HALL–LITTLEWOOD POLYNOMIALS AND CHARACTERS OF AFFINE LIE ALGEBRAS

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Abstract. The Weyl–Kac character formula gives a beautiful closed-form expression for the characters of integrable highest-weight modules of Kac–Moody algebras. It is not, however, a formula that is combinatorial in nature, obscuring positivity. In this paper we show that the theory of Hall–Littlewood polynomials may be employed to prove Littlewood-type combinatorial formulas for the characters of certain highest weight modules of the affine Lie algebras $\mathrm{C}_n^{(1)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$. Through specialisation this yields generalisations for $\mathrm{B}_n^{(1)}$, $\mathrm{C}_n^{(1)}$, $\mathrm{A}_{2n-1}^{(2)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$ of Macdonald's identities for powers of the Dedekind eta-function. These generalised eta-function identities include the Rogers–Ramanujan, Andrews–Gordon and Göllnitz–Gordon q-series as special, low-rank cases.

1. Introduction

Let $\mathfrak{g}$ be a symmetrisable Kac–Moody Lie algebra and $\mathfrak{h}^*$ the dual of the Cartan subalgebra of $\mathfrak{g}$. If P_+ denotes the set of dominant integral weights, then the character of an irreducible $\mathfrak{g}$-module $V(\Lambda)$ of highest weight $\Lambda \in P_+$ is defined as

$$\operatorname{ch} V(\Lambda) = \sum_{\mu \in \mathfrak{h}^*} \dim(V_\mu)\, \mathrm{e}^{\mu} .$$

Here e^μ is a formal exponential and $\dim(V_\mu)$ the dimension of the weight space V_μ in the weight-space decomposition of $V(\Lambda)$. The celebrated Weyl–Kac formula gives a closed-form formula for the character of $V(\Lambda)$ as [27, 28]

$$\operatorname{ch} V(\Lambda) = \frac{\sum_{w\in W} \operatorname{sgn}(w)\, \mathrm{e}^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha>0}(1-\mathrm{e}^{-\alpha})^{\operatorname{mult}(\alpha)}}, \tag{1.1}$$

where W is the Weyl group of $\mathfrak{g}$, $\operatorname{sgn}(w)$ the signature of $w \in W$ and ρ the Weyl vector. The product over $\alpha > 0$ is shorthand for a product over the set of positive roots of $\mathfrak{g}$, and $\operatorname{mult}(\alpha)$ is the dimension of the root space corresponding to α. If $\mathfrak{g}$ is of classical type, then $\operatorname{mult}(\alpha) = 1$ and (1.1) simplifies to the Weyl character formula.

One feature of characters not evident from the Weyl–Kac formula is positivity, and a natural question is whether other closed-form expressions exist that are manifestly positive. The purpose of this paper is to show that for the affine Lie algebras $\mathrm{C}_n^{(1)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$, there is an affirmative answer to this question. The main player in these manifestly-positive formulas is the modified Hall–Littlewood polynomial Q'_μ indexed by the partition (as opposed to weight) μ. The Q'_μ is a

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symmetric function with nonnegative coefficients in $\mathbb{Z}[q]$ admitting a purely combinatorial description. For example, for $x=(x_1,\dots,x_n)$,

$$Q'_\mu(x;q)=\sum_{T\in\mathrm{Tab}(\cdot,\mu)} q^{\mathrm{c}(T)}s_{\mathrm{shape}(T)}(x)=\sum_\lambda K_{\lambda\mu}(q)s_\lambda(x), \tag{1.2}$$

where $\mathrm{Tab}(\lambda,\mu)$ is the set of semistandard Young tableaux of shape λ and weight μ, $s_\lambda(x)$ is the classical Schur function, $\mathrm{c}(T)$ the Lascoux–Schützenberger charge [37] and $K_{\lambda\mu}=\sum_{T\in\mathrm{Tab}(\lambda,\mu)}q^{\mathrm{c}(T)}$ the Kostka–Foulkes polynomial [14, 49].

To give an example of the type of results obtained in this paper we need some more notation. For λ a partition, let $|\lambda|=\sum_{i\geq 1}\lambda_i$ and $b_\lambda(q)=\prod_{i\geq 1}(q)_{m_i(\lambda)}$, where $m_i(\lambda)$ is the multiplicity of parts of size i in λ and $(q)_k=(1-q)\cdots(1-q^k)$. For example, if $\lambda=(4,4,2,1,1,1)=\big(4^2 2^1 1^3\big)$ then $b_\lambda(q)=(q)_2(q)_1(q)_3$. If all parts of λ are even we say that λ is even. Now define a second modified Hall–Littlewood polynomial P'_λ by

$$P'_\lambda(x;q)=Q'_\lambda(x;q)/b_\lambda(q), \tag{1.3}$$

so that its coefficients are in $\mathbb{Q}(q)$ with nonnegative power series expansion. For $\mathfrak{g}$ one of $\mathrm{C}_n^{(1)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$ with labelling of the Dynkin diagram as shown in Figure 2.1, let $\{\alpha_0,\dots,\alpha_n\}$, $\{\Lambda_0,\dots,\Lambda_n\}$ and $\{a_0,\dots,a_n\}$ be the set of simple roots, fundamental weights and marks of $\mathfrak{g}$, and let $\delta=\sum_{i=0}^n a_i\alpha_i$ be the null root. Finally, for $x=(x_1,\dots,x_n)$ define $f\big(x^{\pm}\big):=f(x_1,x_1^{-1},\dots,x_n,x_n^{-1})$.

Theorem 1.1. *Fix a nonnegative integer m and let*

$$q=\mathrm{e}^{-\delta} \quad\text{and}\quad x_i=\mathrm{e}^{-\alpha_i-\cdots-\alpha_{n-1}-\alpha_n/2}.$$

Then, for $\mathfrak{g}=\mathrm{C}_n^{(1)}$ and $\Lambda=m\Lambda_0$,

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)=\sum_{\substack{\lambda \text{ even}\\ \lambda_1\leq 2m}} q^{|\lambda|/2}P'_\lambda\big(x^{\pm};q\big) \tag{1.4a}$$

and, for $\mathfrak{g}=\mathrm{A}_{2n}^{(2)}$ and $\Lambda=2m\Lambda_0$,

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)=\sum_{\substack{\lambda\\ \lambda_1\leq 2m}} q^{|\lambda|/2}P'_\lambda\big(x^{\pm};q\big). \tag{1.4b}$$

We note the remarkable similarity between (1.4) and the following well-known Littlewood-type character identities for the classical groups C_n and B_n:

$$(x_1\cdots x_n)^m\,\mathrm{sp}_{2n,(m^n)}(x)=\sum_{\substack{\lambda \text{ even}\\ \lambda_1\leq 2m}} s_\lambda(x)$$

$$(x_1\cdots x_n)^m\,\mathrm{so}_{2n+1,(m^n)}(x)=\sum_{\substack{\lambda\\ \lambda_1\leq 2m}} s_\lambda(x),$$

where $\mathrm{sp}_{2n,\lambda}$ and $\mathrm{so}_{2n+1,\lambda}$ are the symplectic and odd orthogonal Schur functions (see (2.1) below), and where the second identity also allows for half-integer m. These identities have played an important role in the theory of plane partitions, see e.g., [10, 13, 32, 49, 62, 63, 68, 69].

The map $\exp(-\alpha_i)\mapsto 1$ for all $1\leq i\leq n$ (i.e., $x_i\mapsto 1$) is known as the basic specialisation [28]. Applied to Theorem 1.1, where on the left the Weyl–Kac expression (1.1) is used, leads to the following generalisations of Macdonald's $\mathrm{C}_n^{(1)}$ and $\mathrm{A}_{2n}^{(2)}$ (or affine BC_n) eta-function identities [48]. Let

$$\chi_{\mathrm{B}}(v) := \prod_{i=1}^{n} v_i \prod_{1\le i<j\le n}(v_i^2-v_j^2), \qquad \chi_{\mathrm{B}}(v/w)=\chi_{\mathrm{B}}(v)/\chi_{\mathrm{B}}(w), \tag{1.5}$$

and $(a)_\infty=(a;q)_\infty=(1-a)(1-aq)(1-aq^2)\cdots$.

Corollary 1.2. *Let m be a nonnegative integer and $\rho=(n,\dots,2,1)$ the C_n Weyl vector. Then*

$$\frac{1}{(q)_\infty^{2n^2+n}}\sum \chi_{\mathrm{B}}(v/\rho)\,q^{\frac{\|v\|^2-\|\rho\|^2}{4(m+n+1)}} = \sum_{\substack{\lambda \text{ even}\\ \lambda_1\le 2m}} q^{|\lambda|/2} P'_\lambda(\underbrace{1,\dots,1}_{2n \text{ times}};q), \tag{1.6a}$$

where the sum on the left is over $v\in\mathbb{Z}^n$ such that $v\equiv\rho \pmod{2m+2n+2}$, and

$$\frac{1}{(q^{1/2};q^{1/2})_\infty^{2n}(q^2;q^2)_\infty^{2n}(q)_\infty^{2n^2-3n}}\sum \chi_{\mathrm{B}}(v/\rho)\,q^{\frac{\|v\|^2-\|\rho\|^2}{2(2m+2n+1)}} = \sum_{\substack{\lambda\\ \lambda_1\le 2m}} q^{|\lambda|/2} P'_\lambda(\underbrace{1,\dots,1}_{2n \text{ times}};q), \tag{1.6b}$$

where the sum on the left is over $v\in\mathbb{Z}^n$ such that $v\equiv\rho \pmod{2m+2n+1}$.

Theorem 1.1 and similar combinatorial character formulae such as (5.8) (for the $\mathrm{A}_{2n}^{(2)}$-module $V(m\Lambda_n)$) and (5.10) (for the $\mathrm{D}_{n+1}^{(2)}$-module $V(2m\Lambda_0)$) only deal with a restricted set of weight $\Lambda\in P_+$. We believe however that the type of results obtained in this paper hold more generally. For example, computer experiments suggest that for $\mathrm{C}_n^{(1)}$ we have

$$\mathrm{e}^{-\Lambda_1}\,\mathrm{ch}\,V(\Lambda_1)=x_1\sum_{k=0}^{\infty}\frac{q^k}{(q)_k}\,Q'_{(2^k1)}\big(x^{\pm};q\big).$$

The remainder of this paper is organised as follows. In the next section, after reviewing some standard material from the theory of affine Kac–Moody algebras, we rewrite the Weyl–Kac formula (1.1) for $\mathfrak{g}=\mathrm{C}_n^{(1)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$ as a sum over symplectic or odd orthogonal Schur functions. In Section 3 we use Jing's vertex operators to prove a new basic hypergeometric formula for modified Hall–Littlewood polynomials P'_λ, and apply this to obtain a Littlewood-type summation formula for modified Hall–Littlewood polynomials. We further connect these results with Rogers–Ramanujan and Nahm–Zagier-type q-series. In Section 4 we employ the Milne–Lilly Bailey lemma for the C_n root system to prove a C_n analogue of Andrews' well-known multiple series transformation. Then, in Section 5, it is shown that after specialisation, and a somewhat intricate limiting procedure, one side of the C_n Andrews transformation corresponds to certain characters in their Weyl–Kac representation. Furthermore, applying the Littlewood-type summation formula from Section 3 we show that the other side is expressible in terms of P'_λ, resulting in a proof of our combinatorial character formulas. In Section 6 we provide a compendium to Macdonald's famous list of identities for powers of the Dedekind eta-function, extending his identities for affine B_n, C_n, D_n and BC_n to infinite families of such identities. Finally, in Section 7, we make some concluding remarks in response to questions posed by one of the referees. This includes a brief discussion of an alternative approach to combinatorial character identities recently developed by Eric Rains and the second author.

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2. Affine Kac–Moody algebras

In order to prove the main results of this paper, such as Theorem 1.1, we require a simple rewriting of the Weyl–Kac formula (1.1) for $\mathfrak{g}$ one of $\mathrm{C}_n^{(1)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$ in terms of the odd orthogonal and symplectic Schur functions [47]

$$\mathrm{so}_{2n+1,\lambda}(x)=\frac{\det_{1\le i,j\le n}\big(x_i^{j-1-\lambda_j}-x_i^{2n-j+\lambda_j}\big)}{\Delta_{\mathrm{B}}(x)}, \tag{2.1a}$$

$$\mathrm{sp}_{2n,\lambda}(x)=\frac{\det_{1\le i,j\le n}\big(x_i^{j-1-\lambda_j}-x_i^{2n-j+1+\lambda_j}\big)}{\Delta_{\mathrm{C}}(x)}. \tag{2.1b}$$

Here Δ_{B} and Δ_{C} are the generalised Vandermonde products

$$\Delta_{\mathrm{B}}(x):=\prod_{i=1}^n(1-x_i)\prod_{1\le i<j\le n}(x_i-x_j)(x_ix_j-1)$$
$$\Delta_{\mathrm{C}}(x):=\prod_{i=1}^n(1-x_i^2)\prod_{1\le i<j\le n}(x_i-x_j)(x_ix_j-1).$$

In Section 2.2 will give the full details of this rewrite for $\mathrm{C}_n^{(1)}$ and then state the remaining cases without proof.

First however, we need to recall some basic notions from the general theory of affine Kac–Moody algebras. For more details and background material we refer the reader to the monographs by Kac [28] and Wakimoto [71].

2.1. General definitions and notation.

Let $\mathfrak{g}=\mathfrak{g}(A)$ be an affine Kac–Moody algebra with generalised Cartan matrix $A=(a_{ij})_{i,j\in I}$, $I:=\{0,1,\dots,n\}$. We are primarily interested in $\mathfrak{g}$ of type $\mathrm{C}_n^{(1)}$ $(n\ge 1)$, $\mathrm{A}_{2n}^{(2)}$ $(n\ge 1)$ and $\mathrm{D}_{n+1}^{(2)}$ $(n\ge 2)$, although most of this section applies to arbitrary type. Let $\mathfrak{h}$ and $\mathfrak{h}^*$ be the $(n+2)$-dimensional Cartan subalgebra and its dual. Fix linearly independent elements $\alpha_0^\vee,\dots,\alpha_n^\vee$ and $\alpha_0,\dots,\alpha_n$ of $\mathfrak{h}$ and $\mathfrak{h}^*$, called simple coroots and simple roots, such that $\langle\alpha_i^\vee,\alpha_j\rangle=a_{ij}$. Extend the above to a basis of $\mathfrak{h}$ and $\mathfrak{h}^*$ by choosing the additional elements $d\in\mathfrak{h}$ and $\Lambda_0\in\mathfrak{h}^*$ such that $\langle\alpha_i^\vee,\Lambda_0\rangle=\langle d,\alpha_i\rangle=\delta_{i,0}$ and $\langle d,\Lambda_0\rangle=0$. The marks and comarks (also known as labels and colabels) $a_0,\dots,a_n$ and $a_0^\vee,\dots,a_n^\vee$ are positive integers, uniquely determined by $\sum_{i\in I}a_{ij}a_j=\sum_{i\in I}a_i^\vee a_{ij}=0$ such that

$$\gcd(a_0,\dots,a_n)=\gcd(a_0^\vee,\dots,a_n^\vee)=1.$$

The sum of the marks and comarks are known as the Coxeter and dual Coxeter number respectively, $h=\sum_{i\in I}a_i$ and $h^\vee=\sum_{i\in I}a_i^\vee$. The Dynkin diagrams of the three infinite series of interest are given in Figure 2.1, together with a labelling of the vertices by simple roots α_i and marks a_i.

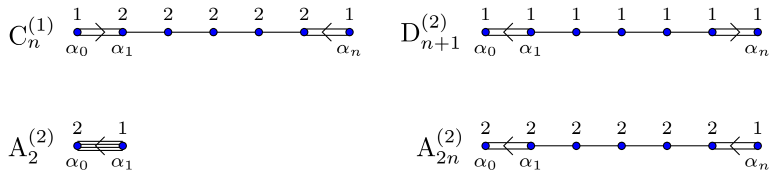

FIGURE 2.1. The Dynkin diagrams of the three infinite series of affine Lie algebras of interest, together with a labelling of vertices by simple roots and by the marks $a_0,\dots,a_n$. $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$ are dual and the comarks of $\mathfrak{g}$ are the marks of its dual. The comarks of $\mathrm{A}_{2n}^{(2)}$ are its marks read in reverse order.

We now fix what is known as the standard non-degenerate bilinear form on $\mathfrak{h}$ by setting

$$(\alpha_i^\vee|\alpha_j^\vee)=\frac{a_j}{a_j^\vee}\,a_{ij},\qquad (\alpha_i^\vee|d)=a_0\delta_{i,0},\qquad (d|d)=0.$$

We adopt the natural identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ by identifying d with $a_0\Lambda_0$ and $\alpha_i^\vee$ with $a_i\alpha_i/a_i^\vee$. Then

$$(\alpha_i|\alpha_j)=\frac{a_i^\vee}{a_i}\,a_{ij},\qquad (\alpha_i|\Lambda_0)=\frac{1}{a_0}\,\delta_{i,0},\qquad (\Lambda_0|\Lambda_0)=0.$$

Before we turn to the Weyl–Kac formula a few more definitions are needed. The null root or fundamental imaginary root δ is defined as $\delta=\sum_{i\in I}a_i\alpha_i$. Then $\mathfrak{h}^*=\mathbb{C}\Lambda_0\oplus\bar{\mathfrak{h}}^*\oplus\mathbb{C}\delta$ where $\overline{\mathfrak{h}}^*=\sum_{i\in\bar{I}}\mathbb{C}\alpha_i$ for $\bar{I}:=\{1,2,\dots,n\}$ is the finite part of $\mathfrak{h}^*$. We complement Λ_0 to a full set of fundamental weights $\Lambda_0,\dots,\Lambda_n\in\mathfrak{h}^*$ by

$$\langle\Lambda_i,\alpha_j^\vee\rangle=\delta_{ij},\qquad \langle\Lambda_i,d\rangle=0.$$

The Weyl vector $\rho\in\mathfrak{h}^*$ is given by $\langle\rho,\alpha_i^\vee\rangle=1$ for all $i\in I$ and $\langle\rho,d\rangle=0$. If K is the canonical central element $K=\sum_{i\in I}a_i^\vee\alpha_i^\vee$ then the level $\operatorname{lev}(\lambda)$ of $\lambda\in\mathfrak{h}^*$ is given by $\operatorname{lev}(\lambda)=\langle\lambda,K\rangle$. Note that $\operatorname{lev}(\Lambda_0)=1$ and $\operatorname{lev}(\rho)=h^\vee$.

The root and coroot lattices Q and $Q^\vee$ are defined by the integer span of the simple roots and simple coroots respectively. Similarly, $\overline{Q}=\sum_{i\in\bar{I}}\mathbb{Z}\alpha_i$ and $\overline{Q}^\vee=\sum_{i\in\bar{I}}\mathbb{Z}\alpha_i^\vee$. One further lattice that will play an important role is

$$M=\begin{cases}\overline{Q}^\vee & \text{if } \mathfrak{g}=\mathrm{X}_n^{(1)} \text{ or } \mathfrak{g}=\mathrm{A}_{2n}^{(2)},\\ \overline{Q} & \text{otherwise.}\end{cases}\tag{2.2}$$

To conclude our string of definitions we let P_+ denote the set of dominant integral weights

$$P_+=\{\lambda\in\mathfrak{h}^*:\ \langle\lambda,\alpha_i^\vee\rangle\in\mathbb{Z}_+:\ \text{for all } i\in I\},$$

where throughout this paper, $\mathbb{Z}_+$ denotes the set of nonnegative integers.

2.2. **The Weyl–Kac formula.** To achieve the desired rewriting of the Weyl–Kac formula we first follow Kac and Peterson [29]. Let $\overline{W}$ be the finite Weyl group corresponding to the Cartan matrix $\bar{A}$ obtained from A by deleting the zeroth row and column; $\bar{A}=(a_{ij})_{i,j\in\bar{I}}$. Then the affine Weyl group W of $\mathfrak{g}$ is given by

$W=\overline{W}\ltimes M$ with M the lattice (2.2). This allows (1.1) to be restated as

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)=\prod_{\alpha>0}(1-\mathrm{e}^{-\alpha})^{-\operatorname{mult}(\alpha)} \times\sum_{\gamma\in M}\sum_{w\in\overline{W}}\operatorname{sgn}(w)q^{\frac{1}{2}\kappa(\gamma|\gamma)-(\gamma|w(\bar\Lambda+\bar\rho))}\,\mathrm{e}^{-\kappa\gamma+w(\bar\Lambda+\bar\rho)-\bar\Lambda-\bar\rho}, \tag{2.3}$$

where $\kappa=\operatorname{lev}(\Lambda+\rho)=\operatorname{lev}(\Lambda)+h^{\vee}$, $q=\exp(-\delta)$ and where $\bar\lambda$ again denotes the finite part.

Next we focus on $\mathfrak{g}=\mathrm{C}_n^{(1)}$ with generalised Cartan matrix A given by the tridiagonal matrix with $d_{-1}=(-2,-1,\dots,-1)$, $d_0=(2,\dots,2)$ and $d_1=(-1,\dots,-1,-2)$. The set of positive roots Δ_+ consists of the disjoint subsets of positive imaginary and positive real roots, given by

$$\Delta_+^{\mathrm{im}}=\big\{m\delta:\ m\in\mathbb{Z}_+\setminus\{0\}\big\},$$

each root occurring with multiplicity n, and

$$\Delta_+^{\mathrm{re}}=\left\{m\delta+\alpha:\ \alpha\in\bar\Delta,\ m\in\begin{cases}\mathbb{Z}_+ & \text{if } \alpha\in\bar\Delta_+\\ \mathbb{Z}_+\setminus\{0\} & \text{otherwise}\end{cases}\right\},$$

of multiplicity 1. Here $\bar\Delta$ is the root system of $\mathfrak{g}(\bar A)$ with base $\overline{\Pi}$. In terms of the standard Euclidean description[1] of $\overline{\Pi}$ and $\bar\Delta_+=\bar\Delta_{s,+}\cup\bar\Delta_{\ell,+}$ we have

$$\overline{\Pi}=\{\alpha_1,\dots,\alpha_n\}=\{\epsilon_1-\epsilon_2,\dots,\epsilon_{n-1}-\epsilon_n,2\epsilon_n\}$$

and

$$\bar\Delta_{s,+}=\{\epsilon_i\pm\epsilon_j:\ 1\le i<j\le n\},\qquad \bar\Delta_{\ell,+}=\{2\epsilon_i:\ 1\le i\le n\}.$$

Setting $x_i=\exp(-\epsilon_i)$ we thus get

$$\prod_{\alpha>0}(1-\mathrm{e}^{-\alpha})^{\operatorname{mult}(\alpha)}=(q)_\infty^n\,\Delta_{\mathrm{C}}(x)\prod_{i=1}^n x_i^{1-i}(qx_i^{\pm 2})_\infty\prod_{1\le i<j\le n}(qx_i^{\pm}x_j^{\pm})_\infty,$$

where $(au^{\pm})_\infty=(au,au^{-1})_\infty$ and $(au^{\pm}v^{\pm})_\infty=(auv,auv^{-1},au^{-1}v,au^{-1}v^{-1})_\infty$ for $(a_1,\dots,a_k)_\infty=(a_1)_\infty\cdots(a_k)_\infty$.

Next we consider the numerator of (2.3). The lattice $M=\overline{Q}^{\vee}$ is spanned by

$$2\{\epsilon_1-\epsilon_2,\dots,\epsilon_{n-1}-\epsilon_n,\epsilon_n\},$$

i.e., M is the classical B_n root lattice scaled by a factor of two

$$M=\left\{2\sum_{i=1}^n r_i\epsilon_i:\ (r_1,\dots,r_n)\in\mathbb{Z}^n\right\}.$$

We also use that $\overline{W}$ is the hyperoctahedral group (or the group of signed permutations) $\overline{W}=\mathfrak{S}_n\ltimes(\mathbb{Z}/2\mathbb{Z})^n$ with natural action on $\mathbb{R}^n$, see e.g., [23]. Finally, for $\Lambda=c_0\Lambda_0+\dots+c_n\Lambda_n\in P_+$ define the partition $\lambda=(\lambda_1,\dots,\lambda_n)$ by $\lambda_i=c_i+\dots+c_n$. Hence, since $\bar\Lambda_i=\epsilon_1+\dots+\epsilon_i$, we have $\bar\Lambda+\bar\rho=\sum_{i=1}^n(\lambda_i+\rho_i)\epsilon_i$, where $\rho_i:=n-i+1$. Also note that

$$\kappa=\sum_{i=0}^n a_i^{\vee}(c_i+1)=h^{\vee}+c_0+\dots+c_n=n+1+c_0+\lambda_1.$$

[1] We deviate from Kac's convention that $(\alpha|\alpha)=2$ for α a long root. This comes at the cost of introducing the factor $1/2$ in $(\epsilon_i|\epsilon_j)=\delta_{ij}/2$ but avoids the occurrence of $\sqrt{2}$ in some of our formulae.

Therefore, the double sum in (2.3) yields

$$\sum_{r\in\mathbb{Z}^n}\sum_{w\in\overline{W}}\operatorname{sgn}(w)\prod_{i=1}^{n} q^{\kappa r_i^2-2r_i\sum_{j=1}^n(\lambda_j+\rho_j)(\epsilon_i|w(\epsilon_j))}x_i^{2\kappa r_i+\lambda_i+\rho_i}w\big(x_i^{-\lambda_i-\rho_i}\big)$$
$$=\sum_{r\in\mathbb{Z}^n}\prod_{i=1}^{n} q^{\kappa r_i^2}x_i^{2\kappa r_i+\lambda_i+\rho_i}\sum_{w\in\overline{W}}\operatorname{sgn}(w)w\Big(\prod_{i=1}^{n}y_i^{-\lambda_i-\rho_i}\Big),$$

where $y_i:=x_iq^{r_i}$. By (2.1b) the sum over $\overline{W}$ is given by

$$\Delta_{\mathrm{C}}(y)\operatorname{sp}_{2n,\lambda}(y)\prod_{i=1}^{n}y_i^{-n}$$

so that we obtain the next lemma.

Lemma 2.1 ($\mathrm{C}_n^{(1)}$ character formula)**.** *For $q=\exp(-\delta)$, $\lambda=(\lambda_1,\dots,\lambda_n)$ a partition and*

$$\Lambda=c_0\Lambda_0+(\lambda_1-\lambda_2)\Lambda_1+\cdots+(\lambda_{n-1}-\lambda_n)\Lambda_{n-1}+\lambda_n\Lambda_n\in P_+, \tag{2.4a}$$

$$x_i=\mathrm{e}^{-\alpha_i-\cdots-\alpha_{n-1}-\alpha_n/2}, \tag{2.4b}$$

we have

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)=\frac{1}{(q)_\infty^n\prod_{i=1}^n(qx_i^{\pm 2})_\infty\prod_{1\le i<j\le n}(qx_i^{\pm}x_j^{\pm})_\infty}\times\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^{n}q^{\kappa r_i^2-nr_i}x_i^{2\kappa r_i+\lambda_i}\operatorname{sp}_{2n,\lambda}\big(xq^r\big), \tag{2.5}$$

where $\kappa=n+1+c_0+\lambda_1$.

In much the same way we can rewrite the other characters of interest.

Lemma 2.2 ($\mathrm{A}_{2n}^{(2)}$ character formula, I)**.** *With the same assumptions as in Lemma 2.1,*

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)=\frac{1}{(q)_\infty^n\prod_{i=1}^n(q^{1/2}x_i^{\pm})_\infty(q^2x_i^{\pm 2};q^2)_\infty\prod_{1\le i<j\le n}(qx_i^{\pm}x_j^{\pm})_\infty}\times\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^{n}q^{\frac{1}{2}\kappa r_i^2-nr_i}x_i^{\kappa r_i+\lambda_i}\operatorname{sp}_{2n,\lambda}(xq^r), \tag{2.6}$$

where $\kappa=2n+1+c_0+2\lambda_1$.

Viewing the Dynkin diagram of $\mathrm{A}_{2n}^{(2)}$ in a mirror leads to an alternative, B-type expression for the above character.

Lemma 2.3 ($\mathrm{A}_{2n}^{(2)}$ character formula, II)**.** *For $q=\exp(-\delta)$, $\mu=(\mu_1,\dots,\mu_n)$ a partition or half-partition, and*

$$\Lambda=2\mu_n\Lambda_0+(\mu_{n-1}-\mu_n)\Lambda_1+\cdots+(\mu_1-\mu_2)\Lambda_{n-1}+c_n\Lambda_n\in P_+,$$

$$y_i=\mathrm{e}^{-\alpha_0-\cdots-\alpha_{n-i}},$$

(so that $y_i = q^{1/2}x_{n-i+1}^{-1}$ *and* $\mu_i = c_0/2 + \lambda_1 - \lambda_{n-i+1}$ *compared to* (2.6)*),*

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda) = \frac{1}{(q)_\infty^n \prod_{i=1}^n (qy_i^{\pm})_\infty (qy_i^{\pm 2};q^2)_\infty \prod_{1\le i<j\le n}(qy_i^{\pm}y_j^{\pm})_\infty} \\ \times \sum_{r\in\mathbb{Z}^n} \frac{\Delta_{\mathrm{B}}(yq^r)}{\Delta_{\mathrm{B}}(y)} \prod_{i=1}^n q^{\frac{1}{2}\kappa r_i^2-(n-\frac{1}{2})r_i} y_i^{\kappa r_i+\mu_i} \operatorname{so}_{2n+1,\mu}(yq^r),$$

where $\kappa = 2n+1+2c_n+2\mu_1$*.*

Here a half-partition $\mu = (\mu_1,\mu_2,\dots,\mu_n)$ is a sequence of weakly decreasing positive numbers such that $\mu_i + 1/2 \in \mathbb{Z}$ for all i.

Lemma 2.4 ($\mathrm{D}_{n+1}^{(2)}$ character formula)**.** *For* $q = \exp(-\delta)$*,* $\lambda = (\lambda_1,\dots,\lambda_n)$ *a partition or half-partition, and*

$$\Lambda = c_0\Lambda_0 + (\lambda_1-\lambda_2)\Lambda_1 + \dots + (\lambda_{n-1}-\lambda_n)\Lambda_{n-1} + 2\lambda_n\Lambda_n \in P_+, \tag{2.7a}$$

$$x_i = \mathrm{e}^{-\alpha_i-\dots-\alpha_n}, \tag{2.7b}$$

we have

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda) = \frac{1}{(q^2;q^2)_\infty^{n-1}(q)_\infty \prod_{i=1}^n (qx_i^{\pm})_\infty \prod_{1\le i<j\le n}(q^2x_i^{\pm}x_j^{\pm};q^2)_\infty} \\ \times \sum_{r\in\mathbb{Z}^n} \frac{\Delta_{\mathrm{B}}(xq^{2r})}{\Delta_{\mathrm{B}}(x)} \prod_{i=1}^n q^{\kappa r_i^2-(2n-1)r_i} x_i^{\kappa r_i+\lambda_i} \operatorname{so}_{2n+1,\lambda}\big(xq^{2r}\big),$$

where $\kappa = 2n+c_0+2\lambda_1$*.*

3. Modified Hall–Littlewood polynomials

3.1. **Preliminaries.** The Hall–Littlewood polynomials are an important family of symmetric functions generalising the well-known Schur functions. Our main interest will be the modified Hall–Littlewood polynomials, for which we shall give a new, closed-form formula as a multiple basic hypergeometric series. It is this formula that will ultimately allow us to express characters of affine Lie algebras in terms of modified Hall–Littlewood polynomials.

For standard notation and terminology from the theory of partitions and symmetric functions we refer the reader to [49].

Fix a positive integer n. For a partition λ of length $l(\lambda) \le n$ let $m_0(\lambda) = n - l(\lambda)$ and $m_i(\lambda)$ for $i \ge 1$ the multiplicity of parts of size i. Define $v_\lambda(q) = \prod_{i\ge 0}(q)_{m_i(\lambda)}/(1-q)^{m_i(\lambda)}$. If $\mathfrak{S}_n$ denotes the symmetric group on n letters and $\mathfrak{S}_n^\lambda$ the stabilizer of λ, then $v_\lambda(q)$ may be identified as the Poincaré polynomial $\sum_{w\in\mathfrak{S}_n^\lambda} t^{\ell(w)}$. For $x = (x_1,\dots,x_n)$ the Hall–Littlewood polynomial P_λ is the symmetric function [49]

$$P_\lambda(x;q) = \frac{1}{v_\lambda(q)} \sum_{w\in\mathfrak{S}_n} w\bigg(x^\lambda \prod_{i<j}\frac{x_i-qx_j}{x_i-x_j}\bigg) = \sum_{w\in\mathfrak{S}_n/\mathfrak{S}_n^\lambda} w\bigg(x^\lambda \prod_{\lambda_i>\lambda_j}\frac{x_i-qx_j}{x_i-x_j}\bigg).$$

Here the symmetric group $\mathfrak{S}_n$ acts on functions $f(x)$ by permuting the x_i.

The Hall–Littlewood polynomial P_λ interpolates between the Schur function s_λ and the monomial symmetric function m_λ, corresponding to $q = 0$ and $q = 1$

respectively. The P_λ, where λ ranges over all partitions of length at most n, form a basis of the ring of symmetric functions in n variables. There is a second Hall–Littlewood polynomial defined as

$$Q_\lambda(x;q) = b_\lambda(q) P_\lambda(x;q), \tag{3.1}$$

where $b_\lambda(q) = \prod_{i\geq 1}(q)_{m_i(\lambda)} = \prod_{i\geq 1}(q)_{\lambda'_i-\lambda'_{i+1}}$, for λ' the conjugate of λ.

The modified Hall–Littlewood polynomial Q'_λ of equation (1.2) is a variant of Q_λ which interpolates between the Schur function s_λ, obtained for $q=0$, and the complete symmetric function h_λ, obtained for $q=1$. Unlike the literature on the ordinary Hall–Littlewood polynomials, where the pair P_λ and Q_λ are usually given equal prominence, the polynomial P'_λ defined in (1.3) usually does not feature in work on the modified polynomials, see e.g., [14, 16, 17, 30, 36, 56]. There are a number of reasons for this. Q'_λ has coefficients in $\mathbb{Z}[q]$, is Schur positive, and has several combinatorial, representation theoretic and geometric interpretations. P'_λ on the other hand, has coefficients in $\mathbb{Q}(q)$ and its $q\to 1$ limit does not exist due to $b_\lambda(1) = \delta_{\lambda,0}$. Nonetheless, most of our results are simplest when expressed in terms of the P'_λ and we will use the two families of modified polynomials interchangeably.

Besides (1.2) there exist numerous other descriptions of the modified Hall–Littlewood polynomials, three of which will be discussed below. First of all, using the notation of λ-rings [22, 35],

$$Q'_\lambda(x;q) = Q_\lambda(x/(1-q);q) \quad \text{and} \quad P'_\lambda(x;q) = P_\lambda(x/(1-q);q),$$

where $x/(1-q)$ is shorthand for the infinite alphabet obtained from x be replacing each x_i by $x_i, x_iq, x_iq^2, \dots$. A second description of the modified Hall–Littlewood polynomials uses the Hall inner product on the ring of symmetric functions, defined by $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$. Then

$$\langle P_\lambda, Q'_\mu\rangle = \langle P'_\lambda, Q_\mu\rangle = \delta_{\lambda\mu}.$$

Finally, and most important for our purposes, the Q'_λ can be computed using Jing's q-Bernstein operators [25] (see also [16, 75]). Let Λ be the ring of symmetric functions. For $f\in\Lambda$, denote by $f^\perp \in \mathrm{End}(\Lambda)$ the operator (also known as Foulkes derivative) which acts as the adjoint of multiplication by f:

$$\langle f^\perp(g), h\rangle = \langle g, fh\rangle \quad \text{for } g,h\in\Lambda.$$

For m an integer the q-Bernstein operator $B_m = B_m(x;q)$ is defined as

$$B_m = \sum_{r,s=0}^{\infty} (-1)^r q^s h_{m+r+s}(x) e_r^\perp h_s^\perp = \sum_{r=0}^{\infty} h_{m+r}(x) h_r^\perp\big(x(q-1)\big),$$

where h_r and e_r are the rth complete and elementary symmetric functions, and where the rightmost expression again uses λ-rings. Alternatively, if $B(z) = B(z;x;q)$ is the vertex operator $B(z) = \sum_m z^m B_m$, then

$$B(z)(f) = f\Big(x - \frac{1-q}{z}\Big)\prod_{i\geq 1}\frac{1}{1-zx_i}.$$

For a partition $\lambda = (\lambda_1,\dots,\lambda_k)$ Jing [25] showed that

$$Q'_\lambda(x;q) = B_{\lambda_1}\cdots B_{\lambda_k}(1), \tag{3.2}$$

or, equivalently, $Q'_0(x;q) = 1$ and

$$Q'_\nu(x;q) = B_m\big(Q'_\lambda(x;q)\big), \tag{3.3}$$

where $\nu=(m,\lambda_1,\lambda_2,\dots,\lambda_k)$ for $m\geq\lambda_1$. We note that although B_0 is not the identity operator, $B_0(1)=1$ so that (3.2) is true regardless of whether $l(\lambda)=k$ or $l(\lambda)<k$. In [16] Garsia expressed the B_m in more explicit form as

$$B_m(x;q)=\sum_{i=1}^n x_i^m\Biggl(\prod_{\substack{j=1\\ j\neq i}}^n\frac{x_i}{x_i-x_j}\Biggr)T_{q,x_i}, \tag{3.4}$$

where $(T_{q,x_i}f)(x)=f(x_1,\dots,x_{i-1},qx_i,x_{i+1},\dots,x_n)$. It is this representation that will be important in proving our hypergeometric formula for P'_λ.

3.2. **The modified Hall–Littlewood polynomial as q-hypergeometric multisum.** Define the q-shifted factorial $(a)_n=(a;q)_n$ indexed by an arbitrary integer n as $(a)_n=(a)_\infty/(aq^n)_\infty$, where $(a)_\infty=(1-a)(1-aq)\cdots$. For $r,s\in\mathbb{Z}_+^n$, τ an integer and $x=(x_1,\dots,x_n)$ we define the q-hypergeometric term

$$f_{r,s}^{(\tau)}(x;q):=\prod_{i=1}^n\Bigl(x_i^{r_i}q^{\binom{r_i}{2}}\Bigr)^\tau\prod_{i,j=1}^n\frac{(qx_i/x_j)_{r_i-r_j}}{(qx_i/x_j)_{r_i-s_j}}. \tag{3.5}$$

Since $1/(q)_n=0$ for $n<0$, it follows that $f_{r,s}^{(\tau)}(x;q)=0$ unless $r_i\geq s_i$ for all $1\leq i\leq n$, or more succinctly, $r\supseteq s$ for r and s viewed as compositions.

Theorem 3.1. *The modified Hall–Littlewood polynomial P'_λ is given by*

$$P'_\lambda(x;q)=\sum\prod_{\ell\geq1}f^{(1)}_{r^{(\ell)},r^{(\ell+1)}}(x;q), \tag{3.6}$$

where the sum is over $r^{(1)}\supseteq r^{(2)}\supseteq\cdots\in\mathbb{Z}_+^n$ such that $|r^{(\ell)}|=\lambda'_\ell$.

Of course, since $|r^{(\ell)}|=0$ for $\ell>l(\lambda')=\lambda_1$, all $r^{(\ell)}$ for $\ell>\lambda_1$ are equal to $0:=(0^n)$ and the product $\prod_{\ell\geq1}$ may be replaced by a finite product $\prod_{\ell=1}^m$ where m is an integer such that $m\geq\lambda_1$.

Proof of Theorem 3.1. Throughout the proof we write P_λ, $f_{r,s}$ and b_λ for $P_\lambda(x;q)$, $f_{r,s}(x;q)$ and $b_\lambda(q)$.

For $\lambda=0$ all $r^{(\ell)}$ in (3.6) are equal to 0, resulting in $P'_0=1$.

It remains to show that for $\lambda\neq0$ our theorem is consistent with the action of the q-Bernstein operators. Before we do so, we translate (3.3) into a statement for P'_λ instead of Q'_λ. First, by (1.3), we get $b_\lambda B_m\bigl(P'_\lambda\bigr)=b_\nu P'_\nu$. But, since $\nu=(m,\lambda_1,\dots,\lambda_k)$ with $m\geq\lambda_1$, we have $b_\nu/b_\lambda=(1-q^{\lambda'_m+1})$. Hence, for $m\geq1$,

$$B_m\bigl(P'_\lambda\bigr)=\bigl(1-q^{\lambda'_m+1}\bigr)P'_\nu. \tag{3.7}$$

We now compute the left-hand side of (3.7) using the claimed expression for P'_λ. Let m be an integer such that $m\geq\lambda_1$. Recalling the remark after Theorem 3.1, we replace the product in (3.6) by $\prod_{\ell=1}^m$ and sum over $r^{(1)}\supseteq\cdots\supseteq r^{(m)}\in\mathbb{Z}_+^n$ such that $|r^{(\ell)}|=\lambda'_\ell$ for $\ell=1,\dots,m$, and $r^{(m+1)}:=0$. By a simple calculation it follows that

$$T_{q,x_i}\bigl(f_{r,s}^{(\tau)}\bigr)=x_i^{-\tau}f^{(\tau)}_{r+\epsilon_i,s+\epsilon_i},$$

where ϵ_i is the ith standard unit vector in $\mathbb{Z}^n$. Hence

$$T_{q,x_i}\bigl(P'_\lambda\bigr)=x_i^{-m}\sum\prod_{\ell=1}^m f^{(1)}_{r^{(\ell)}+\epsilon_i,r^{(\ell+1)}+\epsilon_i}.$$

Making the variable change $r^{(\ell)} \mapsto r^{(\ell)} - \epsilon_i$ for $\ell = 1, \dots, m$ while recalling that $r^{(m+1)} := 0$, this yields

$$T_{q,x_i}\big(P'_\lambda\big) = x_i^{-m} \sum \bigg(\prod_{\ell=1}^{m-1} f^{(1)}_{r^{(\ell)},r^{(\ell+1)}} \bigg) f^{(1)}_{r^{(m)},\epsilon_i} \tag{3.8a}$$

$$= x_i^{-m} \sum \prod_{j=1}^{n} \big(1 - q^{r_j^{(m)}} x_j/x_i\big) \prod_{\ell=1}^{m} f^{(1)}_{r^{(\ell)},r^{(\ell+1)}}, \tag{3.8b}$$

where the second equality follows from

$$f^{(\tau)}_{r,\epsilon_i}(x;q) = f^{(\tau)}_{r,0}(x;q) \prod_{j=1}^{n} \big(1 - q^{r_j} x_j/x_i\big).$$

Both sums in (3.8) are over $r^{(1)} \supseteq \dots \supseteq r^{(m)} \in \mathbb{Z}_+^n$ such that $|r^{(\ell)}| = \lambda'_\ell + 1$ for $\ell = 1, \dots, m$, and $r^{(m+1)} := 0$. (The variable change actually leads to $r^{(m)} \supseteq \epsilon_i$ but this may be relaxed to $r^{(m)} \supseteq 0$ since the summands vanish when $r_i^{(m)} = 0$.) Therefore, by (3.4),

$$B_m\big(P'_\lambda\big) = \sum \bigg(\prod_{\ell=1}^{m} f^{(1)}_{r^{(\ell)},r^{(\ell+1)}} \bigg) \sum_{i=1}^{n} \big(1 - q^{r_i^{(m)}}\big) \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{x_i - q^{r_j^{(m)}} x_j}{x_i - x_j}.$$

Recalling the summation [55, Lemma 1.33]

$$\sum_{i=1}^{n} (1 - y_i) \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{x_i - y_j x_j}{x_i - x_j} = 1 - y_1 \cdots y_n$$

and using that $q^{|r^{(m)}|} = q^{\lambda'_m + 1}$, we finally arrive at

$$B_m\big(P'_\lambda\big) = (1 - q^{\lambda'_m + 1}) \sum \prod_{\ell=1}^{m} f^{(1)}_{r^{(\ell)},r^{(\ell+1)}}, \tag{3.9}$$

summed over $r^{(1)} \supseteq \dots \supseteq r^{(m)} \in \mathbb{Z}_+^n$ such that $|r^{(\ell)}| = \lambda'_\ell + 1$ for $\ell = 1, \dots, m$.

To complete the proof we note that if we introduce the new partition $\nu = (m, \lambda_1, \lambda_2, \dots)$ then $\nu'_\ell = \lambda'_\ell + 1$ for $\ell = 1, \dots, m$ (and $\nu'_\ell = \lambda'_\ell = 0$ for $\ell > m$). Hence the sum on the right of (3.9) yields exactly P'_ν, resulting in (3.7). □

The hypergeometric formula (3.6) may be restated by eliminating redundant summation indices; since $r^{(1)} \supseteq r^{(2)} \supseteq \dots \in \mathbb{Z}_+^n$ such that $|r^{(l)}| = \lambda'_l$, it follows that $r^{(l)} = r^{(l+1)}$ if $\lambda'_l = \lambda'_{l+1}$. But $f^{(1)}_{r,r} f^{(\tau)}_{r,s} = f^{(\tau+1)}_{r,s}$ so that we obtain the following equivalent formulation.

Corollary 3.2. *Let $\lambda' = (M_1^{\tau_1} M_2^{\tau_2} \dots M_m^{\tau_m})$ for $M_1 \geq M_2 \geq \dots \geq M_m \geq 0$ and $\tau_1, \dots, \tau_m > 0$. Then*

$$P'_\lambda(x;q) = \sum \prod_{\ell=1}^{m} f^{(\tau_\ell)}_{r^{(\ell)},r^{(\ell+1)}}(x;q), \tag{3.10}$$

where the sum is over $r^{(1)} \supseteq \dots \supseteq r^{(m)} \in \mathbb{Z}_+^n$ such that $|r^{(\ell)}| = M_\ell$, and $r^{(m+1)} := 0$.

For $m = 1$ this simplifies to Milne's expression for P'_λ indexed by a rectangular partition λ, as implied by equating (2.7) and (2.17) of [57]. To compute P'_λ as efficiently as possible we should take $M_1 > M_2 > \cdots > M_m > 0$. The result, however, is true if some of the M_i are equal and/or zero (in which case further summation indices may be eliminated). The advantage of the stated form is that for $\tau_1 = \tau_2 = \cdots = \tau_m = 1$ we recover Theorem 3.1 provided we rename M_i as λ'_i.

3.3. A Littlewood identity for modified Hall–Littlewood polynomials.

In this section we give an important application of Theorem 3.1, key in proving our combinatorial character formulas.

To state our main result we first need the definition of the Rogers–Szegő polynomials. For m a nonnegative integer, the mth Rogers–Szegő polynomial H_m is given by [3]

$$H_m(z;q) = \sum_{i=0}^{m} z^i \begin{bmatrix} m \\ i \end{bmatrix}, \tag{3.11}$$

where $\begin{bmatrix} m \\ i \end{bmatrix}$ is a q-binomial coefficient. Following [73] we extend the above to partitions by

$$h_\lambda(z;q) = \prod_{i\geq 1} H_{m_i(\lambda)}(z;q).$$

Let $\begin{bmatrix} \infty \\ k \end{bmatrix} := 1/(q)_k$ and let λ_{o} denote the partition containing the odd-sized parts of λ. For example, if $\lambda = (6,4,3,3,2,1,1,1)$ then $\lambda_{\mathrm{o}} = (3,3,1,1,1)$.

Theorem 3.3. *For $M = (M_1,\dots,M_m) \in \mathbb{Z}_+^m$ and $m_0(\lambda) := \infty$*

$$\begin{aligned} \sum_{\substack{\lambda \\ \lambda_1\leq 2m}} z^{\ell(\lambda_{\mathrm{o}})} P'_\lambda(x;q) h_{\lambda_{\mathrm{o}}}(w/z;q) \prod_{\ell=1}^{m} (wz)^{M_\ell-\lambda'_{2\ell-1}} \begin{bmatrix} m_{2\ell-2}(\lambda) \\ M_\ell - \lambda'_{2\ell-1} \end{bmatrix} \\ = \sum \prod_{i=1}^{n} \big({-q^{1-r_i^{(1)}}} w/x_i, -q^{1-r_i^{(1)}} z/x_i\big)_{r_i^{(1)}} \prod_{\ell=1}^{m} f^{(2)}_{r^{(\ell)},r^{(\ell+1)}}(x;q), \end{aligned} \tag{3.12}$$

where the sum on the right is over $r^{(1)},\dots,r^{(m)} \in \mathbb{Z}_+^n$ such that $|r^{(\ell)}| = M_\ell$, and $r^{(m+1)} := 0$.

For actual applications as well as aesthetic reasons we should sum this over the sequence M. To this end we introduce the generalised Rogers–Szegő polynomial

$$h_\lambda^{(m)}(w,z;q) = \prod_{\substack{i=1 \\ i \text{ odd}}}^{2m-1} z^{m_i(\lambda)} H_{m_i(\lambda)}(w/z;q) \prod_{\substack{i=1 \\ i \text{ even}}}^{2m-1} H_{m_i(\lambda)}(wz;q).$$

For example, if $\lambda = (6,4,3,3,2,1,1,1)$ and $m = 3$ then

$$h_\lambda^{(2)}(w,z;q) = z^5 H_1^2(wz;q) H_2(w/z;q) H_3(w/z;q).$$

From $H_m(z;q) = z^m H_m(z^{-1};q)$ it is easily seen that $h_\lambda^{(m)}(w,z;q) = h_\lambda^{(m)}(z,w;q)$. Now taking the M-sum in (3.12), interchanging the sums over λ and M, shifting $M_\ell \to M_\ell + \lambda'_{2\ell-1}$ and finally performing the M_1-sum using the q-binomial theorem [3, Eq. (2.2.5)], (3.12) simplifies to the following identity.

Corollary 3.4 (Littlewood-type identity)**.** *Let $|wz|<1$. Then*

$$\sum_{\substack{\lambda\\ \lambda_1\le 2m}} h_\lambda^{(m)}(w,z;q)P'_\lambda(x;q) \tag{3.13}$$
$$= (wz)_\infty \sum \prod_{i=1}^n \big(-q^{1-r_i^{(1)}}w/x_i, -q^{1-r_i^{(1)}}z/x_i\big)_{r_i^{(1)}} \prod_{\ell=1}^m f^{(2)}_{r^{(\ell)},r^{(\ell+1)}}(x;q),$$

where the sum on the right is over $r^{(1)},\dots,r^{(m)}\in\mathbb{Z}_+^n$, and $r^{(m+1)}:=0$.

It will be convenient later to also consider (3.13) for $m=0$. For this purpose we define $h_0^{(0)}(w,z;q)=(wz)_\infty$, so that for $m=0$ both sides trivialise to $(wz)_\infty$.

For $z=0,-1,-q^{\pm1},q^{\pm1/2}$ the Rogers–Szegő polynomial (3.11) completely factorises. In terms of $h_\lambda^{(m)}(w,z;q)$ (and up to symmetry) this corresponds to $(w,z)=(0,z),(1,q^{1/2}),(-1,-q^{1/2}),(q^{1/2},-q^{1/2})$, the case $(w,z)=(1,-1)$ being ruled out for convergence reasons. Surprisingly, it is precisely these special cases that correspond to characters of affine Lie algebras.

Before proving Theorem 3.3 we prepare three simple identities satisfied by the q-hypergeometric term $f^{(\tau)}_{r,s}(x;q)$.

Proposition 3.5. *Let $N=(N_1,\dots,N_n)\in\mathbb{Z}^n$, $s=(s_1,\dots,s_n)\in\mathbb{Z}^n$ such that $s\subseteq N$, and let $M\ge|s|$ be an integer. Then*

$$\sum_{r\in\mathbb{Z}^n} f^{(\tau)}_{N,r}(x;q)f^{(1)}_{r,s}(x;q) = f^{(\tau)}_{N,s}(x;q)\prod_{i=1}^n x_i^{s_i}q^{\binom{s_i}{2}}\frac{(-x_i)_{N_i}}{(-x_i)_{s_i}}, \tag{3.14a}$$

$$\sum_{\substack{r\in\mathbb{Z}^n\\ |r|=M}} f^{(\tau)}_{N,r}(x;q)f^{(0)}_{r,s}(x;q) = f^{(\tau)}_{N,s}(x;q)\begin{bmatrix}|N|-|s|\\ M-|s|\end{bmatrix}, \tag{3.14b}$$

$$\sum_{r\in\mathbb{Z}^n} z^{|r|}f^{(\tau)}_{N,r}(x;q)f^{(0)}_{r,s}(x;q) = z^{|s|}f^{(\tau)}_{N,s}(x;q)\,H_{|N|-|s|}(z;q). \tag{3.14c}$$

Note that in all three cases we may restrict the sum over r to $s\subseteq r\subseteq N$.

Proof. We first prove (3.14a). Shifting the summation index $r\mapsto r+s$ and using that $f^{(1)}_{s,s}(x;q)=\prod_i x_i^{s_i}q^{\binom{s_i}{2}}$, we get

$$\sum_{r\in\mathbb{Z}^n}\frac{f^{(\tau)}_{N,r+s}(x;q)f^{(1)}_{r+s,s}(x;q)}{f^{(\tau)}_{N,s}(x;q)f^{(1)}_{s,s}(x;q)} = \frac{(-x_i)_{N_i}}{(-x_i)_{s_i}}.$$

Replacing $N\mapsto N+s$ followed by $x\mapsto -xq^{-|N|-s}$, and then using

$$f^{(\tau)}_{N,r+s}(x;q) = f^{(\tau)}_{s,s}(x;q)f^{(\tau)}_{N,r}(xq^s;q) \tag{3.15}$$

and $(aq)_{n+k}=(aq)_k(aq^k)_n$, the s dependence drops out and the resulting identity can be recognised as Milne's terminating q-binomial theorem [58, Theorem 5.46]

$${}_1\Phi_0\big(q^{-N};-;q,x\big)=\prod_{i=1}^n(q^{-|N|}x_i)_{N_i},$$

where $N_1,\dots,N_n\geq 0$ and

$$ {}_1\Phi_0\big(q^{-N};-;q,x\big):=\sum_{r\in\mathbb{Z}_+^n}\frac{f^{(0)}_{N,r}\big(xq^{-|N|};q\big)f^{(1)}_{r,0}\big(xq^{-|N|};q\big)}{f^{(0)}_{N,0}\big(xq^{-|N|};q\big)}. \tag{3.16}$$

To prove the second claim we proceed in almost identical fashion. We first write (3.14b) as

$$\sum_{\substack{r\in\mathbb{Z}^n\\|r|=M-|s|}}\frac{f^{(\tau)}_{N,r+s}(x;q)f^{(0)}_{r+s,s}(x;q)}{f^{(\tau)}_{N,s}(x;q)}=\begin{bmatrix}|N|-|s|\\M\ -|s|\end{bmatrix}$$

and then make substitutions $M\mapsto M+|s|$, $N\mapsto N+s$ and $x\mapsto xq^{-s}$. By (3.15) this yields

$$\sum_{\substack{r\in\mathbb{Z}^n\\|r|=M}}\frac{f^{(0)}_{N,r}(x;q)f^{(0)}_{r,0}(x;q)}{f^{(0)}_{N,0}(x;q)}=\begin{bmatrix}|N|\\M\end{bmatrix}$$

which again is independent of s. By the easy to verify

$$f^{(0)}_{r,0}(x;q)=(-1)^{|r|}q^{-\binom{|r|}{2}}\prod_{1\leq i<j\leq n}\frac{x_iq^{r_i}-x_jq^{r_j}}{x_i-x_j}\prod_{i,j=1}^n q^{\binom{r_j}{2}}\Big(-\frac{x_i}{x_j}\Big)^{r_i}\frac{1}{(qx_i/x_j)_{r_i}}$$

and

$$\frac{f^{(0)}_{N,r}(x;q)}{f^{(0)}_{N,0}(x;q)}=q^{|N||r|}\prod_{i,j=1}^n q^{-\binom{r_i}{2}}\Big(-\frac{x_j}{x_i}\Big)^{r_i}(q^{-N_j}x_i/x_j)_{r_i},$$

this is Milne's [54, Theorem 1.49]

$$\sum_{\substack{r\in\mathbb{Z}^n\\|r|=M}}\prod_{1\leq i<j\leq n}\frac{x_iq^{r_i}-x_jq^{r_j}}{x_i-x_j}\prod_{i,j=1}^n\frac{(a_jx_i/x_j)_{r_i}}{(qx_i/x_j)_{r_i}}=\frac{(a_1\cdots a_n)_M}{(q)_M}$$

for $a_{ii}\mapsto q^{-N_i}$.

Finally, (3.14c) follows after multiplying both sides of (3.14b) by z^M and then summing over M using

$$\sum_{M=|s|}^\infty z^M\begin{bmatrix}|N|-|s|\\M\ -|s|\end{bmatrix}=z^{|s|}\sum_{M=0}^\infty z^M\begin{bmatrix}|N|-|s|\\M\end{bmatrix}=z^{|s|}H_{|N|-|s|}(z;q). \qquad\square$$

We are now prepared to prove Theorem 3.3.

Proof. We will show how to transform the left-hand side of (3.12)—denoted below by LHS—into the right-hand side.

As a first step we apply Theorem 3.1 with λ a partition such that $\lambda_1\leq 2m$, and replace $(r_{2\ell-1},r_{2\ell})\mapsto(u_\ell,v_\ell)$ for all $\ell=1,\dots,m$. This yields

$$P'_\lambda(x;q)=\sum\prod_{\ell=1}^m f^{(1)}_{u^{(\ell)},v^{(\ell)}}(x;q)f^{(1)}_{v^{(\ell)},u^{(\ell+1)}}(x;q), \tag{3.17}$$

summed over $u^{(1)}\supseteq v^{(1)}\supseteq\cdots\supseteq u^{(m)}\supseteq v^{(m)}\in\mathbb{Z}_+^n$ such that $|u^{(\ell)}|=\lambda'_{2\ell-1}$ and $|v^{(\ell)}|=\lambda'_{2\ell}$ (and as usual, $u^{(m+1)}:=0$). Also using

$$h_{\lambda_{\mathrm{o}}}(w/z;q)=\prod_{\ell=1}^m H_{m_{2\ell-1}(\lambda)}(w/z;q),$$

as well as $l(\lambda_{\mathrm{o}})=\sum_{i=1}^m m_{2\ell-1}(\lambda)$ and $m_i(\lambda)=\lambda'_i-\lambda'_{i+1}$, we obtain

$$\text{LHS}=\sum_{\{u^{(\ell)},v^{(\ell)}\}}\prod_{\ell=1}^m\bigg\{w^{M_\ell-|u^{(\ell)}|}z^{M_\ell-|v^{(\ell)}|}\begin{bmatrix}|v^{(\ell-1)}|-|u^{(\ell)}|\\ M_\ell-|u^{(\ell)}|\end{bmatrix}\\ \times H_{|u^{(\ell)}|-|v^{(\ell)}|}(w/z;q)f^{(1)}_{u^{(\ell)},v^{(\ell)}}(x;q)f^{(1)}_{v^{(\ell)},u^{(\ell+1)}}(x;q)\bigg\}, \tag{3.18}$$

where $\sum_{\{u^{(\ell)},v^{(\ell)}\}}$ is shorthand for a sum over $u^{(1)}\supseteq v^{(1)}\supseteq\cdots\supseteq u^{(m)}\supseteq v^{(m)}\in\mathbb{Z}^n_+$. In the above $|v^{(0)}|$ should be interpreted as ∞. Concerning this occurrence of ∞ in one of the q-binomial coefficients, we remark that although $\lim_{N\to(\infty^n)}f_{N,r}(x;q)$ does not exist, $\lim_{N\to(\infty^n)}f_{N,r}(x;q)/f_{N,s}(x;q)$ does and is given by 1. In the next step of our proof we write, by abuse of notation,

$$\sum_{\substack{r\in\mathbb{Z}^n\\|r|=M}}f^{(0)}_{r,s}(x;q)=\frac{1}{(q)_{M-|s|}}$$

as

$$\sum_{\substack{r\in\mathbb{Z}^n\\|r|=M}}f^{(\tau)}_{(\infty^n),r}(x;q)f^{(0)}_{r,s}(x;q)=f^{(\tau)}_{(\infty^n),s}(x;q)\begin{bmatrix}\infty\\ M-|s|\end{bmatrix}.$$

With this in mind we we apply (3.14b) and (3.14c) with $\tau=1$ to expand (3.18) as

$$\text{LHS}=\sum_{\substack{\{r^{(\ell)},s^{(\ell)},u^{(\ell)},v^{(\ell)}\}\\|r^{(\ell)}|=M_\ell}}\prod_{\ell=1}^m\bigg\{w^{M_\ell+|s^{(\ell)}|-|u^{(\ell)}|-|v^{(\ell)}|}z^{M_\ell-|s^{(\ell)}|}\\ \times f^{(0)}_{r^{(\ell)},u^{(\ell)}}(x;q)f^{(1)}_{u^{(\ell)},s^{(\ell)}}(x;q)f^{(0)}_{s^{(\ell)},v^{(\ell)}}(x;q)f^{(1)}_{v^{(\ell)},r^{(\ell+1)}}(x;q)\bigg\},$$

where $\sum_{\{r^{(\ell)},s^{(\ell)},u^{(\ell)},v^{(\ell)}\}}$ stands for a sum over

$$r^{(1)}\supseteq u^{(1)}\supseteq s^{(1)}\supseteq v^{(1)}\supseteq\cdots\supseteq r^{(m)}\supseteq u^{(m)}\supseteq s^{(m)}\supseteq v^{(m)}\in\mathbb{Z}^n_+.$$

By

$$f^{(\tau)}_{N,r}(ax;q)=a^{|N|\tau}f^{(\tau)}_{N,r}(x;q) \tag{3.19}$$

for a a scalar, this is also

$$\text{LHS}=\sum_{\substack{\{r^{(\ell)},s^{(\ell)},u^{(\ell)},v^{(\ell)}\}\\|r^{(\ell)}|=M_\ell}}\prod_{\ell=1}^m\bigg\{w^{M_\ell+|s^{(\ell)}|}z^{M_\ell-|s^{(\ell)}|}\\ \times f^{(0)}_{r^{(\ell)},u^{(\ell)}}\Big(\frac{x}{w};q\Big)f^{(1)}_{u^{(\ell)},s^{(\ell)}}\Big(\frac{x}{w};q\Big)f^{(0)}_{s^{(\ell)},v^{(\ell)}}\Big(\frac{x}{w};q\Big)f^{(1)}_{v^{(\ell)},r^{(\ell+1)}}\Big(\frac{x}{w};q\Big)\bigg\}.$$

By (3.14a) we can now perform the sums over $\{u^{(\ell)}\}$ and $\{v^{(\ell)}\}$, so that

$$\mathrm{LHS}=\sum_{\substack{\{r^{(\ell)},s^{(\ell)}\}\\ |r^{(\ell)}|=M_\ell}}\prod_{\ell=1}^m\biggl\{w^{M_\ell+|s^{(\ell)}|}z^{M_\ell-|s^{(\ell)}|}f^{(0)}_{r^{(\ell)},s^{(\ell)}}\Bigl(\frac{x}{w};q\Bigr)f^{(0)}_{s^{(\ell)},r^{(\ell+1)}}\Bigl(\frac{x}{w};q\Bigr)$$
$$\times\prod_{i=1}^n\Bigl(\frac{x_i}{w}\Bigr)^{s_i^{(\ell)}}q^{\binom{s_i^{(\ell)}}{2}}\frac{(-x_i/w)_{r_i^{(\ell)}}}{(-x_i/w)_{s_i^{(\ell)}}}\cdot\Bigl(\frac{x_i}{w}\Bigr)^{r_i^{(\ell+1)}}q^{\binom{r_i^{(\ell+1)}}{2}}\frac{(-x_i/w)_{s_i^{(\ell)}}}{(-x_i/w)_{r_i^{(\ell+1)}}}\biggr\}.$$

By some telescoping, and the use of

$$(a;q)_k=(-a)^kq^{\binom{k}{2}}(q^{1-k}/a)_k, \tag{3.20}$$

(3.19) and $M_l=|r^{(\ell)}|$, this may be simplified to

$$\mathrm{LHS}=\sum_{\substack{\{r^{(\ell)},s^{(\ell)}\}\\ |r^{(\ell)}|=M_\ell}}\prod_{i=1}^n(-q^{1-r_i^{(1)}}w/x_i)_{r_i^{(1)}}$$
$$\times\prod_{\ell=1}^m\biggl\{z^{2|r^{(\ell)}|}f^{(1)}_{r^{(\ell)},s^{(\ell)}}\Bigl(\frac{x}{z};q\Bigr)f^{(1)}_{s^{(\ell)},r^{(\ell+1)}}\Bigl(\frac{x}{z};q\Bigr)\biggr\}.$$

Now using (3.14a) to sum over $\{s^{(l)}\}$ results in

$$\mathrm{LHS}=\sum_{\substack{\{r^{(\ell)}\}\\ |r^{(\ell)}|=M_\ell}}\prod_{i=1}^n(-q^{1-r_i^{(1)}}w/x_i)_{r_i^{(1)}}\prod_{\ell=1}^m\biggl\{z^{2|r^{(\ell)}|}f^{(1)}_{r^{(\ell)},r^{(\ell+1)}}\Bigl(\frac{x}{z};q\Bigr)$$
$$\times\prod_{i=1}^n\Bigl(\frac{x_i}{z}\Bigr)^{r_i^{(\ell+1)}}q^{\binom{r_i^{(\ell+1)}}{2}}\frac{(-x_i/z)_{r_i^{(\ell)}}}{(-x_i/z)_{r_i^{(\ell+1)}}}\biggr\}.$$

Again using telescoping plus (3.19) and (3.20) this simplifies to

$$\mathrm{LHS}=\sum_{\substack{\{r^{(\ell)}\}\\ |r^{(\ell)}|=M_\ell}}\prod_{i=1}^n(-q^{1-r_i^{(1)}}w/x_i,-q^{1-r_i^{(1)}}z/x_i)_{r_i^{(1)}}\prod_{\ell=1}^m f^{(2)}_{r^{(\ell)},r^{(\ell+1)}}(x;q)$$

which is the desired right-hand side of (3.12). □

3.4. **Rogers–Ramanujan-type q-series.** To conclude the section on modified Hall–Littlewood polynomials, we present a conjecture which will be important in our discussion of Macdonald-type eta-function identities in Section 6.

We begin by defining a very general q-series of Rogers–Ramanujan or Nahm–Zagier-type [1, 61, 76]. Let C_n be the $n\times n$ Cartan matrix of A_n, i.e., $(C_n^{-1})_{ab}=\min\{a,b\}-ab/(n+1)$ and let T_m be the $m\times m$ Cartan-type matrix of the tadpole graph of m vertices, i.e., $(T_m^{-1})_{ij}=\min\{i,j\}$. Then

$$F_{m,n}(u,w,z;q):=\sum\prod_{a,b=1}^n\prod_{i,j=1}^m q^{\frac{1}{2}(C_n)_{ab}(T_m^{-1})_{ij}r_i^{(a)}r_j^{(b)}} \tag{3.21}$$
$$\times\bigl(-zq^{1/2-r_1^{(1)}-\cdots-r_m^{(1)}}/u\bigr)_{r_1^{(1)}+\cdots+r_m^{(1)}}\prod_{a=1}^n\bigl(-u_awq^{r_m^{(a)}+1/2}\bigr)_\infty\prod_{a=1}^n\prod_{i=1}^m\frac{u_a^{2ir_i^{(a)}}}{(q)_{r_i^{(a)}}},$$

where the sum is over $r_i^{(a)} \in \mathbb{Z}_+$ for all $1 \leq a \leq n$ and $1 \leq i \leq m$, and $u_a := u^{(-1)^{a-1}}$. In particular, if $Q_+ = \sum_{i=1}^n \mathbb{Z}_+ \alpha_i$ with $\alpha_1, \dots, \alpha_n$ the simple roots of A_n, then

$$F_{1,n}(u,w,z;q) = \sum_{\alpha \in Q_+} q^{\frac{1}{2}(\alpha|\alpha)} \big(-zq^{1/2-(\alpha|\Lambda_1)}/u\big)_{(\alpha|\Lambda_1)} \prod_{a=1}^{n} \frac{u_a^{2(\alpha|\Lambda_a)} \big(-u_a w q^{1/2+(\alpha|\Lambda_a)}\big)_\infty}{(q)_{(\alpha|\Lambda_a)}}. \tag{3.22}$$

Several important q-series arise as special cases: $F_{1,1}(1,0,0;q)$ and $F_{1,1}(q^{1/2},0,0;q)$ are the Rogers–Ramanujan q-series, $F_{k-1,1}(1,w,0;q)$ for $w=0$ and $w=q^{1/2}$ are the (first) Andrews–Gordon q-series [1] and its even modulus generalisation due to Bressoud [8], and $F_{k-1,1}(1,w^{1/2},1;q)$ for $w=0$ and $w=q^{1/2}$ are the generalised Göllnitz–Gordon q-series [2] and its even modulus variant [9].

Conjecture 3.6. *Let $m,n \geq 1$ and $F_{m,n}(u,w,z;q)$ as defined in* (3.21) *Specialising $x = q^{1/2}(u,u^{-1},u,u^{-1},\dots)$ in the left-hand side of* (3.13) *yields*

$$\sum_{\substack{\lambda \\ \lambda_1 \leq 2m}} q^{|\lambda|/2} h_\lambda^{(m)}(w,z;q) P'_\lambda\big(\underbrace{u,u^{-1},u,u^{-1},\dots}_{n \text{ terms}};q\big) = F_{m,n}(u,w,z;q). \tag{3.23}$$

We note that for $(w,z) \neq (0,0)$ we effectively have two conjectures since the symmetry $F_{m,n}(u,w,z;q) = F_{m,n}(u,z,w;q)$ implied by the conjecture is not at all evident. In the rank-1 case the conjecture is easily proved using standard manipulations for q-hypergeometric series. The conjecture also holds for $u=1$, $w=z=0$ and n even thanks to (6.2) below, and for $u = 1, q^{1/2}$, $w=z=0$ and $m=1$ by [74, Theorem 4.1]. The proof of that theorem only requires minor modifications to settle the conjecture for $m=1$ and arbitrary u, w and z.

Theorem 3.7. *Equation* (3.23) *holds for $m=1$.*

Proof. One possible approach would be to specialise $x = q^{1/2}(u,u^{-1},u,u^{-1},\dots)$ in the $m=1$ case of Corollary 3.4 and prove that

$$\begin{aligned}(wz)_\infty \sum \prod_{i=1}^n u_i^{2r_i} q^{r_i^2} \big(-q^{1/2-r_i} w/u_i, -q^{1/2-r_i} z/u_i\big)_{r_i} \prod_{i,j=1}^n \frac{(qu_i/u_j)_{r_i-r_j}}{(qu_i/u_j)_{r_i}} \\ = \sum \big(-zq^{1/2-r_1}/u\big)_{r_1} \prod_{i=1}^n u_i^{2r_i} q^{r_i^2 - r_i r_{i+1}} \frac{(-u_i w q^{r_i+1/2})_\infty}{(q)_{r_i}},\end{aligned}$$

where $u_i = u^{(-1)^{i-1}}$ and $r_{n+1} := 0$. Using standard basic hypergeometric notation, for $n=1$ this is equivalent to the $c \to 0$ limit of Heine's transformation [18, Equation (III.2)]

$${}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,z\right] = \frac{(c/b,bz)_\infty}{(c,z)_\infty}\, {}_2\phi_1\left[\begin{matrix}abz/c,b\\bz\end{matrix};q,\frac{c}{b}\right]$$

with $(a,b,z) \mapsto (-q^{1/2}u/w, -q^{1/2}u/z, wz)$. For $n>1$, however, proving the above appears rather nontrivial.

There is however a second approach based on the following formula for modified Hall–Littlewood polynomials [30, 74]:

$$P'_\lambda(x;q)=\sum\prod_{j\geq 1}\biggl(\frac{1}{(q)_{\mu_j^{(0)}-\mu_{j+1}^{(0)}}}\prod_{i=1}^n x_i^{\mu_j^{(i-1)}-\mu_j^{(i)}}q^{\binom{\mu_j^{(i-1)}-\mu_j^{(i)}}{2}}\begin{bmatrix}\mu_j^{(i-1)}-\mu_{j+1}^{(i)}\\ \mu_j^{(i-1)}-\mu_j^{(i)}\end{bmatrix}\biggr),$$

where the sum is over $0=\mu^{(n)}\subseteq\cdots\subseteq\mu^{(1)}\subseteq\mu^{(0)}=\lambda'$. This can be used to compute the left-hand side of (3.23) for $m=1$ as follows. Introduce new summation indices $k_0,\dots,k_{n-1}$ and $r_1,\dots,r_n$ by

$$\mu^{(i)}=(r_{i+1}+\cdots+r_n-k_{i+1}-\cdots-k_n,r_{i+1}+\cdots+r_n-k_i-\cdots-k_n),$$

where $k_n:=0$. Then

$$\begin{aligned}\mathrm{LHS}=\sum_{r\in\mathbb{Z}_+^n}\frac{\prod_{i=1}^n u^{2(-1)^{i-1}r_i}q^{r_i^2}}{(q)_{r_n}}\biggl(\sum_{\ell,k_0\geq 0}(zq^{1/2-r_1}/u)^{k_0}\Bigl(\frac{w}{z}\Bigr)^\ell q^{\binom{k_0}{2}}\begin{bmatrix}r_1\\k_0\end{bmatrix}\begin{bmatrix}k_0\\ \ell\end{bmatrix}\biggr)&\\ \times\prod_{i=1}^{n-1}\biggl(\sum_{k_i\geq 0}\frac{q^{k_i(k_i-r_i-r_{i+1})}}{(q)_{r_i-k_i}}\begin{bmatrix}r_{i+1}\\k_i\end{bmatrix}\biggr).&\end{aligned}$$

The sum over k_i (for $1\leq i\leq n-1$) can be carried out by the q-Chu–Vandermonde sum [3, Equation (3.3.10)] to yield $q^{-r_ir_{i+1}}/(q)_{r_i}$. Then shifting $(r_1,k_0)\mapsto(r_1+\ell,k_0+\ell)$ we can successively sum over k_0 and ℓ by the q-binomial theorem, resulting in

$$\mathrm{LHS}=\sum_{r\in\mathbb{Z}_+^n}\bigl(-zq^{1/2-r_1}/u\bigr)_{r_1}\bigl(-uwq^{1/2+r_1-r_2}\bigr)_\infty\prod_{i=1}^n\frac{u^{2(-1)^{i-1}r_i}q^{r_i^2-r_ir_{i+1}}}{(q)_{r_i}}.$$

This is also

$$\begin{aligned}\mathrm{LHS}=\sum_{r_1,r_3,\dots,r_n=0}^\infty\prod_{\substack{i=1\\ i\neq 2}}^n\frac{u^{2(-1)^{i-1}r_i}q^{r_i^2-r_ir_{i+1}}}{(q)_{r_i}}&\\ \times\bigl(-zq^{1/2-r_1}/u\bigr)_{r_1}\bigl(-uwq^{1/2+r_1}\bigr)_\infty\,{}_1\phi_1\biggl[\begin{matrix}-q^{1/2-r_1}/(uw)\\0\end{matrix};q,-\frac{wq^{1/2-r_3}}{u}\biggr].&\end{aligned}$$

By ${}_1\phi_1(a;0;q,z)=(z)_\infty\,{}_0\phi_1(-;z;q,az)$ [18, Equation (III.4)] this can be transformed into

$$\begin{aligned}\mathrm{LHS}=\sum_{r\in\mathbb{Z}_+^n}\prod_{i=1}^n\frac{u^{2(-1)^{i-1}r_i}q^{r_i^2-r_ir_{i+1}}}{(q)_{r_i}}&\\ \times\bigl(-zq^{1/2-r_1}/u\bigr)_{r_1}\bigl(-uwq^{1/2+r_1}\bigr)_\infty\bigl(-wq^{1/2+r_2-r_3}/u\bigr)_\infty.&\end{aligned}$$

We now simply keep iterating the above transformation, first on r_3, then on r_4 and so on, until we arrive at

$$\mathrm{LHS}=\sum_{r\in\mathbb{Z}_+^n}\bigl(-zq^{1/2-r_1}/u\bigr)_{r_1}\prod_{i=1}^n\frac{u^{2(-1)^{i-1}r_i}q^{r_i^2-r_ir_{i+1}}}{(q)_{r_i}}\bigl(-u_iwq^{1/2+r_i}\bigr)_\infty.$$

This is equivalent to (3.22), completing the proof. □

4. The C_n Andrews transformation

Andrews' multiple series transformation [2] is one of the most complicated results in all of the theory of basic hypergeometric series. It is also one of the most useful; it implies many important partition and Rogers–Ramanujan-type identities [2] and has recently played a major role in answering deep arithmetic questions related to the Riemann zeta function, see e.g., [26, 33, 34, 77].

In this section we apply the Milne–Lilly C_n Bailey lemma to prove a C_n-analogue of Andrews' transformation. This result in itself is too complicated to be of much independent interest, but as we will see in Section 5, characters of affine Lie algebras arise through specialisation, allowing us to prove the claims of the introduction.

4.1. **The Milne–Lilly C_n Bailey lemma.** The Bailey lemma is a standard tool in the theory of basic hypergeometric series, see e.g., [4–7, 72]. The generalisation of the Bailey machinery to the C_n (as well as A_n) root system was developed by Milne and Lilly in a series of papers [46, 59, 60]. (Quite a different Bailey lemma for the non-reduced root system BC_n was recently discovered by Coskun [12].) We begin with the definition of a C_n Bailey pair, albeit using a slightly different normalisation than Milne and Lilly. Two sequences $\alpha=(\alpha_N)_{N\in\mathbb{Z}_+^n}$ and $\beta=(\beta_N)_{N\in\mathbb{Z}_+^n}$ are said to form a C_n Bailey pair if

$$\beta_N=\sum_{0\subseteq r\subseteq N}\alpha_r\prod_{i,j=1}^n\frac{1}{(qx_i/x_j)_{N_i-r_j}(qx_ix_j)_{N_i+r_j}},\tag{4.1}$$

where we remind the reader that $0\subseteq r\subseteq N$ stands for $0\leq r_i\leq N_i$ for $i=1,\dots,n$. The above definition may be inverted, expressing α in terms of β:

$$\alpha_N=\frac{\Delta_{\mathrm{C}}(xq^N)}{\Delta_{\mathrm{C}}(x)}\sum_{0\subseteq r\subseteq N}\beta_r\,q^{-(n-1)|r|}\prod_{1\leq i<j\leq n}\frac{x_iq^{r_i}-x_jq^{r_j}}{x_i-x_j}\cdot\frac{1-x_ix_jq^{r_i+r_j}}{1-x_ix_j}\\
\times\prod_{i,j=1}^n\Big(-\frac{x_i}{x_j}\Big)^{N_i-r_j}q^{\binom{N_i-r_j}{2}}\frac{(x_ix_j)_{N_i+r_j}}{(qx_i/x_j)_{N_i-r_j}}.\tag{4.2}$$

The most important ingredient of the theory is the Bailey lemma, which generates an infinite sequence of Bailey pairs from a given seed. Unfortunately Milne and Lilly's C_n Bailey lemma, first stated as [59, Equation 2.5] and copied verbatim in [46] and [60] contains a minor typographical error in the expression for β'_N. In the following this has been corrected.

Lemma 4.1 (C_n Bailey lemma)**.** *If (α,β) is a C_n Bailey pair, then so is the new pair (α',β') given by*

$$\alpha'_N=\alpha_N\prod_{i=1}^n\frac{(bx_i,cx_i)_{N_i}}{(qx_i/b,qx_i/c)_{N_i}}\Big(\frac{q}{bc}\Big)^{N_i},$$

$$\beta'_N=\sum_{0\subseteq r\subseteq N}\beta_r\,(q/bc)_{|N|-|r|}\Big(\frac{q}{bc}\Big)^{|r|}\prod_{i=1}^n\frac{(bx_i,cx_i)_{r_i}}{(qx_i/b,qx_i/c)_{N_i}}\\
\times\prod_{1\leq i<j\leq n}\frac{(qx_ix_j)_{r_i+r_j}}{(qx_ix_j)_{N_i+N_j}}\prod_{i,j=1}^n\frac{(qx_i/x_j)_{r_i-r_j}}{(qx_i/x_j)_{N_i-r_j}},$$

where b,c are indeterminates.

Equipped with the above lemma it is straightforward to obtain the C_n-analogue of Andrews' transformation formula.

Theorem 4.2 (C_n Andrews transformation)**.** *For m a nonnegative integer and $N\in\mathbb{Z}_+^n$,*

$$
\begin{aligned}
\sum_{0\subseteq r\subseteq N}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n\Bigg[\prod_{\ell=1}^{m+1}\frac{(b_\ell x_i,c_\ell x_i)_{r_i}}{(qx_i/b_\ell,qx_i/c_\ell)_{r_i}}\Big(\frac{q}{b_\ell c_\ell}\Big)^{r_i}& \\
\times\prod_{j=1}^n\frac{(q^{-N_j}x_i/x_j,x_ix_j)_{r_i}}{(qx_i/x_j,q^{N_j+1}x_ix_j)_{r_i}}\,q^{N_jr_i}\Bigg]& \\
=\prod_{i,j=1}^n(qx_ix_j)_{N_i}\prod_{1\le i<j\le n}\frac{1}{(qx_ix_j)_{N_i+N_j}}& \\
\times\sum_{r^{(1)},\dots,r^{(m)}\in\mathbb{Z}_+^n}\prod_{i,j=1}^n\frac{(qx_i/x_j)_{N_i}}{(qx_i/x_j)_{N_i-r_j^{(1)}}}\prod_{\ell=1}^m f^{(0)}_{r^{(\ell)},r^{(\ell+1)}}(x;q)& \\
\times\prod_{\ell=1}^{m+1}\Bigg[(q/b_\ell c_\ell)_{|r^{(\ell-1)}|-|r^{(\ell)}|}\Big(\frac{q}{b_\ell c_\ell}\Big)^{|r^{(\ell)}|}\prod_{i=1}^n\frac{(b_\ell x_i,c_\ell x_i)_{r_i^{(\ell)}}}{(qx_i/b_\ell,qx_i/c_\ell)_{r_i^{(\ell-1)}}}\Bigg],&
\end{aligned}
\tag{4.3}
$$

where $r^{(0)}:=N$ and $r^{(m+1)}:=0$.

For $m=0$ this is Lilly and Milne's C_n analogue of Jackson's ${}_6\phi_5$ summation [46, Theorem 2.11] and for $m=1$ it is Milne's C_n analogue of Watson's q-Whipple transformation [57, Theorem A.3] (see also [60, Theorem 6.6]).

Proof of Theorem 4.2. Taking $\beta_N=\delta_{N,0}=\prod_{i=1}^n\delta_{N_i,0}$ in (4.2) yields the C_n unit Bailey pair

$$
\alpha_N=\frac{\Delta_{\mathrm{C}}(xq^N)}{\Delta_{\mathrm{C}}(x)}\prod_{i,j=1}^n\Big(-\frac{x_i}{x_j}\Big)^{N_i}q^{\binom{N_i}{2}}\frac{(x_ix_j)_{N_i}}{(qx_i/x_j)_{N_i}}\quad\text{and}\quad\beta_N=\delta_{N,0}.
$$

Iterating this using the Bailey lemma and induction we obtain the new Bailey pair

$$
\begin{aligned}
\alpha_N&=\frac{\Delta_{\mathrm{C}}(xq^N)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n\Bigg[\prod_{\ell=1}^{m+1}\frac{(b_\ell x_i,c_\ell x_i)_{N_i}}{(qx_i/b_\ell,qx_i/c_\ell)_{N_i}}\Big(\frac{q}{b_\ell c_\ell}\Big)^{N_i} \\
&\qquad\times\prod_{j=1}^n\frac{(x_ix_j)_{N_i}}{(qx_i/x_j)_{N_i}}\Big(-\frac{x_i}{x_j}\Big)^{N_i}q^{\binom{N_i}{2}}\Bigg], \\
\beta_N&=\prod_{1\le i<j\le n}\frac{1}{(qx_ix_j)_{N_i+N_j}} \\
&\quad\times\sum_{r^{(1)},\dots,r^{(m)}\in\mathbb{Z}_+^n}\prod_{i,j=1}^n\frac{1}{(qx_i/x_j)_{N_i-r_j^{(1)}}}\prod_{\ell=1}^m f^{(0)}_{r^{(\ell)},r^{(\ell+1)}}(x;q) \\
&\qquad\times\prod_{\ell=1}^{m+1}\Bigg[(q/b_\ell c_\ell)_{|r^{(\ell-1)}|-|r^{(\ell)}|}\Big(\frac{q}{b_\ell c_\ell}\Big)^{|r^{(\ell)}|}\prod_{i=1}^n\frac{(b_\ell x_i,c_\ell x_i)_{r_i^{(\ell)}}}{(qx_i/b_\ell,qx_i/c_\ell)_{r_i^{(\ell-1)}}}\Bigg].
\end{aligned}
$$

After substitution in (4.1) the claim follows. ∎

5. The C_n Andrews transformation and character formulas

Isolating the variables b_1, c_1, we write the C_n Andrews transformation (4.3) as

$$L_N(x;b_1,c_1;b_2,\dots,c_{m+1};q)=R_N(x;b_1,c_1;b_2,\dots,c_{m+1};q) \tag{5.1}$$

where L_N stands for the left-hand side of (4.3) and R_N for the right-hand side. The aim of this section is to show that (5.1) implies Theorem 1.1 of the introduction. After first showing that

$$R_m(x;b,c;q):=R_{(\infty^n)}(x;b,c;\underbrace{\infty,\dots,\infty}_{2m\text{ times}};q)$$

can be expressed in terms of the modified Hall–Littlewood polynomials P'_λ, we will prove that if

$$L_m(x;b,c;q):=L_{(\infty^n)}(x;b,c;\underbrace{\infty,\dots,\infty}_{2m\text{ times}};q), \tag{5.2}$$

then $L_m\big(x^{\pm};b,c;q\big)$ is a function which unifies certain characters of $\mathrm{C}_n^{(1)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$ in their in Weyl–Kac representation. In particular, the identity

$$L_m\big(x^{\pm};b,c;q\big)=R_m\big(x^{\pm};b,c;q\big) \tag{5.3}$$

includes (1.4a) and (1.4b) of the introduction as special limiting cases.

5.1. The right-hand side of the C_n Andrews transformation. Since the right-hand side of (5.1) is a rational function we may let $b_2,c_2,\dots,b_{m+1},c_{m+1}$ tend to infinity for fixed $N\in\mathbb{Z}_+^n$. To then take the large N limit we need to assume that $|q/b_1c_1|<1$. By an appeal to dominated convergence this yields

$$\begin{aligned}R_m(x;b,c;q)&=(q/bc)_\infty D(x;b,c;q)\\&\quad\times\sum_{r^{(1)},\dots,r^{(m)}\in\mathbb{Z}_+^n}\prod_{i=1}^n\big(q^{1-r_i^{(1)}}/bx_i,q^{1-r_i^{(1)}}/cx_i\big)_{r_i^{(1)}}\prod_{\ell=1}^m q^{|r^{(\ell)}|}f^{(2)}_{r^{(\ell)},r^{(\ell+1)}}(x;q),\end{aligned}$$

where $r^{(m+1)}:=0$, $|q/bc|<1$ and

$$D(x;b,c;q):=\prod_{i=1}^n\frac{(qx_i^2)_\infty}{(qx_i/b,qx_i/c)_\infty}\prod_{1\le i<j\le n}(qx_ix_j)_\infty.$$

If we now take (3.13), replace $(x,w,z)\mapsto(q^{1/2}x,-q^{1/2}/b,-q^{1/2}/c)$ and use that $f^{(2)}_{r,s}(q^{1/2}x;q)=q^{|r|}f^{(2)}_{r,s}(x;q)$, then the right-hand side of (3.13) matches the above expression for $R_m(x;b,c;q)$, except for the prefactor $D(x;b,c;q)$. Hence, for $|q/bc|<1$,

$$R_m(x;b,c;q)=D(x;b,c;q)\sum_{\substack{\lambda\\ \lambda_1\le 2m}}q^{|\lambda|/2}h_\lambda^{(m)}\big({-q^{1/2}/b},-q^{1/2}/c;q\big)P'_\lambda(x;q). \tag{5.4}$$

5.2. **The left-hand side of the C_n Andrews transformation.** Because in our initial considerations the parameters $b_1, c_1, \dots, b_{m+1}, c_{m+1}$ and q play a passive role we suppress their dependence, writing $L_N(x)$ instead of $L_N(x; b_1, c_1; b_2, \dots, c_k; q)$. To transform $L_N(x)$ into a function that resembles the Weyl–Kac character formula we must achieve the appropriate Weyl group symmetry. As will be shown below, this can be realised by doubling the rank to $2n$ and by then reducing this back to n by taking a limit in which n distinct pairs of variables tend to 1 as follows:

$$\lim_{y_1\to x_1^{-1},\dots,y_n\to x_n^{-1}} L_{(N_1,M_1,\dots,N_n,M_n)}\big(x_1,y_1,\dots,x_n,y_n\big) =: L_{M,N}(x).$$

We remark that this limiting process is highly non-trivial due to the occurrence of the denominator term $\Delta_{\mathrm{C}}(x)$ in the summand of $L_N(x)$. Indeed, $\Delta_{\mathrm{C}}(x)$ vanishes whenever the product of two of its variables equals 1. For later purposes we will also consider the following limit in the case of an odd number of variables:

$$\lim_{y_1\to x_1^{-1},\dots,y_{n-1}\to x_{n-1}^{-1},x_n\to 1} L_{(N_1,M_1,\dots,N_{n-1},M_{n-1},N_n)}\big(x_1,y_1,\dots,x_{n-1},y_{n-1},x_n\big) \\ =: \hat{L}_{M,N}(\hat{x}),$$

where $\hat{x}=(x_1,\dots,x_{n-1})$.

Proposition 5.1. *For $x=(x_1,\dots,x_n)$ and $M,N\in\mathbb{Z}_+^n$,*

$$\text{(5.5a)}\quad L_{M,N}(x)=\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n\Bigg[\prod_{\ell=1}^{m+1}\frac{(b_\ell x_i,c_\ell x_i)_{r_i}}{(qx_i/b_\ell,qx_i/c_\ell)_{r_i}}\Big(\frac{q}{b_\ell c_\ell}\Big)^{r_i} \\ \times\prod_{j=1}^n\frac{(q^{-N_j}x_i/x_j,q^{-M_j}x_ix_j)_{r_i}}{(q^{M_j+1}x_i/x_j,q^{N_j+1}x_ix_j)_{r_i}}\,q^{(M_j+N_j)r_i}\Bigg],$$

and for $\hat{x}=(x_1,\dots,x_{n-1})$, $x=(x_1,\dots,x_{n-1},1)$, $M\in\mathbb{Z}_+^{n-1}$ and $N\in\mathbb{Z}_+^n$,

$$\text{(5.5b)}\quad \hat{L}_{M,N}(\hat{x})=\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{B}}(-xq^r)}{\Delta_{\mathrm{B}}(-x)}\prod_{i=1}^n\Bigg[\prod_{\ell=1}^{m+1}\frac{(b_\ell x_i,c_\ell x_i)_{r_i}}{(qx_i/b_\ell,qx_i/c_\ell)_{r_i}}\Big(\frac{q}{b_\ell c_\ell}\Big)^{r_i} \\ \times\prod_{j=1}^{n-1}\frac{(q^{-M_j}x_ix_j)_{r_i}}{(q^{M_j+1}x_i/x_j)_{r_i}}\,q^{M_jr_i}\prod_{j=1}^n\frac{(q^{-N_j}x_i/x_j)_{r_i}}{(q^{N_j+1}x_ix_j)_{r_i}}\,q^{N_jr_i}\Bigg].$$

A number of remarks are in order. First of all we note that both summands vanish unless $-M\subseteq r\subseteq N$, i.e., $-M_i\leq r_i\leq N_i$ for all i (where $M_n:=N_n$ in the case of (5.5b)). Moreover, if we set $M_1=\dots=M_n=0$ in (5.5a) we recover $L_N(x)$. Finally we note that the series on the right of (5.5a) exhibits the desired symmetry, in that it is invariant under the natural action of the hyperoctahedral group. For example, for $n=2$,

$$\begin{aligned} L_{(M_1,M_2),(N_1,N_2)}(x_1,x_2) &= L_{(M_2,M_1),(N_2,N_1)}(x_2,x_1) = \\ L_{(M_1,N_2),(N_1,M_2)}(x_1,x_2^{-1}) &= L_{(N_2,M_1),(M_2,N_1)}(x_2^{-1},x_1) = \\ L_{(N_1,M_2),(M_1,N_2)}(x_1^{-1},x_2) &= L_{(M_2,N_1),(N_2,M_1)}(x_2,x_1^{-1}) = \\ L_{(N_1,N_2),(M_1,M_2)}(x_1^{-1},x_2^{-1}) &= L_{(N_2,N_1),(M_2,M_1)}(x_2^{-1},x_1^{-1}). \end{aligned}$$

The proof of Proposition 5.1 is long and technical, and has been relegated to the appendix.

5.3. **Proof of Theorem 1.1 and related results.** Recall that for $x=(x_1,\dots,x_n)$ we abbreviate $f(x_1,x_1^{-1},\dots,x_n,x_n^{-1})$ by $f(x^{\pm})$. By abuse of notation, for $x=(x_1,\dots,x_{n-1},1)$ we also denote $f(x_1,x_1^{-1},\dots,x_{n-1},x_{n-1}^{-1},1)$ as $f(x^{\pm})$ (so that in this case $f(x^{\pm})$ should not be interpreted as $f(x_1,x_1^{-1},\dots,x_{n-1},x_{n-1}^{-1},1,1)$).

To obtain (5.3) in a more explicit form, we let $b_2,c_2,\dots,b_{m+1},c_{m+1}$ tend to infinity in (5.5a) followed by $M,N\to(\infty^n)$, and equate the resulting expression with (5.4) with $x\mapsto x^{\pm}$. This gives (5.6a) below. By a similar computation starting from (5.5b) we obtain (5.6b).

Theorem 5.2. *Let m be a nonnegative integer and $|q/bc|\le 1$. Then the following two identities hold:*

$$\text{(5.6a)}\qquad \frac{1}{D(x^{\pm};b,c;q)}\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^{n}\frac{(bx_i,cx_i)_{r_i}}{(qx_i/b,qx_i/c)_{r_i}}\Big(\frac{q^{1-n}}{bc}\Big)^{r_i}\big(x_i^2q^{r_i}\big)^{Kr_i}$$
$$=\sum_{\substack{\lambda\\ \lambda_1\le 2m}} q^{|\lambda|/2}h_\lambda^{(m)}\big(-q^{1/2}/b,-q^{1/2}/c;q\big)P'_\lambda\big(x^{\pm};q\big),$$

where $x=(x_1,\dots,x_n)$ and $K=m+n$, and

$$\text{(5.6b)}\qquad \frac{1}{D(x^{\pm};b,c;q)}\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{B}}(-xq^r)}{\Delta_{\mathrm{B}}(-x)}\prod_{i=1}^{n}\frac{(bx_i,cx_i)_{r_i}}{(qx_i/b,qx_i/c)_{r_i}}\Big(-\frac{q^{3/2-n}}{bc}\Big)^{r_i}\big(x_i^2q^{r_i}\big)^{Kr_i}$$
$$=\sum_{\substack{\lambda\\ \lambda_1\le 2m}} q^{|\lambda|/2}h_\lambda^{(m)}\big(-q^{1/2}/b,-q^{1/2}/c;q\big)P'_\lambda\big(x^{\pm};q\big),$$

where $x=(x_1,\dots,x_{n-1},1)$ and $K=m+n-1/2$.

Recalling that $h_0^{(0)}(w,z;q)=(wz)_\infty$, we note that for $m=0$ both identities are limiting cases of Gustafson's $\mathrm{C}_n^{(1)}$-analogue of Bailey's sum of a very-well poised ${}_6\psi_6$ series [20]. We also note that for $b\to\infty$ the right-hand side of (5.6a) and (5.6b) simplifies to

$$\text{(5.7)}\qquad \sum_{\substack{\lambda\\ \lambda_1\le 2m}} q^{|\lambda|/2}(-q^{1/2}/c)^{l(\lambda_{\mathrm{o}})}P'_\lambda\big(x^{\pm};q\big).$$

We now consider the various specialisations of Theorem 5.2. Noting that for $x=(x_1,\dots,x_n)$,

$$D\big(x^{\pm};b,c;q\big)=(q)_\infty^n\prod_{i=1}^{n}\frac{(qx_i^{\pm 2})_\infty}{(qx_i^{\pm}/b,qx_i^{\pm}/c)_\infty}\prod_{1\le i<j\le n}(qx_i^{\pm}x_j^{\pm})_\infty,$$

and recalling Lemma 2.1, it follows that in the $b,c\to\infty$ limit the left-hand side of (5.6a) yields the $\mathrm{C}_n^{(1)}$ character (2.5) for $\Lambda=m\Lambda_0$. (Note in particular that for this highest weight the partition λ in Lemma 2.1 is 0 so that the symplectic Schur function in (2.4) trivialises to 1.) But when $c\to\infty$ the summand of (5.7) vanishes unless $l(\lambda_{\mathrm{o}})=0$, i.e., unless λ is even. We thus obtain (1.4a). Similarly, for $b\to\infty$ and $c\to -q^{1/2}$, and by appeal to Lemma 2.2 and $(aq)_\infty/(-aq^{1/2})_\infty=(aq^{1/2})_\infty(aq^2;q^2)_\infty$, we arrive at (1.4b). This completes our proof of Theorem 1.1.

If we take $b\to\infty$ and $c=-1$ in (5.6a), and use Lemma 2.3 as well as $(a^2q)_\infty/(-aq)_\infty=(aq)_\infty(a^2q;q^2)_\infty$, we obtain our next theorem.

Theorem 5.3. *Let $\mathfrak{g} = \mathrm{A}_{2n}^{(2)}$, $\Lambda = m\Lambda_n$ for m a nonnegative integer, and*

$$q = \mathrm{e}^{-\delta} \quad \text{and} \quad x_i = \mathrm{e}^{-\alpha_0 - \cdots - \alpha_{n-i}}.$$

Then

$$\mathrm{e}^{-\Lambda} \operatorname{ch} V(\Lambda) = \sum_{\substack{\lambda \\ \lambda_1 \leq 2m}} q^{(|\lambda| + l(\lambda_{\mathrm{o}}))/2} P'_\lambda\big(x^{\pm}; q\big). \tag{5.8}$$

Our next result corresponds to (5.6a) for $b = -1$ and $c = -q^{1/2}$. Then the summand on the right simplifies, since

$$\begin{aligned} h_\lambda^{(m)}(q^{1/2}, 1; q) &= \prod_{i=1}^{2m-1} (-q^{1/2}; q^{1/2})_{m_i(\lambda)} \\ &= \frac{1}{(-q^{1/2}; q^{1/2})_\infty} \prod_{i=0}^{2m-1} (-q^{1/2}; q^{1/2})_{m_i(\lambda)}, \end{aligned} \tag{5.9}$$

by $H_m(q^{1/2}; q) = (-q^{1/2}; q^{1/2})_m$ [73]. If on the left we use Lemma 2.4 and the simple identity $(a^2 q)_\infty / (-aq^{1/2}, -aq)_\infty = (aq^{1/2}; q^{1/2})_\infty$, we obtain the following theorem.

Theorem 5.4. *Let $\mathfrak{g} = \mathrm{D}_{n+1}^{(2)}$, $\Lambda = 2m\Lambda_0$ for m a nonnegative integer, and*

$$q = \mathrm{e}^{-\delta} \quad \text{and} \quad x_i = \mathrm{e}^{-\alpha_i - \cdots - \alpha_n}.$$

Then

$$\mathrm{e}^{-\Lambda} \operatorname{ch} V(\Lambda) = \sum_{\substack{\lambda \\ \lambda_1 \leq 2m}} q^{|\lambda|} \bigg(\prod_{i=0}^{2m-1} (-q)_{m_i(\lambda)} \bigg) P'_\lambda\big(x^{\pm}; q^2\big). \tag{5.10}$$

6. Dedekind η-function identities

In the appendix of his paper [48] Macdonald gave his now famous list of identities for powers of the Dedekind η-function $\eta(\tau) = q^{1/24} \prod_{j=1}^\infty (1 - q^j)$, where $q = \exp(2\pi \mathrm{i}\, \tau)$ for $\mathrm{Im}(\tau) > 0$. The simplest of his identities correspond to the non-twisted affine Lie algebras $\mathfrak{g} = \mathrm{X}_n^{(1)}$ and yield expansions of $\eta(\tau)^{\dim(\mathrm{X}_n)}$. For example, Macdonald's formula for $\mathrm{C}_n^{(1)}$ generalises Jacobi's well known identity for the third power of the η-function to

$$\eta(\tau)^{2n^2+n} = c_0 \sum q^{\frac{\|v\|^2}{4(n+1)}} \prod_{i=1}^n v_i \prod_{1 \leq i < j \leq n} \big(v_i^2 - v_j^2\big), \tag{6.1}$$

where $c_0 = 1/(1!3! \cdots (2n-1)!)$ and where the sum is over $v \in \mathbb{Z}^n$ such that $v_i \equiv n - i + 1 \pmod{2n+2}$.

In this final section we extend many of Macdonald's identities by specialising our character formulae. To facilitate comparison with Macdonald's results we adopt his definitions of χ_{B} and χ_{D} as given by (1.5) and

$$\chi_{\mathrm{D}}(v) = \prod_{1 \leq i < j \leq n} \big(v_i^2 - v_j^2\big).$$

We also write $\chi_{\mathfrak{g}}(v/w)=\chi_{\mathfrak{g}}(v)/\chi_{\mathfrak{g}}(w)$ and define the classical $\mathfrak{g}$-Weyl vectors $\rho_{\mathfrak{g}}$ by

$$\rho_{\mathrm{B}}=(n-1/2,\dots,3/2,1/2),\quad \rho_{\mathrm{C}}=(n,\dots,2,1),\quad \rho_{\mathrm{D}}=(n-1,\dots,1,0).$$

Since carrying out the required specialisations in the Weyl–Kac formula is standard, see e.g., [28, 48], we only list the final η-function identities below. For $m=0$ these correspond to Macdonald's results. In the identities below we also give alternative expressions for the right-hand side as implied by Theorem 3.7 ($m=1$) and Conjecture 3.6 ($m\geq 2$). This equality will be written as $\overset{?_{m\geq 2}}{=\!=}$. Because in each case we have $u=1$ we will write $F_{m,n}(w,z;q)$ for $F_{m,n}(1,w,z;q)$.

Type $\mathrm{C}_n^{(1)}$. If we specialise $x=(x_1,\dots,x_n)$ to $(1,\dots,1)$ in (1.4a) we obtain a generalisation of (6.1) (or [48, p. 136, (6)]):

$$\frac{1}{\eta(\tau)^{2n^2+n}}\sum_v \chi_{\mathrm{B}}(v/\rho)\,q^{\frac{\|v\|^2-\|\rho\|^2}{4(m+n+1)}+\frac{\|\rho\|^2}{4(n+1)}} = \sum_{\substack{\lambda \text{ even}\\ \lambda_1\leq 2m}} q^{|\lambda|/2}P'_\lambda(\underbrace{1,\dots,1}_{2n \text{ times}};q)=F_{m,2n}(0,0;q), \tag{6.2}$$

where $\rho=\rho_{\mathrm{C}}$, $v\in\mathbb{Z}^n$ such that $v\equiv\rho \pmod{2m+2n+2}$ and $m\geq 0$. The equality between the first and last expression was proved by Feigin and Stoyanovsky [15] ($n=1$) and Stoyanovsky [70] ($n>1$). The implied equality between the two expressions in the second line proves Conjecture 3.6 for n even, $u=1$ and $w=z=0$.

Type $\mathrm{A}_{2n}^{(2)}$ (or affine BC_n). If we specialise $x=(x_1,\dots,x_n)$ to $(1,\dots,1)$ in (5.8) we obtain a generalisation of [48, page 138, (6a)]:

$$\frac{\eta(2\tau)^{2n}}{\eta(\tau)^{2n^2+3n}}\sum_v \chi_{\mathrm{B}}(v/\rho)\,q^{\frac{\|v\|^2-\|\rho\|^2}{2(2m+2n+1)}+\frac{\|\rho\|^2}{2(2n+1)}} = \sum_{\substack{\lambda\\ \lambda_1\leq 2m}} q^{(|\lambda|+l(\lambda_{\mathrm{o}}))/2}P'_\lambda(\underbrace{1,\dots,1}_{2n \text{ times}};q) \overset{?_{m\geq 2}}{=\!=} F_{m,2n}(0,q^{1/2};q),$$

where $\rho=\rho_{\mathrm{B}}$ and $v\in(\mathbb{Z}/2)^n$ such that $v\equiv\rho \pmod{2m+2n+1}$.

If we specialise $x=(x_1,\dots,x_n)$ to $(1,\dots,1)$ in (1.4b) we obtain a generalisation of [48, p. 138, (6b)]:

$$\frac{1}{\eta(\tau/2)^{2n}\eta(2\tau)^{2n}\eta(\tau)^{2n^2-3n}}\sum_v \chi_{\mathrm{B}}(v/\rho)\,q^{\frac{\|v\|^2-\|\rho\|^2}{2(2m+2n+1)}+\frac{\|\rho\|^2}{2(2n+1)}} = \sum_{\substack{\lambda\\ \lambda_1\leq 2m}} q^{|\lambda|/2}P'_\lambda(\underbrace{1,\dots,1}_{2n \text{ times}};q) \overset{?_{m\geq 2}}{=\!=} F_{m,2n}(0,1;q),$$

where $\rho=\rho_{\mathrm{C}}$ and $v\in\mathbb{Z}^n$ such that $v\equiv\rho \pmod{2m+2n+1}$.

If we let $b,c\to\infty$ in (5.6b) and then specialise $x=(x_1,\dots,x_{n-1},1)$ to $(1,\dots,1)$ we obtain a generalisation of [48, page 138, (6c)]:

$$\text{(6.3)}\qquad \frac{1}{\eta(\tau)^{2n^2-n}}\sum_{v}(-1)^{|v|-|\rho|}\chi_{\mathrm{D}}(v/\rho)\,q^{\frac{\|v\|^2-\|\rho\|^2}{2(2m+2n+1)}+\frac{\|\rho\|^2}{2(2n+1)}} = \sum_{\substack{\lambda \text{ even}\\ \lambda_1\le 2m}} q^{|\lambda|/2}P'_{\lambda}(\underbrace{1,\dots,1}_{2n-1\text{ times}};q) \overset{?_{m\geq 2}}{=\!=} F_{m,2n-1}(0,0;q),$$

where $\rho=\rho_{\mathrm{B}}$ and v is summed over $(\mathbb{Z}/2)^n$ such that $v\equiv\rho \pmod{2m+2n+1}$. By $P'_{2\lambda}(x;q)=x^{2|\lambda|}q^{2n(\lambda)}/b_{\lambda}(q)$ it follows that for $n=1$ the two expressions on the second line are identically the same and (after replacing m by $k-1$) are given by the famous Rogers–Ramanujan–Andrews–Gordon series [1,19]

$$\sum_{n_1,\dots,n_{k-1}\geq 0}\frac{q^{N_1^2+\cdots+N_{k-1}^2}}{(q)_{n_1}\cdots(q)_{n_{k-1}}},$$

where $N_i=n_i+\cdots+n_{k-1}$. Of course, by the Jacobi triple product identity the left hand side for $n=1$ can be written in the familiar product form

$$\frac{(q^k,q^{k+1},q^{2k+1};q^{2k+1})_\infty}{(q)_\infty}.$$

We may thus view (6.3) as an $\mathrm{A}_{2n}^{(2)}$ analogue of these famous q-series identities. In [74, Conjecture 1.1 and Theorem 1.2] the equality between the left-most and right-most expressions in (6.3) was conjectured and proved for $m=1$. The connection between the Rogers–Ramanujan partition identities and the representation theory of Kac–Moody algebras is certainly not new, and we refer the interested reader to [11,28,41–45,52,53] and references therein.

Type $\mathrm{B}_n^{(1)}$**.** If we set $b=-1$, $c=-q^{1/2}$ in (5.6b) and then specialise $x=(x_1,\dots,x_{n-1},1)$ to $(1,\dots,1)$, we obtain a generalisation of [48, p. 135, (6c)]:

$$\begin{aligned}
&\frac{1}{\eta(\tau/2)^{2n}\eta(\tau)^{2n^2-3n}}\sum_{v}(-1)^{|v|-|\rho|}\chi_{\mathrm{D}}(v/\rho)\,q^{\frac{\|v\|^2-\|\rho\|^2}{2(2m+2n-1)}+\frac{\|\rho\|^2}{2(2n-1)}}\\
&\quad=\sum_{\substack{\lambda\\ \lambda_1\le 2m}} q^{|\lambda|/2}\bigg(\prod_{i=0}^{2m-1}(-q^{1/2};q^{1/2})_{m_i(\lambda)}\bigg)P'_{\lambda}(\underbrace{1,\dots,1}_{2n-1\text{ times}};q)\\
&\quad\overset{?_{m\geq 2}}{=\!=}(-q^{1/2};q^{1/2})_\infty\,F_{m,2n-1}(q^{1/2},1;q),
\end{aligned}$$

where $\rho=\rho_{\mathrm{D}}$, $v\in\mathbb{Z}^n$ such that $v\equiv\rho\pmod{2m+2n-1}$ and $m_0(\lambda):=\infty$. The second equality assumes $m\geq 1$.

Type $\mathrm{A}_{2n-1}^{(2)}$ **(or** $\mathrm{B}_n^{\vee}$**).** If we let $b\to\infty$, $c\to -1$ in (5.6b) and then specialise $x=(x_1,\dots,x_{n-1},1)$ to $(1,\dots,1)$ we obtain a generalisation of [48, page 136 (6b)]

$$\text{(6.4)}\qquad \frac{\eta(2\tau)^{2n-1}}{\eta(\tau)^{2n^2+n-1}}\sum(-1)^{\frac{|v|-|\rho|}{2(m+n)}}\chi_{\mathrm{D}}(v/\rho)\,q^{\frac{\|v\|^2-\|\rho\|^2}{4(m+n)}+\frac{\|\rho\|^2}{4n}} = \sum_{\substack{\lambda\\ \lambda_1\le 2m}} q^{(|\lambda|+l(\lambda_{\mathrm{o}}))/2}P'_{\lambda}(\underbrace{1,\dots,1}_{2n-1\text{ times}};q) \overset{?_{m\geq 2}}{=\!=} F_{m,2n-1}(0,q^{1/2};q),$$

where $\rho=\rho_{\mathrm{D}}$, $v\in\mathbb{Z}^n$ such that $v\equiv\rho \pmod{2m+2n}$. A somewhat different generalisation of the same η-function identity arises if we take $b=-c=1$ in (5.6a), then use [73]

$$h_\lambda^{(m)}(-q^{1/2},q^{1/2};q)=\begin{cases} q^{l(\lambda_{\mathrm{o}})/2}\displaystyle\prod_{i=1}^{2m-1}(q;q^2)_{\lceil m_i(\lambda)/2\rceil} & \text{for } m_{2i-1}(\lambda) \text{ even} \\ 0 & \text{otherwise,}\end{cases}$$

and $h_0^{(0)}(-q^{1/2},q^{1/2};q)=(-q)_\infty=(q^2;q^2)_\infty/(q)_\infty$, and finally specialise $x=(x_1,\dots,x_n)$ to $(1,\dots,1)$. Then

$$\begin{aligned}\mathrm{LHS}(6.4)&=\sum_{\substack{\lambda\\ \lambda_1\le 2m\\ (\lambda_{\mathrm{o}})'\text{ is even}}} q^{(|\lambda|+l(\lambda_{\mathrm{o}}))/2}\bigg(\prod_{i=0}^{2m-1}(q;q^2)_{\lceil m_i(\lambda)/2\rceil}\bigg)P'_\lambda(\underbrace{1,\dots,1}_{2n\text{ times}};q)\\ &\overset{?_{m\geq 2}}{=\!=}(q;q^2)_\infty F_{m,2n}(-q^{1/2},q^{1/2};q),\end{aligned}$$

where $m_0(\lambda):=\infty$ and the second equality assumes $m\ge 1$.

Type $\mathrm{D}_{n+1}^{(2)}$ (or $\mathrm{C}_n^\vee$). If we specialise $x=(x_1,\dots,x_n)$ to $(1,\dots,1)$ in (5.10) we obtain a generalisation of [48, page 137, (6a)]:

$$\begin{aligned}&\frac{1}{\eta(\tau)^{2n+1}\eta(2\tau)^{2n^2-n-1}}\sum_v \chi_{\mathrm{B}}(v/\rho)\,q^{\frac{\|v\|^2-\|\rho\|^2}{2(m+n)}+\frac{\|\rho\|^2}{2n}} \\ &\quad=\sum_{\substack{\lambda\\ \lambda_1\le 2m}} q^{|\lambda|}\bigg(\prod_{i=0}^{2m-1}(-q)_{m_i(\lambda)}\bigg)P'_\lambda(\underbrace{1,\dots,1}_{2n\text{ times}};q^2)\overset{?_{m\geq 2}}{=\!=}(-q)_\infty F_{m,2n}(q,1;q^2),\end{aligned}\tag{6.5}$$

where $\rho=\rho_{\mathrm{B}}$, $v\in(\mathbb{Z}/2)^n$ such that $v\equiv\rho\pmod{2m+2n}$ and second equality assumes $m\ge 1$.

Finally, if we let $b\to\infty$ and $c=-q^{1/2}$ in (5.6b), then specialise $x=(x_1,\dots,x_{n-1},1)=(1,\dots,1)$ and replace $q\mapsto q^2$ we obtain

$$\begin{aligned}&\frac{1}{\eta(\tau)^{2n-1}\eta(4\tau)^{2n-1}\eta(2\tau)^{2n^2-5n+2}}\sum_v(-1)^{\frac{|v|-|\rho|}{2(m+n)}}\chi_{\mathrm{D}}(v/\rho)\,q^{\frac{\|v\|^2-\|\rho\|^2}{2(m+n)}+\frac{\|\rho\|^2}{2n}}\\ &\qquad=\sum_{\substack{\lambda\\ \lambda_1\le 2m}} q^{|\lambda|}P'_\lambda(\underbrace{1,\dots,1}_{2n-1\text{ times}};q^2)\overset{?_{m\geq 2}}{=\!=}F_{m,2n-1}(0,1;q^2),\end{aligned}$$

with v as in (6.5). For $m=0$ (and after replacing q by $-q$) we recover [48, page 137, (6b)]. For $m>0$ the above should be viewed as a generalisation of Andrews' generalised Göllnitz–Gordon q-series [2].

To conclude this section we remark that Leininger and Milne employed multiple basic hypergeometric series for A_n (as opposed to the C_n series used in this paper) to derive other infinite families of identities for powers of the η-function, see [38], [39, Theorem 2.4] and [40, Theorems 2.3 and 3.2].

7. Concluding remarks

We end the paper with some comments in response to two questions raised by one of the referees.

The first question asked why our results do not include combinatorial character formulas for what is perhaps the simplest affine Lie algebra, $\mathrm{A}_{n-1}^{(1)}$. Using the Milne–Lilly Bailey lemma for A_{n-1} [59, 60] it is indeed possible to prove an A_{n-1} counterpart of the C_n Andrews transformation of Theorem 4.2. Specialising sufficiently many of the free parameters, the right-hand side of this transformation can again be expressed in terms of modified Hall–Littlewood polynomials. Unfortunately, we have been unable to recognise (or rewrite) the left-hand side as the Weyl–Kac expression for $\operatorname{ch} V(\Lambda)$ where $\mathfrak{g} = \mathrm{A}_{n-1}^{(1)}$ and Λ is an appropriately chosen highest weight. However, recently in [21, Section 4] Griffin, Ono and the second author used Corollary 3.2 to prove a formula for characters of $\mathrm{A}_{n-1}^{(1)}$ of highest weight $\Lambda = (m-k)\Lambda_0 + k\Lambda_1$ in terms of modified Hall–Littlewood polynomials. This formula is somewhat different in nature from the identities of Theorem 1.1 in that it involves a limit. For example, when $k = 0$ it takes the form

$$\mathrm{e}^{-\Lambda} \operatorname{ch} V(\Lambda) = \lim_{r\to\infty} q^{-mn\binom{r}{2}} \frac{Q'_{(m^{nr})}(x;q)}{(x_1\cdots x_n)^{mr}},$$

where $q = \mathrm{e}^{-\alpha_0-\alpha_1-\cdots-\alpha_{n-1}}$ and $x_i/x_{i+1} = \mathrm{e}^{-\alpha_i}$ for $1 \le i \le n-1$. For $m = 1$ this is Kirillov's formula [30] for the basic representation of $\mathrm{A}_{n-1}^{(1)}$.

The second question concerned the possibility of simpler proofs of the combinatorial character formulas using either representation-theoretic ideas (utilising, for example, the connection between affine Demazure characters and Macdonald polynomials [24, 67]) or combinatorial methods. In fact, Rains and the second author have recently developed an alternative, more conceptual approach in [66]. In particular, using Macdonald–Koornwinder theory [31, 50, 51] and virtual Koornwinder integrals [64, 65], we show that Theorems 1.1, 5.3 and (5.4) as well as additional identities follow by specialising decomposition or branching formulas for Hall–Littlewood polynomials of type R into Hall–Littlewood polynomials of type A. The results of [66] still depend crucially on Proposition 5.1 of this paper but do not rely on the C_n Bailey lemma.

Appendix A. Proof of Proposition 5.1

Before proving the proposition we prepare a key lemma. For p an integer such that $0 \le p \le n$, let $M = (M_1, \dots, M_p) \in \mathbb{Z}_+^p$, $N = (N_1, \dots, N_n) \in \mathbb{Z}_+^n$ and $r \in \mathbb{Z}^n$, and define

$$\text{(A.1a)}\quad L_{M,N;r}^{(p)}(x) := \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^n \Biggl[\prod_{\ell=1}^{m+1} \frac{(b_\ell x_i, c_\ell x_i)_{r_i}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i}} \Bigl(\frac{q}{b_\ell c_\ell}\Bigr)^{r_i} \times \prod_{j=1}^n \frac{(q^{-N_j} x_i/x_j, q^{-M_j} x_i x_j)_{r_i}}{(q^{M_j+1} x_i/x_j, q^{N_j+1} x_i x_j)_{r_i}}\, q^{(M_j+N_j)r_i} \Biggr],$$

and

$$\text{(A.1b)}\quad L_{M,N}^{(p)}(x) := \sum_{r_1=-M_1}^{N_1} \cdots \sum_{r_n=-M_n}^{N_n} L_{M,N;r}^{(p)}(x),$$

where $M_{p+1}=\cdots=M_n:=0$. Recalling that $L_N(x)$ denotes the left-hand side of (4.3), we note that

$$L_N(x)=L^{(0)}_{-,N}(x). \tag{A.2}$$

We further observe that $L^{(n)}_{M,N}(x)$ coincides with the expression for $L_{M,N}(x)$ as claimed in (5.5a).

Given $x=(x_1,\dots,x_n)$ we set $x^{(i)}=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$.

Lemma A.1. *Let $M=(M_1,\dots,M_{p-1})$ and $M'=(M_1,\dots,M_{p-1},N_{p+2})$. For $1\le p\le n-1$*

$$\lim_{x_{p+1}\to x_p^{-1}} L^{(p-1)}_{M,N}(x)=L^{(p)}_{M',N^{(p+1)}}\big(x^{(p+1)}\big).$$

Proof. Let us first focus on the numerator and denominator terms of $L^{(p-1)}_{M,N}(x)$ that vanish when $x_{p+1}\to 1/x_p$. By $\prod_{i=1}^n\prod_{j=p}^n(x_ix_j)_{r_i}$ the numerator contains the factor $(x_px_{p+1})_{r_p}(x_px_{p+1})_{r_{p+1}}$, which in turn results in a factor $(1-x_px_{p+1})^2$ if r_p and r_{p+1} are both positive, $1-x_px_{p+1}$ if only one of these is positive and 1 if both are zero. From $\Delta_{\mathrm{C}}(xq^r)/\Delta_{\mathrm{C}}(x)$ we pick up the contribution

$$\frac{1-x_px_{p+1}q^{r_p+r_{p+1}}}{1-x_px_{p+1}},$$

which is 1 if both r_p and r_{p+1} are zero, but leads to a factor $(1-x_px_{p+1})$ in the denominator if (at least) one of r_p,r_{p+1} is positive. As a result, $L^{(p-1)}_{M,N;r}(x)$ vanishes in the limit $x_{p+1}\to 1/x_p$ unless one of r_p,r_{p+1} is zero.

It is now a somewhat tedious, but elementary exercise to show that

$$\lim_{x_{p+1}\to x_p^{-1}}\Big(L^{(p-1)}_{M,N;r}(x)\big|_{r_{p+1}=0}\Big)=L^{(p)}_{M',N^{(p+1)};r^{(p+1)}}\big(x^{(p+1)}\big),$$

where $r^{(i)}:=(r_1,\dots,r_{i-1},r_{i+1},\dots,r_n)$. Again elementary, although now requiring

$$\frac{(a)_{-n}}{(b)_{-n}}=\frac{(q/b)_n}{(q/a)_n}\Big(\frac{b}{a}\Big)^n, \tag{A.3}$$

is to show that

$$\lim_{x_{p+1}\to x_p^{-1}}\Big(L^{(p-1)}_{M,N;r}(x)\big|_{r_p=0}\Big)=L^{(p)}_{M',N^{(p+1)};\hat{r}^{(p)}}\big(x^{(p+1)}\big),$$

where $\hat{r}^{(i)}:=(r_1,\dots,r_{i-1},-r_{i+1},r_{i+2},\dots,r_n)$. Consequently,

$$\begin{aligned}\lim_{x_{p+1}\to x_p^{-1}} L^{(p-1)}_{M,N}(x)&=\sum_{\substack{-M_i\le r_i\le N_i\\ i=1,\dots,n\\ i\neq p,p+1}}\Bigg(\sum_{\substack{r_p=0\\ r_{p+1}=0}}^{N_p}+\sum_{\substack{r_{p+1}=1\\ r_p=0}}^{N_{p+1}}\Bigg)\lim_{x_{p+1}\to x_p^{-1}} L^{(p-1)}_{M,N;r}(x)\\
&=\sum_{\substack{-M_i\le r_i\le N_i\\ i=1,\dots,n\\ i\neq p,p+1}}\Bigg(\sum_{r_p=0}^{N_p}L^{(p)}_{M',N^{(p+1)};r^{(p+1)}}\big(x^{(p+1)}\big)\\
&\qquad\qquad+\sum_{r_{p+1}=1}^{N_{p+1}}L^{(p)}_{M',N^{(p+1)};\hat{r}^{(p)}}\big(x^{(p+1)}\big)\Bigg),\end{aligned}$$

where $M_{p+2}=\dots=M_n:=0$. Renaming the summation index r_{p+1} as $-r_p$, this yields

$$\lim_{x_{p+1}\to x_p^{-1}} L_{M,N}^{(p-1)}(x)=\sum_{\substack{-M_i'\le r_i\le N_i\\ i=1,\dots,n\\ i\neq p+1}} L_{M',N^{(p+1)};r^{(p+1)}}^{(p)}\big(x^{(p+1)}\big)=L_{M',N^{(p+1)}}^{(p)}\big(x^{(p+1)}\big),$$

where $M'_{p+1}=\dots=M'_n:=0$. □

Equipped with Lemma A.1, the proof of Proposition 5.1 is straightforward.

Proof. According to Lemma A.1

$$\lim_{y_p\to x_p^{-1}} L_{M,N}^{(p-1)}(x_1,\dots,x_p,y_p,x_{p+1},\dots,x_n)=L_{M',N^{(p+1)}}^{(p)}(x).$$

Iterating this equation and recalling (A.2) gives

$$\lim_{y_1\to x_1^{-1},\dots,y_p\to x_p^{-1}} L_{(N_1,M_1,\dots,N_p,M_p,N_{p+1},\dots,N_n)}(x_1,y_1,\dots,x_p,y_p,x_{p+1},\dots,x_n)\\ =L_{M,N}^{(p)}(x).$$

Recalling the remark made immediately after (A.2) this yields (5.5a) when $p=n$. If $p=n-1$, however, we obtain

$$\begin{aligned}\lim_{y_1\to x_1^{-1},\dots,y_{n-1}\to x_{n-1}^{-1}} & L_{(N_1,M_1,\dots,N_{n-1},M_{n-1},N_n)}(x_1,y_1,\dots,x_{n-1},y_{n-1},x_n)\\ &=\sum_{-M\subseteq r\subseteq N}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n\Bigg[\prod_{\ell=1}^{m+1}\frac{(b_\ell x_i,c_\ell x_i)_{r_i}}{(qx_i/b_\ell,qx_i/c_\ell)_{r_i}}\Big(\frac{q}{b_\ell c_\ell}\Big)^{r_i}\\ &\qquad\times\prod_{j=1}^n\frac{(q^{-N_j}x_i/x_j,q^{-M_j}x_ix_j)_{r_i}}{(q^{M_j+1}x_i/x_j,q^{N_j+1}x_ix_j)_{r_i}}\,q^{(M_j+N_j)r_i}\Bigg],\end{aligned}$$

where $M_n:=0$. Letting x_n tend to 1, treating the $r_n=0$ and $r_n>0$ cases of the summand separately, results in

$$\begin{aligned}\hat{L}_{M,N}(\hat{x})=\sum_{-M\subseteq r\subseteq N}u_{r_n}\frac{\Delta_{\mathrm{B}}(-xq^r)}{\Delta_{\mathrm{B}}(-x)}\prod_{i=1}^n\Bigg[\prod_{\ell=1}^{m+1}\frac{(b_\ell x_i,c_\ell x_i)_{r_i}}{(qx_i/b_\ell,qx_i/c_\ell)_{r_i}}\Big(\frac{q}{b_\ell c_\ell}\Big)^{r_i}\\ \times\prod_{j=1}^{n-1}\frac{(q^{-M_j}x_ix_j)_{r_i}}{(q^{M_j+1}x_i/x_j)_{r_i}}\,q^{M_jr_i}\prod_{j=1}^n\frac{(q^{-N_j}x_i/x_j)_{r_i}}{(q^{N_j+1}x_ix_j)_{r_i}}\,q^{N_jr_i}\Bigg],\end{aligned}$$

where $x=(x_1,\dots,x_{n-1},1)$ (so that $x_n:=1$), $M_n:=0$, $u_0=1$ and $u_i=2$ for $1\le i\le N_n$. Using (A.3) and the fact that for $x_n=1$

$$\frac{\Delta_{\mathrm{B}}(-xq^r)}{\Delta_{\mathrm{B}}(-x)}\bigg|_{r_n\mapsto -r_n}=q^{-(2n-1)r_n}\frac{\Delta_{\mathrm{B}}(-xq^r)}{\Delta_{\mathrm{B}}(-x)},$$

this can be rewritten in exactly the same functional form as the above but now with $M_n:=N_n$ and $u_i=1$ for all $-M_n\le i\le N_n$. □

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