M-r_1\\ L\end{bmatrix}\begin{bmatrix}L-r_2\\ r_1\end{bmatrix}\begin{bmatrix}r_1+r_{k-1}-\sum_{j=2}^{k-2}r_j\\ 2r_{k-1}\end{bmatrix}_{q^{1/2}}\prod_{i=2}^{k-2}\begin{bmatrix}r_1-r_{i+1}-\sum_{j=2}^{i-1}r_j\\ r_i-r_{i+1}\end{bmatrix},\end{aligned}$$

where $k\geq 2$ and where the sum on the right is over $r_1,\dots,r_{k-1}\geq 0$. Being of special interest, we separately state the $k=2$ case

$$(5.8)\qquad \sum_{j=-\infty}^{\infty}(-1)^j q^{j(15j+2)}\mathcal{V}(L,M,5j,3j;q^2)=\sum_{n=0}^{M}q^{n^2}\begin{bmatrix}L+M-n\\ L\end{bmatrix}_{q^2}\begin{bmatrix}2L-n\\ n\end{bmatrix}.$$

Applying (2.20) to (5.5) we also have

$$(5.9)\qquad \sum_{j=-\infty}^{\infty}(-1)^j q^{j(15j+2)}\mathcal{V}(L,M,5j+1,3j;q^2)=\sum_{n=0}^{M}q^{n^2}\begin{bmatrix}L+M-n\\ L\end{bmatrix}_{q^2}\begin{bmatrix}2L-n-1\\ n\end{bmatrix},$$

for $M\geq 1$. These two formulas provide our second and third doubly-bounded analogue of the first Rogers–Ramanujan identity. If we follow Schur [25] and define the polynomials e_n recursively as $e_n=e_{n-1}+q^{n-2}e_{n-2}$ with $e_0=0$ and $e_1=1$, then the $M\to\infty$ limits of (5.8) and (5.9) can be written as

$$(5.10)\qquad e_{2L+\tau}=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(15j+2)}\left\{\begin{bmatrix}L\\ 5j+1-\tau\end{bmatrix}_{2;q^2}+q^{6j+1}\begin{bmatrix}L\\ 5j+2-\tau\end{bmatrix}_{2;q^2}\right\}.$$

where $\tau\in\{0,1\}$. For $\tau=1$ this is equivalent to [4, Eqs. (6.16) and (6.17)] of Andrews, who remarked that on the right one can easily discern even and odd powers of q. Even powers follow from j even (odd) in the first (second) term and odd powers follow from j odd (even) in the first (second) term. From the well-known combinatorial interpretation of the Schur polynomial e_n as the generating function of partitions with difference between parts at least two and largest part not exceeding $n-2$, it follows that the right-hand side of (5.10) can be dissected to give the generating function of partitions of m with the parity of m fixed, difference between parts at least two and largest part at most $2L+\tau-2$. It seems an interesting combinatorial problem to also interpret the identities (5.8) and (5.9) in terms of restricted Rogers–Ramanujan partitions.

Letting not M but L tend to infinity in (5.8) and (5.9) leads to the somewhat unusual Rogers–Ramanujan polynomial identity

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j(15j+2)}\Big\{\begin{bmatrix}2M\\M-3j\end{bmatrix}_{q^2}+q^{6j+1}\begin{bmatrix}2M\\M-3j-1\end{bmatrix}_{q^2}\Big\}=\sum_{n=0}^{M}q^{n^2}\begin{bmatrix}M\\n\end{bmatrix}(-q^{M-n+1})_n.$$

A slight modification of the previous derivations leads to analogous results for the second Rogers–Ramanujan identity. To avoid repeating ourselves we have chosen to only state the counterparts of (5.8) and (5.9), given by

(5.11)

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j(15j+4)}\big\{\mathcal{S}(L,M,5j+\tau,3j;q^2)+q^{10j+3}\mathcal{S}(L,M,5j+3-\tau,3j+1;q^2)\big\}$$
$$=\sum_{n=0}^{M}q^{n(n+1)}\begin{bmatrix}L+M-n\\L\end{bmatrix}_{q^2}\begin{bmatrix}2L+\tau-n-2\\n\end{bmatrix},$$

for $\tau\in\{0,1\}$, $L\geq 1-\tau$ and $M\geq 2-\tau$. This can be proven using

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j+2)}\big\{\mathcal{T}(L,M,3j,2j+\tau)-\mathcal{T}(L,M,3j+2,2j+\tau)\big\}$$
$$=q^{L/2}\begin{bmatrix}L+2M+\tau-2\\L\end{bmatrix}_{q^{1/2}},$$

for $L+M\geq\tau$, which follows from (5.4) and (5.5) using the recurrences (4.6) and (4.9). When M tends to infinity in (5.11) one obtains q-trinomial identities for $d_{2L+\tau}$ where d_n is again a Schur polynomial, this time defined by $d_n=d_{n-1}+q^{n-2}d_{n-2}$ with $d_0=1$ and $d_1=0$. For $\tau=0$ these q-trinomial identities are equivalent to [4, Eqs. (6.18) and (6.19)].

After this intermezzo on polynomial analogues of the Rogers–Ramanujan identities we return to the general result (5.7), send L and M to infinity and repeat the exercise of separating even and odd powers of $q^{1/2}$. This yields the first identity stated in the introduction of this section.

5.2. **A generalization of Bailey pair G(4).** Our final example before we come to the generalized Borwein conjecture takes the following identity as starting point.

Lemma 5.3. *For $L,M\in\mathbb{Z}_+$*

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac14 j(j+1)}\mathcal{U}(L,M,\lfloor\tfrac12 j\rfloor,j)=(-1)^M q^{\frac12 M^2}\begin{bmatrix}\lfloor(L+M+1)/2\rfloor\\M\end{bmatrix}_{q^2}.$$

When L becomes large this leads to

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac14 j(j+1)}\begin{bmatrix}2M\\M-j\end{bmatrix}=(-1)^M q^{\frac12 M^2}(q^{1/2})_M,$$

which follows from the terminating q-Dougall sum [16, Eq. (II.21)] and is equivalent to the Bailey pair G(4) of Rogers [23, 27].

Proof. We proceed as in the proof of Lemma 5.1 and, assuming $M \geq 1$, compute the generating function of the identity in the lemma. Using the relation $(-zq^{1/2})_M(z^2;q^2)_{M+1} = (z^2)_{2M+1}/(zq^{1/2})_M$ this yields

$$\sum_{j=-\infty}^{\infty}\sum_{L=0}^{\infty}\sum_{\tau=0}^{1}(-1)^\tau z^{2L+j}q^{\frac{1}{2}j(2j+1)}\Big\{z\begin{bmatrix}L+M+2j+\tau\\L\end{bmatrix}+\begin{bmatrix}L+M+2j+\tau-1\\L-1\end{bmatrix}\Big\}\begin{bmatrix}L+M-j-\tau\\L+j\end{bmatrix}$$
$$=\frac{(-1)^M z^M(1+z)q^{\frac{1}{2}M^2}(zq^{1/2})_M}{(z^2)_{2M+1}}.$$

To the second term on the left we add the trivial identity

$$\sum_{j=-\infty}^{\infty}\sum_{L=0}^{\infty}\sum_{\tau=0}^{1}(-1)^\tau z^{2L+j}q^{L+\frac{1}{2}j(2j+1)}\begin{bmatrix}L+M+2j+\tau-1\\L\end{bmatrix}\begin{bmatrix}L+M-j-\tau\\L+j\end{bmatrix}=0.$$

(To prove this make the replacements $\tau \to 1-\tau$ and $L \to L-j$ followed by $j \to -j$.) By the recurrence (4.9) the second term then becomes equal to the first but without the z, so that both sides of the identity can be divided by $1+z$. Equating coefficients of z^a and renaming L as j and M as L then gives

$$\sum_{j=0}^{a}\sum_{\tau=0}^{1}(-1)^{a+\tau}q^{\frac{1}{2}(2j-a)(4j-2a-1)}\begin{bmatrix}L-3j+2a+\tau\\j\end{bmatrix}\begin{bmatrix}L+3j-a-\tau\\a-j\end{bmatrix}$$
$$=q^{\frac{1}{2}L^2}\sum_{i=0}^{\infty}q^{\frac{1}{2}(2i+L-a)^2}\begin{bmatrix}2L+i\\i\end{bmatrix}\begin{bmatrix}L\\2i+2L-a\end{bmatrix},$$

for $a \geq 0$ and $L \geq 1$. Setting $a = 2M+\sigma$ for $\sigma \in \{0,1\}$ and changing $j \to j+M+\sigma$ on the left and $i \to M-i$ on the right results in

$$\sum_{j=-\infty}^{\infty}\sum_{\tau=0}^{1}(-1)^\tau q^{j(4j+2\sigma+1)}\mathcal{B}_{0,\sigma}(L,M,4j+\sigma+\tau,j)$$
$$=q^{\frac{1}{2}L^2}\sum_{i=0}^{M}q^{\frac{1}{2}(2i-L+\sigma)^2}\begin{bmatrix}2L+M-i\\2L\end{bmatrix}\begin{bmatrix}L\\2L-2i-\sigma\end{bmatrix}.$$

This identity is simple consequence of Theorem 2.6 applied to

$$\sum_{j=-\infty}^{\infty}\sum_{\tau=0}^{1}(-1)^\tau q^{j(3j+\sigma+1)}\mathcal{B}_{\sigma,\sigma}(L,M,3j+\sigma+\tau,j)=q^{L(L-\sigma)}\begin{bmatrix}L+M+\sigma\\2L+\sigma\end{bmatrix},$$

which for $\sigma = 0$ is due to Burge [13, p. 217] and for $\sigma = 1$ to Gessel and Krattenthaler [17, Thm. 9; $r = 2$, $a = M$, $c = L$, $m = \epsilon = 1$, $n = 0$]. □

Applying the transformation (2.19) and carrying out the resulting sum on the right using (A.3) leads to

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{4}j(3j+1)}\mathcal{U}(L,M,\lfloor\tfrac{3}{2}j\rfloor,j)=\begin{bmatrix}\lfloor L/2\rfloor+M\\M\end{bmatrix}_{q^2}, \tag{5.12}$$

which is a refinement of the Bailey pair G(1) [23, 27]. We remark that a calculation very similar to the one that led to Lemma 5.2 shows that (5.12) implies a generalization of the Bailey pair G(3);

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{3}{4}j(j+1)}\mathcal{U}(L,M,3\lfloor\tfrac{1}{2}(j+1)\rfloor,j)=q^M\begin{bmatrix}\lfloor L/2\rfloor+M\\ M\end{bmatrix}_{q^2}.$$

We will not pursue the consequences of this identity but content ourselves with iterating just (5.12). By (2.19) and induction this gives

$$\begin{aligned}(5.13)\qquad &\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{4}j((2k+1)j+1)}\mathcal{U}(L,M,\lfloor\tfrac{1}{2}(2k+1)j\rfloor,j)\\ &\quad=\sum_{r_1,\dots,r_{k-1}\geq 0} q^{\frac{1}{2}\sum_{i=1}^{k-1}r_i^2}\begin{bmatrix}\lfloor(L+r_{k-1}-\sum_{j=1}^{k-2}r_j)/2\rfloor\\ r_{k-1}\end{bmatrix}_{q^2}\prod_{i=0}^{k-2}\begin{bmatrix}L-r_{i+1}-\sum_{j=1}^{i-1}r_j\\ r_i-r_{i+1}\end{bmatrix},\end{aligned}$$

where $r_0:=M$ and $k\geq 2$. We immediately consider the case when both L and M become large. To shorten some of the resulting equations we also replace k by $k+1$. Thanks to the triple product identity (4.10) this yields

$$(5.14)\qquad \sum_{n_1,\dots,n_k\geq 0}\frac{q^{\frac{1}{2}(N_1^2+\cdots+N_k^2)}}{(q)_{n_1}\cdots(q)_{n_{k-1}}(q^2;q^2)_{n_k}}=\frac{(q^{k/2+1/2},q^{k/2+1},q^{k+3/2};q^{k+3/2})_\infty}{(-q)_\infty(q^{1/2};q^{1/2})_\infty},$$

with $N_i=n_i+\cdots+n_k$. For $k=1$ this is [22, page 330], given as entry (20) in Slater's list.

Once more we utilize Lemma A.1. When k in (5.14) is even this implies

$$\sum_{n_1,\dots,n_k\geq 0}\frac{q^{N_1^2+\cdots+N_k^2}}{(q)_{n_1}\cdots(q)_{n_k}(-q^{1/2};q^{1/2})_{2n_k}}=\frac{(q^{k+1/2},q^{k+1},q^{2k+3/2};q^{2k+3/2})_\infty}{(q)_\infty}.$$

For $k=1$ this is the first Rogers–Selberg identity [23, p. 339, Eq. (6.1)], [26, Eq. (29)] (or item (33) in Slater's list), and for general k it is Paule's generalized Rogers–Selberg identity [21, Eq. (45); $r\to k+1$]. Similarly, when k is odd in (5.14) we get

$$\sum_{n_1,\dots,n_k\geq 0}\frac{q^{N_1^2+\cdots+N_k^2}(-q^{1/2-N_1})_{N_1}}{(q)_{n_1}\cdots(q)_{n_k}(-q^{1/2};q^{1/2})_{2n_k}}=\frac{(q^k,q^{k+1/2},q^{2k+1/2};q^{2k+1/2})_\infty}{(q^{1/2})_\infty(q^2;q^2)_\infty}.$$

Finally, applying (2.20) to (5.13) and replacing k by $k-1$ yields

$$\begin{aligned}&\sum_{j=-\infty}^{\infty}\bigl\{q^{\frac{1}{2}j((2k-1)(2k+1)j+1)}\mathcal{V}(L,M,(2k+1)j,(2k-1)j)\\ &\qquad -q^{\frac{1}{2}((2k-1)j+k-1)((2k+1)j+k)}\mathcal{V}(L,M,(2k+1)j+k,(2k-1)j+k-1)\bigr\}\\ &=\sum q^{\frac{1}{2}\sum_{i=1}^{k-1}r_i^2}\begin{bmatrix}L+M-r_1\\ L\end{bmatrix}\begin{bmatrix}L-r_2\\ r_1\end{bmatrix}\begin{bmatrix}\lfloor(r_1+r_{k-1}-\sum_{j=2}^{k-2}r_j)/2\rfloor\\ r_{k-1}\end{bmatrix}_{q^2}\prod_{i=2}^{k-2}\begin{bmatrix}r_1-r_{i+1}-\sum_{j=2}^{i-1}r_j\\ r_i-r_{i+1}\end{bmatrix},\end{aligned}$$

where $k\geq 3$ and where the sum on the right is over $r_1,\dots,r_{k-1}\geq 0$. When $k=2$ the correct right side reads $\sum_{r\geq 0}q^{\frac{1}{2}r^2}\begin{bmatrix}L+M-r\\ L\end{bmatrix}\begin{bmatrix}\lfloor L-r/2\rfloor\\ L-r\end{bmatrix}_{q^2}$. The second identity claimed in the introduction of this section follows in the large L and M limit.

6. The generalized Borwein conjecture

Finally we have come to our main application of the transformations of section 2, leading to proofs of Theorems 1.4–1.6.

We begin by recalling some notation and results of [31]. Assume that a,b are coprime integers such that $0<b<a$, and define a nonnegative integer n and positive integers $a_0,\dots,a_n$ as the order and partial quotients of the continued fraction representation of $(a/b-1)^{\operatorname{sign}(a-2b)}$ ($\operatorname{sign}(0)=0$), i.e.,

$$\Big(\frac{a}{b}-1\Big)^{\operatorname{sign}(a-2b)}=[a_0,\dots,a_n]=a_0+\cfrac{1}{a_1+\cfrac{1}{\cdots+\cfrac{1}{a_n}}}.$$

For simplicity, a_n will be fixed by requiring that $a_n\ge 2$ for $(a,b)\neq(2,1)$. (This is by no means necessary, see [31].) We denote the continued fraction corresponding to a,b by $\operatorname{cf}(a,b)$, and note the obvious symmetry $\operatorname{cf}(a,b)=\operatorname{cf}(a,a-b)$. We further define the partial sums $t_j=\sum_{k=0}^{j-1}a_k$ for $j=1,\dots,n+1$ and introduce $t_0=0$ and $d(a,b)=t_{n+1}$. Finally we need a $d(a,b)\times d(a,b)$ matrix $\mathcal{I}(a,b)$ with entries

$$\mathcal{I}(a,b)_{j,k}=\begin{cases}\delta_{j,k+1}+\delta_{j,k-1} & \text{for } j\neq t_i\\ \delta_{j,k+1}+\delta_{j,k}-\delta_{j,k-1} & \text{for } j=t_i\end{cases}$$

and a corresponding Cartan-type matrix $C(a,b)=2I-\mathcal{I}(a,b)$ where I is the $d(a,b)\times d(a,b)$ identity matrix. Note that the matrix $\mathcal{I}(a,b)$ has the following block-structure:

$$\mathcal{I}(a,b)=\left(\begin{array}{c|c|c} T_{a_0} & \begin{matrix} \\ -1 & \end{matrix} & \\ \hline \begin{matrix} & 1 \\ \\ \end{matrix} & \ddots & \begin{matrix} \\ -1 & \end{matrix} \\ \hline & \begin{matrix} & 1 \\ \\ \end{matrix} & T_{a_n} \end{array}\right)$$

where T_i is the incidence matrix of the tadpole graph with i vertices, i.e., $(T_i)_{j,k}=\delta_{j,k+1}+\delta_{j,k-1}+\delta_{j,k}\delta_{j,i}$.

With the above notation we define a polynomial for each pair of coprime integers a,b such that $0<2b<a$ as follows:

(6.1)
$$F_{a,b}(L,M)=\sum_{m\in\mathbb{Z}_+^{d(a,b)}}q^{L(L-2m_1)+mC(a,b)m}\begin{bmatrix}M+m_1-m_2\\ L\end{bmatrix}\begin{bmatrix}m_1-m_2\\ n_1\end{bmatrix}\prod_{j=2}^{d(a,b)}\begin{bmatrix}\tau_jm_j+n_j\\ \tau_jm_j\end{bmatrix}$$

for $2b<a<3b$, and

$$F_{a,b}(L,M)=\sum_{m\in\mathbb{Z}_+^{d(a,b)}}q^{L(L-2m_1)+mC(a,b)m}\begin{bmatrix}M+m_2\\ L\end{bmatrix}\begin{bmatrix}m_2\\ n_1\end{bmatrix}\prod_{j=2}^{d(a,b)}\begin{bmatrix}\tau_jm_j+n_j\\ \tau_jm_j\end{bmatrix}\tag{6.2}$$

for $0<3b\le a$. Here

$$mC(a,b)m=\sum_{j,k=1}^{d(a,b)} m_j C(a,b)_{j,k} m_k=\sum_{j=0}^{n}\Bigl(m_{t_j+1}^2+\sum_{k=t_j+1}^{t_{j+1}-1}(m_k-m_{k+1})^2\Bigr)$$

and $\tau_j=\tau_j(a,b)=2-\delta_{j,d(a,b)}$. The auxiliary variables n_j in the summand are integers defined by the (m,n)-system

$$n_j=L\delta_{j,1}-\sum_{k=1}^{d(a,b)} C(a,b)_{j,k} m_k \quad \text{for } j=1,\dots,d(a,b). \tag{6.3}$$

We are now prepared to state our first results of this section.

Theorem 6.1. *For $L,M\in\mathbb{Z}_+$ and a,b coprime integers such that $0<2b<a$,*

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{2}j((2ab+1)j+1)}\mathcal{S}(L,M,aj,2bj)=F_{a,b}(L,M). \tag{6.4}$$

A proof of this will be given in the next section.

In order to turn (6.4) into identities for q-binomial coefficients we take the large L limit. On the left this limit is easily computed with the aid of (2.17). On the right some rewritings need to be carried out first to cancel the term $L(L-2m_1)$ in the exponent of q in (6.1) and (6.2). To this end we eliminate $m_1,\dots,m_{a_0}$ in favour of $n_1,\dots,n_{a_0}$. By (6.3) this yields

$$\begin{aligned} m_j&=L-jm_{a_0+1}-\sum_{k=1}^{a_0}\min(j,k)n_k, \qquad j=1,\dots,a_0\\ n_{a_0+1}&=L-a_0m_{a_0+1}-\sum_{k=1}^{a_0}kn_k-\sum_{k=a_0+1}^{d(a,b)}C(a,b)_{a_0+1,k}m_k\\ n_j&=-\sum_{k=1}^{d(a,b)}C(a,b)_{j,k}m_k, \qquad j=a_0+2,\dots,d(a,b), \end{aligned} \tag{6.5}$$

from which it follows that

$$L(L-2m_1)+mC(a,b)m=\sum_{j=1}^{a_0}(N_j+m_{a_0+1})^2+\sum_{j,k=a_0+1}^{d(a,b)} m_jC(a,b)_{j,k}m_k,$$

where $N_j=n_j+\cdots+n_{a_0}$. Taking the large L limit is now straightforward and if we define $F_{a,b}(M)=(q)_M\lim_{L\to\infty}F_{a,b}(L,M)$ then

$$\begin{aligned} F_{a,b}(M)=&\sum_{\substack{n_1,\dots,n_{a_0}\ge 0\\ m_{a_0+1},\dots,m_{d(a,b)}\ge 0}}\\ &\times\frac{q^{\sum_{j=1}^{a_0}(N_j+m_{a_0+1})^2+\sum_{j,k=a_0+1}^{d(a,b)}m_jC(a,b)_{j,k}m_k}(q)_M}{(q)_{M-2m_{a_0+1}-N_1-N_2}(q)_{n_1}\cdots(q)_{n_{a_0}}(q)_{2m_{a_0+1}}}\prod_{j=a_0+2}^{d(a,b)}\begin{bmatrix}\tau_jm_j+n_j\\ \tau_jm_j\end{bmatrix} \end{aligned} \tag{6.6}$$

for $a>2b$. Here the auxiliary variables n_j for $j\ge a_0+2$ are given by (6.5). When $2b<a<3b$ there holds $a_0=1$ in which case $N_1=n_1$ and $N_2=0$. When $b=1$ there holds $d(a,1)=a_0=a-1$ in which case $m_{a_0+1}=0$.

Having defined the polynomial $F_{a,b}(M)$ we can now state the identities obtained when L tends to infinity in Theorem 6.1.

Corollary 6.1. *For $L, M \in \mathbb{Z}_+$ and a, b coprime integers such that $0 < 2b < a$,*

$$G(M,M;a/2,(a+1/b)/2,2b) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{1}{2}j((2ab+1)j+1)} \begin{bmatrix} 2M \\ M-2bj \end{bmatrix} = F_{a,b}(M).$$

From (6.6) it follows that for $2b < a < 3b$ the above right-hand side can be written as

$$\sum_{n_1,m_2,\dots,m_{d(a,b)}\geq 0} q^{(n_1+m_2)^2+\sum_{j,k=2}^{d(a,b)} m_j C(a,b)_{j,k} m_k} \begin{bmatrix} M \\ n_1 \end{bmatrix} \begin{bmatrix} M-n_1 \\ 2m_2 \end{bmatrix} \prod_{j=3}^{d(a,b)} \begin{bmatrix} \tau_j m_j + n_j \\ \tau_j m_j \end{bmatrix} \geq 0,$$

leading to our next corollary.

Corollary 6.2. *$G(M,M;b/2,(b+1/a)/2,2a) \geq 0$ for a, b coprime integers such that $2a < b < 3a$.*

The reason for interchanging a and b in comparison with Corollary 6.1 will become clear shortly.

To derive further positivity results form Theorem 6.1 we replace q by $1/q$ in (6.4) using (2.16). Defining the polynomial $f_{a,b}(L,M)$ by (6.1) and (6.2) but with $mC(a,b)m$ replaced by $\bar{m}C(a,b)m$ with $\bar{m} = (m_1,\dots,m_{d(a,b)-1},0)$ (so that $\bar{m}C(a,b)m = mC(a,b)m + m_{d(a,b)}(m_{d(a,b)-1} - m_{d(a,b)})$), the following result arises.

Corollary 6.3. *For $L, M \in \mathbb{Z}_+$ and a, b coprime integers such that $0 < 2b < a$,*

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{1}{2}j((2ab-1)j+1)} \mathcal{S}(L,M,aj,2bj) = f_{a,b}(L,M). \tag{6.7}$$

The large L limit can be taken following the same steps as before, and if we define $f_{a,b}(M) = (q)_M \lim_{L\to\infty} f_{a,b}(L,M)$ then $f_{a,b}(M)$ for $b \neq 1$ is given by (6.6) with $\sum_{j,k=a_0+1}^{d(a,b)} m_j C(a,b)_{j,k} m_k$ replaced by $\sum_{j,k=a_0+1}^{d(a,b)} \bar{m}_j C(a,b)_{j,k} m_k$ where $\bar{m}_j = m_j(1-\delta_{j,d(a,b)})$. This leads to the following analogue of Corollary 6.1.

Corollary 6.4. *For $L, M \in \mathbb{Z}_+$ and a, b coprime integers such that $2 < 2b < a$,*

$$G(M,M;(a-1/b)/2,a/2,2b) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{1}{2}j((2ab-1)j+1)} \begin{bmatrix} 2M \\ M-2bj \end{bmatrix} = f_{a,b}(M).$$

For $2b < a < 3b$ the above right-hand side can be written as

$$\sum_{n_1,m_2,\dots,m_{d(a,b)}\geq 0} q^{(n_1+m_2)^2+\sum_{j,k=2}^{d(a,b)} \bar{m}_j C(a,b)_{j,k} m_k} \begin{bmatrix} M \\ n_1 \end{bmatrix} \begin{bmatrix} M-n_1 \\ 2m_2 \end{bmatrix} \prod_{j=3}^{d(a,b)} \begin{bmatrix} \tau_j m_j + n_j \\ \tau_j m_j \end{bmatrix} \geq 0.$$

Hence $G(M,M;(a-1/b)/2,a/2,2b) \geq 0$ for $2b < a < 3b$. If we apply the symmetry (1.2) followed by the duality

$$G(M,M;\alpha,\beta,K;1/q) = q^{-M^2} G(M,M;K-\alpha,K-\beta,K;q)$$

and make the simultaneous replacements $a \to 4a - b$ and $b \to a$, we obtain the following result.

Corollary 6.5. *$G(M,M;b/2,(b+1/a)/2,2a) \geq 0$ for a, b coprime integers such that $a < b < 2a$.*

In Corollary 6.4 $b=1$ is excluded, requiring a different treatment. Namely, if we express the summand of $f_{k+1,1}(L,M)$ in terms of $n_1,\dots,n_k$ we find that (6.7) becomes

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{2}j((2k+1)j+1)}\mathcal{S}(L,M,(k+1)j,2j) \\ =\sum_{n_1,\dots,n_k\geq 0} q^{N_1^2+\cdots+N_k^2+n_k\tilde{N}_k}\begin{bmatrix}M+\tilde{N}_2\\ L\end{bmatrix}\begin{bmatrix}\tilde{N}_2\\ n_1\end{bmatrix}\begin{bmatrix}n_k+\tilde{N}_k\\ n_k\end{bmatrix}\prod_{j=2}^{k-1}\begin{bmatrix}n_j+2\tilde{N}_j\\ n_j\end{bmatrix},$$

with $N_j=n_j+\cdots+n_k$, $\tilde{N}_j=L-N_1-\cdots-N_j$ and $k\geq 2$. Noting that in the large L limit only the terms with $n_k=0$ contribute to the sum on the right, this can be recognized as a doubly-bounded analogue of the (first) Andrews–Gordon identity, with $k=2$ corresponding to our fourth doubly-bounded version of the first Rogers–Ramanujan identity. Taking $L\to\infty$ yields

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{1}{2}j((2k+1)j+1)}\begin{bmatrix}2M\\ M-2j\end{bmatrix}=\sum_{n_1,\dots,n_{k-1}\geq 0}\frac{q^{N_1^2+\cdots+N_{k-1}^2}(q)_M}{(q)_{M-N_1-N_2}(q)_{n_1}\cdots(q)_{n_{k-1}}},$$

which for $k=2$ is Bressoud's identity (4.3). We remark that it follows from (6.6) that also Theorem 6.1 for $b=1$ is a doubly-bounded analogue of the (first) Andrews–Gordon identity, with the same large L limit as above, but with $k=2$ excluded.

Corollaries 6.2 and 6.5 are the $\bar{a}=(2)$ and $\bar{a}=(0,2)$ instances of Theorem 1.6. We can use the full set of transformations of section 2 to derive polynomial identities that imply all of Theorem 1.6. However, the notation required to describe these more general identities is a lot more complicated than the already involved notation needed in the definitions of $F_{L,M}$, F_M, $f_{L,M}$ and f_M. Since we trust that the previous examples (and the proof of Theorem 6.1 given in the next section) illustrate how one can, in principle, obtain explicit representations for all of the polynomials occurring in Theorem 1.6 we will in the following take a shortcut and prove the remainder of the theorem without first deriving explicit polynomial identities.

We begin by noting that the Corollaries 2.1 and 2.2 imply the following simple lemma.

Lemma 6.1. *If $G(M,M;\alpha,\beta,K)\geq 0$ then $G(M,M;\alpha',\beta',K')\geq 0$ with*

$$\alpha'=\alpha/2+K,\qquad \beta'=\beta/2+K,\qquad K'=2K \tag{6.8}$$

or

$$\alpha'=(\alpha+K)/2,\qquad \beta'=(\beta+K)/2,\qquad K'=2K. \tag{6.9}$$

Proof. By Corollary 2.1 and the assumption that $G(M,M;\alpha,\beta,K)\geq 0$,

$$\begin{aligned}
0 &\leq \sum_{k=0}^{\infty} C_{M,k}(q) G(k,k;\alpha,\beta,K) \\
&= \sum_{j=-\infty}^{\infty} (-1)^j q^{Kj((\alpha+\beta)j+\alpha-\beta)/2} \sum_{k=0}^{\infty} C_{M,k}(q) \left[{2k \atop k-Kj} \right] \\
&= \sum_{j=-\infty}^{\infty} (-1)^j q^{Kj((4K+\alpha+\beta)j+\alpha-\beta)/2} \left[{2M \atop M-2Kj} \right] = G(M,M;\alpha',\beta',K'),
\end{aligned}$$

with α',β' and K' given by (6.8). Using Corollary 2.2 instead of 2.1 and copying the above steps leads to a proof of the second statement. □

Next we iterate Lemma 6.1 to arrive at a binary tree of positivity results.

Proposition 6.1. *For $n\geq 0$, let $\bar{a}=(a_0,\dots,a_n)\in\mathbb{Z}^{n+1}$ such that $a_0\geq 0$, $a_1,\dots,a_{n-1}\geq 1$ and $a_n\geq 2-\delta_{n,0}$. Then $G(M,M;\alpha,\beta,K)\geq 0$ implies that $G(M,M;\alpha',\beta',K')\geq 0$ with*

$$\alpha'=(\alpha+\mu K)/2^{d-1}, \quad \beta'=(\beta+\mu K)/2^{d-1}, \quad K'=2^{d-1}K$$

where $d,\mu\in\mathbb{Z}_+$ are given by (1.3).

The reason for referring to this as a binary tree is that in the proof given below $G(M,M;\alpha,\beta,K)$ corresponds to the "initial condition" $\bar{a}=(1)$ (for which $d=1$ and $\mu=0$), (6.8) corresponds to the transformation $(a_0,a_1,\dots,a_n)\to(a_0+1,a_1,\dots,a_n)$ and (6.9) corresponds to $(a_0,a_1,\dots,a_n)\to(0,a_0+1,a_1,\dots,a_n)$.

Having said this, it is clear that we could equally well have chosen a different labelling of the triples (α',β',K') in the tree. For example, we could have chosen to let (6.9) correspond to the transformation $(a_0,a_1,\dots,a_n)\to(a_0+1,a_1,\dots,a_n)$ and (6.8) to $(a_0,a_1,\dots,a_n)\to(0,a_0+1,a_1,\dots,a_n)$. On the sequences $\bar{a}$ this corresponds to the involution $(1)\to(1)$ and

$$(a_0,a_1,\dots,a_n)\to(0,1^{a_0-1},2,1^{a_1-2},2,\dots,2,1^{a_{n-1}-2},2,1^{a_n-3},2),$$

which leaves d invariant. Here 1^{μ} stands for μ repeated ones, and

$$(0,1^{-1})^{\mu}(2,1^{-1})^{\nu},2$$

(which can occur for $\mu\in\{0,1\}$ and $\nu\geq 0$; and stands for $0,1^{-1}$ repeated μ times followed by $2,1^{-1}$ repeated ν times followed by a 2) has to be identified with the single integer $\nu-\mu+2$. For example, $(1,1,2)\to(0,1^0,(2,1^{-1})^2,2)=(0,4)$ and $(0,4)\to(0,1^{-1},2,1,2)=(1,1,2)$. With this different choice of labelling the statement of the Proposition remains the same except that the expression for μ would then be

$$\mu=\frac{1}{3}(4^{d-1}-1)+\sum_{i=1}^{n}4^{a_i+\cdots+a_n-2}.$$

We do however believe that no relabelling is possible that would simplify the claim of the proposition.

Proof of Proposition 6.1. Given a sequence $\bar{a}=(a_0,\dots,a_n)$ we write $d=a_0+\cdots+a_n=|\bar{a}|$. We will also make the dependence of α',β' and K' on $\bar{a}$ explicit by writing $\alpha'_{\bar{a}},\beta'_{\bar{a}}$ and $K'_{\bar{a}}$. Similarly we will write $\alpha_{\bar{a}}$, $\beta_{\bar{a}}$ and $K_{\bar{a}}$ whenever necessary, but $\alpha_{(1)}$, $\beta_{(1)}$ and $K_{(1)}$ will always be just α, β and K.

As already remarked above, for $\bar{a}=(1)$, which is the only admissible sequence for which $|\bar{a}|=1$, the proposition is trivially true. We will now proceed by induction on d and assume that the proposition is true for sequences $\bar{a}$ with $|\bar{a}|=d$. Now there are two types of (admissible) sequences $\bar{a}'$ that have $|\bar{a}'|=d+1$. Either it is of the form $\bar{a}'=(a_0+1,a_1,\dots,a_n)$ or it is of the form $\bar{a}'=(0,a_0+1,a_1,\dots,a_n)$ where in both cases $a_0\geq 0$. (If $n=0$ in the latter case, then $a_0\geq 1$.)

First assume $\bar{a}'=(a_0+1,a_1,\dots,a_n)$. Since $|\bar{a}'|=d+1$ the sequence $\bar{a}=(a_0,\dots,a_n)$ has $|\bar{a}|=d$ and by our induction hypothesis the proposition holds for this $\bar{a}$. If we now apply the (6.8) case of Lemma 6.1 to $\bar{a}$, and then use the induction hypothesis, we find the proposition to be true for

$$K'=2K_{\bar{a}}=2(2^{d-1}K)=2^dK=K_{\bar{a}'}$$

and

$$\alpha'=\alpha_{\bar{a}}/2+K_{\bar{a}}=(\alpha+\mu_{\bar{a}}K)/2^d+2^{d-1}K=(\alpha+\mu_{\bar{a}'}K)/2^d=\alpha_{\bar{a}'},$$

where we have used that $\mu_{\bar{a}'}-\mu_{\bar{a}}=4^{d-1}$. Repeating the last calculation with α replaced by β also shows that $\beta'=\beta_{\bar{a}'}$.

Next assume $\bar{a}'=(0,a_0+1,a_1,\dots,a_n)$. Since $|\bar{a}'|=d+1$ the sequence $\bar{a}=(a_0,\dots,a_n)$ has $|\bar{a}|=d$ and by our induction hypothesis the proposition holds for this $\bar{a}$. If we now apply the (6.9) case of Lemma 6.1 to $\bar{a}$, and then use the induction hypothesis, we find the proposition to be true for $K'=K_{\bar{a}'}$ and

$$\alpha'=(\alpha_{\bar{a}}+K_{\bar{a}})/2=(\alpha+\mu_{\bar{a}}K)/2^d+2^{d-2}K=(\alpha+\mu_{\bar{a}'}K)/2^d=\alpha_{\bar{a}'},$$

where we have used that $\mu_{\bar{a}'}-\mu_{\bar{a}}=4^{d-1}$. Repeating this with α replaced by β shows that $\beta'=\beta_{\bar{a}'}$. □

Using the Theorems 1.1–1.3 as input to Proposition 6.1 the Theorems 1.4–1.6 easily follow, and we only present the proof of Theorem 1.6.

Proof of Theorem 1.6. By Theorem 1.1 there holds $G(M,M,\tilde{b},\tilde{b}+1/\tilde{a},\tilde{a})\geq 0$ for $\tilde{a},\tilde{b}$ coprime integers such that $0<\tilde{b}<\tilde{a}$. By Proposition 6.1 it is therefore true that $G(M,M,\alpha,\beta,K)\geq 0$ with $\alpha=(\tilde{b}+\mu\tilde{a})/2^{d-1}$, $\beta=(\tilde{b}+1/\tilde{a}+\mu\tilde{a})/2^{d-1}$ and $K=2^{d-1}\tilde{a}$. Defining $a=\tilde{a}$ and $b=\tilde{b}+\mu\tilde{a}$ gives the statement of the theorem. (Since $\tilde{a}$ and $\tilde{b}$ are coprime, so are a and b.) □

A close scrutiny of the set of admissible sequences $\bar{a}$ allows for a slight reformulation of Theorem 1.6. First note there are 2^{d-1} sequences with fixed d. For example, if $d=4$ we have the following eight sequences in reverse lexicographic order: $S_4:=\{(4),(2,2),(1,3),(1,1,2),(0,4),(0,2,2),(0,1,3),(0,1,1,2)\}$. Now observe that these eight sequences form four pairs, with a typical pair given by $(a_0,\dots,a_{n-1},a_n)$ and $(a_0,\dots,a_{n-1},a_n-2,2)$ with $a_n\geq 3$. (The only exception is the pair (2) and $(0,2)$ for $d=2$, which corresponds to $a_n=a_0=2$.) Moreover, if b and b' form such a pair, with $b>b'$ in reverse lexicographical order, then $\mu_b=\mu_{b'}+1$ with $\mu_b\equiv 2 \pmod 4$. For example, the eight values of μ corresponding to the elements of the set S_4 are given by $42,41,38,37,26,25,22,21$. We can thus reformulate the above theorem as follows.

Theorem 6.2. *For $n\geq 0$, let $\bar{a}=(a_0,\dots,a_n)\in\mathbb{Z}^{n+1}$ such that (ii) $a_0\geq 0$, $a_1,\dots,a_{n-1}\geq 1$ and $a_n\geq 3-\delta_{n,0}$. Then*

$$G(M,M;b/2^{d-1},(b+1/a)/2^{d-1},2^{d-1}a)\geq 0,$$

for a,b coprime integers such that $(\mu-1)a<b<(\mu+1)a$, with d and μ given by (1.3).

Note that in comparison with Theorem 1.6 the case $\bar{a}=(1)$ is now excluded. We also note that the discussion leading to the Theorem 6.2 does not quite justify the claim of the theorem. After all, what we really have argued is that a,b must satisfy $(\mu-1)a<b<(\mu+1)a$ and $b\neq\mu a$. The following proof is to show that this latter condition can be dropped.

Proof. Since a,b are positive, coprime integers, the only solution to $b=\mu a$ is given by $(a,b)=(1,\mu)$. So the problem is to show that $G(M,M;\mu/2^{d-1},(\mu+1)/2^{d-1},2^{d-1})\geq 0$ for each admissible sequence $\bar{a}$. Now fix $\bar{a}$. From $\mu_{\bar{a}}\equiv 2 \pmod 4$ it follows that $\mu_{\bar{a}}/2$ is an odd integer. Hence we can apply Theorem 1.6 with $(a,b)=(2,\mu_{\bar{a}}/2)$ and sequence $\bar{a}'=(a_0,\dots,a_n-1)$. Since $d_{\bar{a}'}=d_{\bar{a}}-1$ and $\mu_{\bar{a}'}=(\mu_{\bar{a}}-2)/4$ the inequality $\mu_{\bar{a}'}a<b<(\mu_{\bar{a}'}+1)a$ translates into $\mu_{\bar{a}}-2<\mu_{\bar{a}}<\mu_{\bar{a}}+2$ and is therefore satisfied, as required by Theorem 1.6. But with the above choice for a,b and $\bar{a}'$ Theorem 1.6 tells us that $G(M,M;\mu_{\bar{a}}/2^{d_{\bar{a}}-1},(\mu_{\bar{a}}+1)/2^{d_{\bar{a}}-1},2^{d_{\bar{a}}-1})\geq 0$. $\square$

7. Proof of Theorem 6.1

In addition to our earlier definitions we set $d(1,1)=1/\mathrm{cf}(1,1)=0$. To also facilitate computations involving continued fractions, we sometimes, by abuse of notation, write $\mathrm{cf}(a,b)=(a/b-1)^{\mathrm{sign}(a-2b)}$.

For coprime integers a,b such that $1\leq b\leq a$ define

$$G_{a,b}(L,M)=\sum_{m\in\mathbb{Z}_+^{d(a,b)}} q^{mC(a,b)m}\begin{bmatrix}\tau_0L+M-m_1\\ \tau_0L\end{bmatrix}\prod_{j=1}^{d(a,b)}\begin{bmatrix}\tau_jm_j+n_j\\ \tau_jm_j\end{bmatrix} \tag{7.1}$$

for $1\leq b\leq a\leq 2b$ (so that $G_{1,1}(L,M)=\begin{bmatrix}L+M\\ L\end{bmatrix}$) and

$$G_{a,b}(L,M)=\sum_{m\in\mathbb{Z}_+^{d(a,b)}} q^{L(L-2m_1)+mC(a,b)m}\begin{bmatrix}L+M+m_1\\ 2L\end{bmatrix}\prod_{j=1}^{d(a,b)}\begin{bmatrix}\tau_jm_j+n_j\\ \tau_jm_j\end{bmatrix} \tag{7.2}$$

for $a\geq 2b$.

Our proof of Theorems 6.1 relies on the following identity for the polynomial $G_{a,b}$ [31, Lem. 3.1 and Thm. 3.1].

Theorem 7.1. *For $L,M\in\mathbb{Z}_+$ and a,b coprime integers such that $1\leq b\leq a$,*

$$\sum_{j=-\infty}^{\infty}(-1)^jq^{\frac12 j((2ab+1)j+1)}\mathcal{B}(L,M,aj,bj)=G_{a,b}(L,M).$$

7.1. **Proof of Theorem 6.1 for $2b<a<3b$.** Take Theorem 7.1, replace a,b by $\bar{a},\bar{b}$ and apply (2.22) followed by (2.20). With the notation $a=2\bar{a}+\bar{b}$ and $b=\bar{a}$ this leads to

$$\begin{aligned}\sum_{j=-\infty}^{\infty}&(-1)^jq^{\frac12 j((2ab+1)j+1)}\mathcal{S}(L,M,aj,2bj)\\ &=\sum_{\substack{r_1,r_2\geq 0\\ r_1+r_2\text{ even}}} q^{\frac12(r_1^2+r_2^2)}\begin{bmatrix}L+M-r_1\\ L\end{bmatrix}\begin{bmatrix}L-r_1\\ r_2\end{bmatrix}G_{\bar{a},\bar{b}}(\tfrac12(r_1-r_2),L-r_1).\end{aligned} \tag{7.3}$$

Next insert the expression for $G_{\bar{a},\bar{b}}$ given in (7.1) and (7.2). First, when $1 \le \bar{b} < \bar{a} < 2\bar{b}$ ($\tau_0 = 2$ since $(\bar{a},\bar{b}) \neq (1,1)$),

$$\text{RHS(7.3)} = \sum_{\substack{r_1,r_2\ge 0\\ r_1+r_2 \text{ even}}} \sum_{m\in\mathbb{Z}_+^{d(\bar{a},\bar{b})}} q^{\frac{1}{2}(r_1^2+r_2^2)+mC(\bar{a},\bar{b})m} \times \begin{bmatrix} L+M-r_1 \\ L\end{bmatrix}\begin{bmatrix} L-r_1 \\ r_2\end{bmatrix}\begin{bmatrix} L-r_2-m_1 \\ r_1-r_2\end{bmatrix}\prod_{j=1}^{d(\bar{a},\bar{b})}\begin{bmatrix} \bar{\tau}_j m_j+n_j \\ \bar{\tau}_j m_j\end{bmatrix}, \tag{7.4}$$

with $\bar{\tau}_j = \tau_j(\bar{a},\bar{b})$ and n_j given by (6.3) with $L \to (r_1-r_2)/2$ and $a,b\to \bar{a},\bar{b}$. Now relabel $m_j \to m_{j+2}$ and $n_j\to n_{j+2}$, then replace $r_1 \to L-m_1+m_2$ and $r_2 \to L-m_1-m_2$, and introduce the auxiliary variables $n_1 = L-m_1-m_2$ and $n_2 = m_1-m_2-m_3$. Since $\bar{a}<2\bar{b}$ and $a>2b$ one finds $\text{cf}(a,b) = 1+1/(1+1/\text{cf}(\bar{a},\bar{b}))$ and thus $\text{cf}(a,b) = [1,1,\alpha_0,\dots,\alpha_n]$ with $\text{cf}(\bar{a},\bar{b}) = [\alpha_0,\dots,\alpha_n]$ (with $\alpha_n\ge 2$ since $(\bar{a},\bar{b})\neq(2,1)$). This implies $d(a,b) = d(\bar{a},\bar{b})+2$, $\tau_j = \tau_j(a,b) = \bar{\tau}_{j+2}$ and

$$C(a,b) = \left(\begin{array}{cc|c} 1 & 1 & \\ -1 & 1 & 1 \\ \hline & -1 & \\ & & C(\bar{a},\bar{b}) \\ & & \end{array}\right)$$

and therefore RHS(7.3) = RHS(6.1). Eliminating $\bar{a}$ and $\bar{b}$ in $1\le \bar{b}<\bar{a}<2\bar{b}$ in favour of a and b yields the condition $5b/2 < a < 3b$.

Next, when $2\le 2\bar{b}\le \bar{a}$, one again finds (7.4), but with an additional $(r_1-r_2)(r_1-r_2-4m_1)/4$ in the exponent of q. By the same variable changes as before this yields an extra $m_2(m_2-2m_3)$ in the exponent of q. Since $\bar{a}\ge 2\bar{b}$ and $a>2b$ one finds $\text{cf}(a,b) = 1+1/(1+\text{cf}(\bar{a},\bar{b}))$ and thus $\text{cf}(a,b) = [1,1+\alpha_0,\alpha_1,\dots,\alpha_n]$ with $\text{cf}(\bar{a},\bar{b}) = [\alpha_0,\dots,\alpha_n]$. (For $(\bar{a},\bar{b}) = (2,1)$ one finds $\text{cf}(a,b) = \text{cf}(5,2) = [1,2]$ which has $\alpha_n\ge 2$ as it should.) Hence $d(a,b) = d(\bar{a},\bar{b})+2$, $\tau_j = \tau_j(a,b) = \bar{\tau}_{j+2}$ and

$$C(a,b) = \left(\begin{array}{cc|c} 1 & 1 & \\ -1 & 2 & -1 \\ \hline & -1 & \\ & & C(\bar{a},\bar{b}) \\ & & \end{array}\right)$$

and thus again RHS(7.3) = RHS(6.1). This time, however, $2\le 2\bar{b}\le\bar{a}$ leads to the condition $2b<a\le 5b/2$.

7.2. **Proof of Theorem 6.1 for** $a\ge 3b$**.** Take Theorem 7.1 with a,b replaced by $\bar{a},\bar{b}$, use the symmetry (2.29) and then apply (2.22) followed by (2.20). With the notation $a = \bar{a}+2\bar{b}$ and $b=\bar{b}$ this gives

$$\begin{aligned}\sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{1}{2}j((2ab+1)j+1)}\mathcal{S}(L,M,aj,2bj) \\ = \sum_{\substack{r_1,r_2\ge 0\\ r_1+r_2\text{ even}}} q^{\frac{1}{2}(r_1^2+r_2^2)}\begin{bmatrix} L+M-r_1 \\ L\end{bmatrix}\begin{bmatrix} L-r_1 \\ r_2\end{bmatrix} G_{\bar{a},\bar{b}}(L-r_1,\tfrac{1}{2}(r_1-r_2)).\end{aligned} \tag{7.5}$$

Now insert the explicit expression for $G_{\bar{a},\bar{b}}$. First, when $1 \le \bar{b} \le \bar{a} < 2\bar{b}$,

$$\text{RHS(7.5)} = \sum_{\substack{r_1,r_2\ge 0\\ r_1+r_2 \text{ even}}} \sum_{m\in\mathbb{Z}_+^{d(\bar{a},\bar{b})}} q^{\frac{1}{2}(r_1^2+r_2^2)+mC(\bar{a},\bar{b})m} \times \begin{bmatrix} L+M-r_1 \\ L \end{bmatrix} \begin{bmatrix} L-r_1 \\ r_2 \end{bmatrix} \begin{bmatrix} \tau_0(L-r_1)+\frac{1}{2}(r_1-r_2)-m_1 \\ \tau_0(L-r_1) \end{bmatrix} \prod_{j=1}^{d(\bar{a},\bar{b})} \begin{bmatrix} \bar{\tau}_j m_j+n_j \\ \bar{\tau}_j m_j \end{bmatrix}, \tag{7.6}$$

with n_j given by (6.3) with $L \to L - r_1$ and $(a,b) \to (\bar{a},\bar{b})$.

The next step is to relabel $m_j \to m_{j+2}$ and $n_j \to n_{j+2}$, then to replace $r_1 \to L - m_2$ and $r_2 \to L - 2m_1 + m_2$, and to introduce the auxiliary variables $n_1 = L - 2m_1 + m_2$ and $n_2 = m_1 - m_2 - m_3$. Since $\bar{a} < 2\bar{b}$ and $a > 2b$ one finds $\mathrm{cf}(a,b) = 2 + 1/\mathrm{cf}(\bar{a},\bar{b}))$ and thus $\mathrm{cf}(a,b) = [2,\alpha_0,\dots,\alpha_n]$ with $\mathrm{cf}(\bar{a},\bar{b}) = [\alpha_0,\dots,\alpha_n]$. This implies $d(a,b) = d(\bar{a},\bar{b}) + 2$, $\tau_j = \tau_j(a,b) = \bar{\tau}_{j+2}$ and

$$C(a,b) = \left(\begin{array}{cc|c} 2 & -1 & \\ -1 & 1 & 1 \\ \hline & -1 & \\ & & C(\bar{a},\bar{b}) \\ & & \end{array}\right)$$

$(C(3,1) = ((2,-1),(-1,1)))$ and thus RHS(7.5) = RHS(6.2). Writing $1 \le \bar{b} \le \bar{a} < 2\bar{b}$ in terms of a and b yields the condition $3b \le a < 4b$.

Next, when $\bar{a} \ge 2\bar{b}$, one again finds (7.6), but with an additional $(L-r_1)(L-r_1-2m_1)$ in the exponent of q. By the same variable changes as above this yields an extra $m_2(m_2-2m_3)$ in the exponent of q. Since $\bar{a} \ge 2\bar{b}$ and $a > 2b$ this yields $\mathrm{cf}(a,b) = 2 + \mathrm{cf}(\bar{a},\bar{b}))$ and thus $\mathrm{cf}(a,b) = [2+\alpha_0,\alpha_1,\dots,\alpha_n]$ with $\mathrm{cf}(\bar{a},\bar{b}) = [\alpha_0,\dots,\alpha_n]$. This implies $d(a,b) = d(\bar{a},\bar{b}) + 2$, $\tau_j = \tau_j(a,b) = \bar{\tau}_{j+2}$ and

$$C(a,b) = \left(\begin{array}{cc|c} 2 & -1 & \\ -1 & 2 & -1 \\ \hline & -1 & \\ & & C(\bar{a},\bar{b}) \\ & & \end{array}\right),$$

and once again RHS(7.5) = RHS(6.2). The condition $\bar{a} \ge 2\bar{b}$ implies $a \ge 4b$.

Appendix A. Some simple summation formulas

In this appendix some simple identities used in the main text are proven.

Our first result is nothing but a corollary of the q-Pfaff–Saalschütz sum [16, Eq. (II.12)]

$$\sum_{k=0}^{n} \frac{(a,b,q^{-n})_k\, q^k}{(q,c,abq^{1-n}/c)_k} = \frac{(c/a,c/b)_n}{(c,c/ab)_n}. \tag{A.1}$$

Specializing $n \to M$, $a \to q^{-L/2}$, $b \to q^{-(L+1)/2}$ and $c \to q^{1/2}$, and making some simplifications, yields

$$\sum_{i=0}^{M} q^{\frac{1}{2}i(2i-1)} \begin{bmatrix} L+M-i \\ L \end{bmatrix} \begin{bmatrix} L+1 \\ 2i \end{bmatrix}_{q^{1/2}} = \begin{bmatrix} L+2M \\ L \end{bmatrix}_{q^{1/2}}. \tag{A.2}$$

Also the next identity follows from (A.1), albeit with a bit more effort,

$$\sum_{i=0}^{M}(-1)^i q^{i^2}\begin{bmatrix} L+M-i \\ L \end{bmatrix}\begin{bmatrix} \lfloor (L+1)/2\rfloor \\ i \end{bmatrix}_{q^2} = \begin{bmatrix} \lfloor L/2\rfloor + M \\ M \end{bmatrix}_{q^2}. \tag{A.3}$$

When L is even this follows from the substitutions $n \mapsto M$, $a = -b \mapsto q^{-L/2}$ and $c \mapsto -q$ in (A.1). To obtain (A.3) for L odd we denote the left side of (A.3) by $f_{L,M}$ and note that by (4.9) $f_{2L-1,M} = f_{2L,M} - q^{2L} f_{2L,M-1}$. Since we already proved (A.3) for even L we may on the right replace $f_{2L,M}$ and $f_{2L,M-1}$ using (A.3). By (4.6) this yields $f_{2L-1,M} = \left[\begin{smallmatrix} L+M-1 \\ M \end{smallmatrix}\right]_{q^2}$ completing the proof. We note that (A.3) for odd L can also be viewed as a corollary of a basic hypergeometric summation, given by the $\tau = 2$ instance of

$${}_3\phi_2\left[\begin{matrix} a^{1/2}, -a^{1/2}, q^{-n} \\ -q, aq^{1-n} \end{matrix}; q, q^\tau\right] := \sum_{k=0}^{n} \frac{(a;q^2)_k (q^{-n})_k q^{\tau k}}{(q^2;q^2)_k (aq^{1-n})_k} = \frac{q^{(2-\tau)n}(1/a;q^2)_n}{(-q,1/a)_n}$$

true for $\tau \in \{1,2\}$. The proof of this almost balanced summation proceeds along the same lines as the proof of (A.3).

Simple as it is, our final summation formula — used in the main text to simplify multiple sums — appears to be new. We remark that it can also be used very effectively to reduce the number of (independent) entries in Slater's list of 130 Rogers–Ramanujan-type identities [28]. For $M_i = m_i + \cdots + m_k$ define

$$f_k(m_k) = \sum_{m_1,\dots,m_{k-1}\geq 0} \frac{a^{M_1+\cdots+M_k} q^{\frac{1}{2}(M_1^2+\cdots+M_k^2)}}{(q)_{m_1}\cdots(q)_{m_{k-1}}}.$$

Lemma A.1. *Let $N_i = n_i + \cdots + n_k$. Then*

$$f_{2k-1}(n_k) = \sum_{n_1,\dots,n_{k-1}\geq 0} \frac{a^{2N_1+\cdots+2N_k} q^{N_1^2+\cdots+N_k^2}(-q^{1/2-N_1}/a)_{N_1}}{(q)_{n_1}\cdots(q)_{n_{k-1}}(-aq^{1/2})_{n_k}}$$

$$f_{2k}(n_k) = (-aq^{1/2})_\infty \sum_{n_1,\dots,n_{k-1}\geq 0} \frac{a^{2N_1+\cdots+2N_k} q^{N_1^2+\cdots+N_k^2}}{(q)_{n_1}\cdots(q)_{n_{k-1}}(-aq^{1/2})_{n_k}}.$$

Proof. When k is odd we replace k by $2k-1$ and introduce new summation variables $n_1,\dots,n_{k-1}$ and $t_1,\dots,t_{k-1}$ as $n_i = m_{2i-1}+m_{2i}$ and $t_i = m_{2i}$. Using the notation of the lemma this gives

$$\begin{aligned} f_{2k-1}(n_k) &= \sum_{n_1,\dots,n_{k-1}\geq 0} a^{N_1} q^{\frac{1}{2}N_1^2} \prod_{i=1}^{k-1} a^{2N_{i+1}} q^{N_{i+1}^2} \sum_{t_i=0}^{n_i} \frac{a^{t_i} q^{\frac{1}{2}t_i(t_i+2N_{i+1})}}{(q)_{t_i}(q)_{n_i-t_i}} \\ &= \sum_{n_1,\dots,n_{k-1}\geq 0} \frac{a^{N_1+2N_2+\cdots+2N_k} q^{\frac{1}{2}N_1^2+N_2^2+\cdots+N_k^2}}{(q)_{n_1}\cdots(q)_{n_{k-1}}} \prod_{i=1}^{k-1}(-aq^{N_{i+1}+1/2})_{n_i} \\ &= \sum_{n_1,\dots,n_{k-1}\geq 0} \frac{a^{2N_1+\cdots+2N_k} q^{N_1^2+\cdots+N_k^2}(-q^{1/2-N_1}/a)_{N_1}}{(q)_{n_1}\cdots(q)_{n_{k-1}}(-aq^{1/2})_{n_k}}, \end{aligned}$$

where the second equality follows from the q-binomial theorem (2.3) with $x = -aq^{N_{i+1}+1/2}$ and the last equality follows from

$$\prod_{i=1}^{k-1}(-aq^{N_{i+1}+1/2})_{n_i} = \prod_{i=1}^{k-1}\frac{(-aq^{1/2})_{N_i}}{(-aq^{1/2})_{N_{i+1}}} \\ = \frac{(-aq^{1/2})_{N_1}}{(-aq^{1/2})_{N_k}} = a^{N_1}q^{\frac{1}{2}N_1^2}\frac{(-q^{1/2-N_1}/a)_{N_1}}{(-aq^{1/2})_{n_k}}.$$

Next, when k is even, we replace k by $2k$ and introduce new summation variables $n_1,\dots,n_{k-1}$, $t_1,\dots,t_{k-1}$ and s as $n_i = m_{2i}+m_{2i+1}$, $t_i = m_{2i-1}$ and $s=\sum_{j=1}^{k} m_{2j-1}$. With the notation $T_i = s-t_1-\cdots-t_i$ and $t_k = T_{k-1} = s-t_1-\cdots-t_{k-1}$ this yields

$$f_{2k}(n_k) = \sum_{\substack{n_1,\dots,n_{k-1}\geq 0\\ s,t_1,\dots,t_{k-1}\geq 0}} \frac{a^{s+2N_1+\cdots+2N_k}q^{\frac{1}{2}s^2+sn_k+N_1^2+\cdots+N_k^2+\sum_{i=1}^{k-1}t_i(N_i-n_k-T_i)}}{(q)_{T_{k-1}}\prod_{i=1}^{k-1}(q)_{t_i}(q)_{n_i-t_{i+1}}}.$$

Now define

$$g(n_1,\dots,n_l,s) = \sum_{t_1,\dots,t_l\geq 0}\frac{1}{(q)_{T_l}}\prod_{i=1}^{l}\frac{q^{t_i(n_i+\cdots+n_l-T_i)}}{(q)_{t_i}(q)_{n_i-t_{i+1}}},$$

where $t_{l+1}=T_l$. Obviously, $g(s)=1/(q)_s$. From the q-Chu–Vandermonde sum (2.18) with $a\to\infty$, $n\to T_{l-1}$ and $c\to q^{n_l-T_{l-1}+1}$ it follows that $g(n_1,\dots,n_l,s) = g(n_1,\dots,n_{l-1})/(q)_{n_l}$. Hence

$$g(n_1,\dots,n_l,s) = \frac{1}{(q)_s(q)_{n_1}\cdots(q)_{n_l}}.$$

When $l=k-1$ we insert this in the expression for $f_{2k}(n_k)$ to get

$$f_{2k}(n_k) = \sum_{n_1,\dots,n_{k-1},s\geq 0}\frac{a^{s+2N_1+\cdots+2N_k}q^{\frac{1}{2}s^2+sn_k+N_1^2+\cdots+N_k^2}}{(q)_s(q)_{n_1}\cdots(q)_{n_l}}.$$

Performing the sum over s by the $L\to\infty$ limit of (2.3) settles the second claim of the lemma. □

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Department of Mathematics and Statistics, The University of Melbourne, Vic 3010, Australia