DISCRETE ANALOGUES OF MACDONALD–MEHTA INTEGRALS

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Abstract. We consider discretisations of the Macdonald–Mehta integrals from the theory of finite reflection groups. For the classical groups, A_{r-1}, B_r and D_r, we provide closed-form evaluations in those cases for which the Weyl denominators featuring in the summands have exponents 1 and 2. Our proofs for the exponent-1 cases rely on identities for classical group characters, while most of the formulas for the exponent-2 cases are derived from a transformation formula for elliptic hypergeometric series for the root system BC_r. As a byproduct of our results, we obtain closed-form product formulas for the (ordinary and signed) enumeration of orthogonal and symplectic tableaux contained in a box.

1. Introduction

Motivated by work in [8] concerning the Hadamard maximal determinant problem [16], the recent papers [6, 7] considered various binomial multi-sum identities of which the following two results (the latter being conjectural in [6]) are representative:

$$\sum_{i,j,k=-n}^{n} \big|(i^2-j^2)(i^2-k^2)(j^2-k^2)\big| \binom{2n}{n+i}\binom{2n}{n+j}\binom{2n}{n+k} = 3\cdot 2^{2n-1} n^3 (n-1)\binom{2n}{n}^2 \tag{1.1}$$

and

$$\sum_{i,j=-n}^{n} \big|ij(i^2-j^2)\big| \binom{2n}{n+i}\binom{2n}{n+j} = \frac{2n^3(n-1)}{2n-1}\binom{2n}{n}^2. \tag{1.2}$$

Starting point for the current paper is the observation that these kinds of identities are reminiscent of multiple integral evaluations due to Macdonald and Mehta. To make this more precise, and to allow us to embed (1.1) and (1.2) into larger families of *discrete analogues* of Macdonald–Mehta integrals, we first review the continuous case.

Let G be a finite reflection group consisting of m reflecting hyperplanes $H_1, \dots, H_m$ in $\mathbb{R}^r$, see, e.g., [18]. Let $a_i \in \mathbb{R}^r$ be the normal of H_i normalised up to sign such that $\|a_i\|^2 := a_i \cdot a_i = 2$. For $x \in \mathbb{R}^r$ define the polynomial

$$P(x) = P_G(x) = \prod_{i=1}^{m} (a_i \cdot x). \tag{1.3}$$

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In 1982 Macdonald [30] conjectured that

$$\int_{\mathbb{R}^r} |P(x)|^{2\gamma} \, \mathrm{d}\varphi(x) = \prod_{i=1}^{r} \frac{\Gamma(1+d_i\gamma)}{\Gamma(1+\gamma)}, \tag{1.4}$$

where $\varphi(x)$ is the r-dimensional Gaußian measure

$$\mathrm{d}\varphi(x) = \frac{\mathrm{e}^{-\|x\|^2/2}}{(2\pi)^{r/2}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_r,$$

$d_1, \dots, d_r$ are the degrees of the fundamental invariants of G, and $\mathrm{Re}(\gamma) > -\min\{1/d_i\}$. For $G = \mathrm{A}_{r-1}$ the integral (1.4) had appeared as an earlier conjecture in work of Mehta and Dyson [33, 34] and is commonly referred to as Mehta's integral. It was first proved by Bombieri, who obtained it as a limit of the Selberg integral [45], see [11] for details. For the two other classical series, B_r and D_r, the conjecture also follows from the Selberg integral, as was already noted in Macdonald's original paper.[1] Complete proofs of Macdonald's conjecture were subsequently given in [10, 14, 37, 38].

The above-mentioned three classical series are of particular interest to us here. For these, we have

$$P_{\mathrm{A}_{r-1}}(x) = \prod_{1\leqslant i<j\leqslant r} (x_i - x_j) =: \Delta(x), \tag{1.5a}$$

$$P_{\mathrm{B}_r}(x) = 2^{r/2} \prod_{i=1}^{r} x_i \prod_{1\leqslant i<j\leqslant r} (x_i^2 - x_j^2), \quad \text{and} \quad P_{\mathrm{D}_r}(x) = \prod_{1\leqslant i<j\leqslant r} (x_i^2 - x_j^2), \tag{1.5b}$$

so that we can identify these cases of (1.4) as the $(\alpha, \delta) = (1, 0), (2, 2\gamma), (2, 0)$ instances of the *Macdonald–Mehta integral*

$$\mathcal{S}_r(\alpha, \gamma, \delta) := \int_{\mathbb{R}^r} |\Delta(x^\alpha)|^{2\gamma} \prod_{i=1}^{r} |x_i|^{\delta} \, \mathrm{d}\varphi(x). \tag{1.6}$$

It may now be recognised that (1.1) and (1.2) are discrete analogues of the D_3 and B_2 Macdonald–Mehta integral for $\gamma = 1/2$. This suggests that one should study the more general binomial sums

$$\mathcal{S}_{r,n}(\alpha, \gamma, \delta) := \sum_{k_1,\dots,k_r=-n}^{n} |\Delta(k^\alpha)|^{2\gamma} \prod_{i=1}^{r} |k_i|^{\delta} \binom{2n}{n+k_i}, \tag{1.7}$$

where n is a non-negative integer. It is easy to show that (1.7) is indeed a (scaled) discrete approximation to (1.6) in the sense that

$$\lim_{n\to\infty} 2^{-2rn} \left(\tfrac{1}{2}n\right)^{-\alpha\gamma\binom{r}{2}-\delta r/2} \mathcal{S}_{r,n}(\alpha, \gamma, \delta) = \mathcal{S}_r(\alpha, \gamma, \delta).$$

Using elements from representation theory and from the theory of elliptic hypergeometric series, respectively, we evaluate the *discrete Macdonald–Mehta integral* (1.7) for $\gamma = 1/2$ and $\gamma = 1$ and α, δ corresponding to A_{r-1}, B_r and D_r. By the same methods we can evaluate four additional cases that do not appear to be related to reflection groups (or root systems), and the total of ten evaluations is summarised in the following table:

[1] Macdonald attributes this to A. Regev, unpublished.

α	γ	δ	G
1	1/2	0	A_{r-1}
1	1	$0,1$	A_{r-1}, –
2	1/2	$0,1,2$	D_r, B_r, –
2	1	$0,1,2,3$	D_r, –, B_r, –

Table 1. The ten closed-form evaluations

All of these correspond to discrete analogues of the integrals

$$\mathcal{S}_r(1,\gamma,0)=\int\limits_{\mathbb{R}^r}\prod_{1\leqslant i<j\leqslant r}|x_i-x_j|^{2\gamma}\,\mathrm{d}\varphi(x)=\prod_{i=1}^{r}\frac{\Gamma(1+i\gamma)}{\Gamma(1+\gamma)}$$

for $\mathrm{Re}(\gamma)>-1/r$,

$$\begin{aligned}\mathcal{S}_r(1,1,1)&=\int\limits_{\mathbb{R}^r}\prod_{1\leqslant i<j\leqslant r}|x_i-x_j|^{2}\prod_{i=1}^{r}|x_i|\,\mathrm{d}\varphi(x)\\&=2^{r^2/2}\,\frac{\Gamma(1+r)}{\Gamma(\frac{1}{2})}\prod_{i=1}^{\lfloor\frac{1}{2}r\rfloor}\frac{\Gamma(i)\Gamma(1+i)}{\Gamma(\frac{1}{2})}\prod_{i=1}^{\lceil\frac{1}{2}r\rceil-1}\frac{\Gamma^2(1+i)}{\Gamma(\frac{1}{2})},\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_r(2,\gamma,\delta)&=\int\limits_{\mathbb{R}^r}\prod_{1\leqslant i<j\leqslant r}\big|x_i^2-x_j^2\big|^{2\gamma}\prod_{i=1}^{r}|x_i|^{\delta}\,\mathrm{d}\varphi(x)\\&=2^{2\gamma\binom{r}{2}+\delta r/2}\prod_{i=1}^{r}\frac{\Gamma(1+i\gamma)}{\Gamma(1+\gamma)}\cdot\frac{\Gamma(\frac{1}{2}+(i-1)\gamma+\frac{1}{2}\delta)}{\Gamma(\frac{1}{2})}\end{aligned}$$

for $\mathrm{Re}(\gamma)>-1/r$ and $\mathrm{Re}(\delta/2+(r-1)\gamma)>-1/2$. The first of these is the actual Mehta integral. Also the last integral (which was also considered by Macdonald in [30]) can easily be obtained as a limit of the Selberg integral by a generalisation of Regev's limiting procedure.

As a representative example of our results we state the closed-form evaluation of $\mathcal{S}_{r,n}(2,\frac{1}{2},0)$.

Proposition 1.1 (Discrete Macdonald–Mehta integral for D_r)**.** *Let r be a positive integer and n a non-negative integer. Then*

$$\begin{aligned}\mathcal{S}_{r,n}(2,\tfrac{1}{2},0)&=\sum_{k_1,\dots,k_r=-n}^{n}\prod_{1\leqslant i<j\leqslant r}\big|k_i^2-k_j^2\big|\prod_{i=1}^{r}\binom{2n}{n+k_i}\\&=2^{2rn-r(r-1)}\,\frac{\Gamma(1+\frac{1}{2}r)}{\Gamma(\frac{3}{2})}\cdot\frac{\Gamma(n-\frac{1}{2}r+\frac{3}{2})}{\Gamma(n+1)}\\&\qquad\times\prod_{i=1}^{r-1}\frac{\Gamma(i+1)}{\Gamma(\frac{3}{2})}\cdot\frac{\Gamma(2n+1)\,\Gamma(n-i+\frac{3}{2})}{\Gamma(2n-i+1)\,\Gamma(n-i+1)}.\end{aligned}\tag{1.8}$$

For $r=2$ this is [7, Theorem 1], for $r=3$ it is (1.1) (first proved in [6, Theorem 4.1]) and for $r=4$ this proves Conjecture 4.1 of that same paper. We further remark that both sides of (1.8) trivially vanish unless $n \geqslant r-1$. Indeed, all k_i^2 need to be distinct for the summand to be nonzero, requiring $n \geqslant r-1$. On the right the factor $1/\Gamma(n-i+1)|_{i=r-1}$ is identically zero for $0 \leqslant n \leqslant r-2$, and the poles of $\prod_i \Gamma(n-\frac{1}{2}r+\frac{3}{2})/\Gamma(2n-i+1)$ at $n=0,1,\dots,(r-3)/2$ (these only arise for odd values of r) have zero residue.

In several instances we obtain q-analogues and/or extensions to half-integer values of n (in which case the k_i need to be summed over half-integers so that $n+k_i \in \mathbb{Z}$). Furthermore, when $\gamma=1$ we prove more general summations containing additional free parameters, see Sections 6 and 7.

As a byproduct of our proofs, we obtain some new results on the enumeration of tableaux. A particularly elegant example concerns *Sundaram tableaux* [47]. These are semi-standard Young tableaux on the alphabet $1<\bar{1}<2<\bar{2}<\cdots<n<\bar{n}<\infty$ such that all entries in row k are at least k and with the exceptional rule that ∞ may occur multiple times in each column but at most once in each row. We denote the size (or number of squares) of T by $|T|$ and the number of occurrences of the letter k by $m_k(T)$. Obviously, $\sum_k m_k(T)=|T|$ with k summed over all $2n+1$ letters. For example,

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & \bar{1} & 2 & 5 & \infty \\
\hline
2 & \bar{2} & 3 & \infty \\
\cline{1-4}
3 & \bar{3} & \bar{4} & \infty \\
\cline{1-4}
\bar{5} & \bar{5} \\
\cline{1-2}
\end{array}$$

is a Sundaram tableau of size 15 for all $n \geqslant 5$.

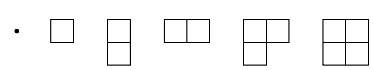
Theorem 1.2. *The number of Sundaram tableaux of height at most n and width at most r is given by*

$$\prod_{i=1}^{n} \frac{2i+r-1}{2i-1} \prod_{i,j=1}^{n} \frac{i+j+r-1}{i+j-1}.$$

Similarly, the number of Sundaram tableaux of height at most n and width at most r such that each tableaux is given a weight $(-1)^{|T|}$ (resp. $(-1)^{m_\infty(T)}$) is given by

$$(-1)^{rn} \prod_{i,j=1}^{n} \frac{i+j+r-1}{i+j-1} \qquad \Bigg(\textit{resp.} \; \prod_{i,j=1}^{n} \frac{i+j+r-1}{i+j-1}\Bigg).$$

For example, when $r=n=2$, there are $(3\cdot5\cdot3\cdot4\cdot4\cdot5)/(1\cdot3\cdot1\cdot2\cdot2\cdot3)=100$ tableaux, with the following break-down according to shape



1 5 10 14 35 35

or according to the multiplicities $m_\infty(T)$:

$$\big|\{T : m_\infty(T)=0\}\big|=50,\ \big|\{T : m_\infty(T)=1\}\big|=40,\ \big|\{T : m_\infty(T)=2\}\big|=10.$$

Moreover, $1-5+10+14-35+35=20=(3\cdot4\cdot4\cdot5)/(1\cdot2\cdot2\cdot3)$, and also $50-40+10=20$.

Our paper is organised as follows. The next, short section summarises the ten key evaluations corresponding to the binomial sums of Macdonald–Mehta-type listed in

Table 1. Section 3 reviews some standard material concerning classical group characters needed in our subsequent computations. Section 4 deals with summation identities for orthogonal and symplectic characters. Although several such identities were derived previously by Okada [36], his results are not sufficient for our purposes, and more refined identities as well as identities in which the summands have alternating signs are added to Okada's list. In Section 5 we then apply the results from Section 4 to evaluate the sums $\mathcal{S}_{r,n}(\alpha,\frac{1}{2},\delta)$ claimed in Section 2. In most cases, we are able to also provide q-analogues. Our evaluations of $\mathcal{S}_{r,n}(\alpha,1,\delta)$ given in Section 2 are dealt with in Sections 6 and 7. All these evaluations result from a single identity, a transformation formula between multiple elliptic hypergeometric series originally conjectured by the third author [48, Conj. 6.1], and proven independently by Rains [43, Theorem 4.9] and by Coskun and Gustafson [9]. We do not present this formula in its full generality here, but restrict ourselves to stating the relevant (q-)special case in Theorem 6.1 at the beginning of Section 6. The remainder of that section is devoted to proving our evaluations of the sums $\mathcal{S}_{r,n}(2,1,\delta)$, while Section 7 is devoted to proving the evaluations of the sums $\mathcal{S}_{r,n}(1,1,\delta)$. In all cases but one, we provide q-analogues which actually contain an additional parameter. The only exception is the sum $\mathcal{S}_{r,n}(1,1,1)$, where we are "only" able to establish a summation containing an additional parameter (see Proposition 7.2), but for which we were not able to find a q-analogue. Moreover, in this case we needed to take recourse to an ad hoc approach, since we could not figure out a way to use the aforementioned transformation formula. The final section, Section 8, discusses some further aspects of the work presented in this article, open problems, and (possible) further avenues.

To conclude the introduction, we point out two further articles addressing the multisums in [6]. First, in [28] the double sums considered in [6] are embedded into a three-parameter family of double sums, and it is shown that all of them can be explicitly computed by using complex contour integrals or by the use of the computer algebra package *Sigma* [44], thus proving in particular all the respective conjectures in [6], including (1.2). Second, Bostan, Lairez and Salvy [3] recently presented an algorithmic approach to finding recurrences for multiple binomial sums of the type considered in this paper. Interestingly, complex contour integrals are again instrumental in this approach. Among other things, it allowed them to prove automatically all the double-sum identities from [6], again including all the conjectures from [6], such as (1.2). Moreover, their algorithmic approach is — in principle — capable of proving any of our r-fold sum identities for *fixed* r. (As usual, "in principle" refers to the fact that today's computers may not actually be able to finish the required computations.) To come up with an automatic proof for any of our identities for *generic* r seems however to be currently out of reach.

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2. Summary of the ten primary identities

Here we summarise as succinctly as possible the ten product formulas for the discrete Macdonald–Mehta integral $\mathcal{S}_{r,n}(\alpha,\gamma,\delta)$ (defined in (1.7)), corresponding to the parameter choices listed in Table 1. Proofs and further generalisations are given in Sections 4–7.

For $\alpha=2$ there are a total of seven cases, given by

$$\begin{aligned}(2.1)\quad &\mathcal{S}_{r,n}(2,\gamma,\delta)\\ &=\prod_{i=1}^{r}\frac{\Gamma(1+i\gamma)}{\Gamma(1+\gamma)}\cdot\frac{\Gamma(2n+1)\,\Gamma(n-i-\gamma+\chi+2)\,\Gamma((i-1)\gamma+\frac{\delta+1}{2})}{\Gamma(n-i+\chi+1)\,\Gamma(n-i\gamma+\chi+1)\,\Gamma(n-(i-1)\gamma-\frac{\delta-3}{2}-\chi)},\end{aligned}$$

where $\chi=1$ if $\delta=0$, and $\chi=0$ otherwise. For $\alpha=1$ and $\delta=0$ there are two cases, given by

$$\mathcal{S}_{r,n}(1,\gamma,0)=2^{2rn-\gamma r(r-1)}\prod_{i=1}^{r}\frac{\Gamma(1+i\gamma)}{\Gamma(1+\gamma)}\cdot\frac{\Gamma(2n+1)\,\Gamma(2n-i+\gamma+2)}{\Gamma(2n-(i-2)\gamma+1)\,\Gamma(2n-i+2)}. \tag{2.2}$$

(This formula remains valid if $\gamma=0$ or n is a half-integer.)

The remaining case is

$$\mathcal{S}_{r,n}(1,1,1)=r!\prod_{i=1}^{\lceil r/2\rceil}\frac{\Gamma^2(i)\,\Gamma(2n+1)}{\Gamma(n-i+1)\,\Gamma(n-i+2)}\prod_{i=1}^{\lfloor r/2\rfloor}\frac{\Gamma(i)\,\Gamma(i+1)\,\Gamma(2n+1)}{\Gamma^2(n-i+1)}. \tag{2.3}$$

3. The Weyl character formula and Schur functions of type G

The purpose of this section is to collect standard material on classical group characters that we use in Sections 4 and 5.

3.1. **Some simple q-functions.** Assume that $0<q<1$ and m,n are integers such that $0\leqslant m\leqslant n$. Then the *q-shifted factorial*, *q-binomial coefficient*, *q-gamma function* and *q-factorial* are given by

$$(a;q)_n=\prod_{k=1}^{n}(1-aq^{k-1}),\qquad (a;q)_\infty=\prod_{k=1}^{\infty}(1-aq^{k-1})$$

$$\begin{bmatrix}n\\m\end{bmatrix}=\begin{bmatrix}n\\m\end{bmatrix}_q=\frac{(q^{n-m+1};q)_m}{(q;q)_m}$$

$$\Gamma_q(x)=(1-q)^{1-x}\,\frac{(q;q)_\infty}{(q^x;q)_\infty}$$

$$[n]_q=\frac{1-q^n}{1-q},\qquad [n]_q!=\Gamma_q(n+1)=[n]_q\,[n-1]_q\cdots[1]_q.$$

We also need some generalisations of the q-shifted factorials to partitions. We use standard terminology for partitions, as for example found in [31, Chapter 1], More precisely, let λ be a *partition*, that is, $\lambda=(\lambda_1,\lambda_2,\dots)$ is a weakly decreasing sequence of non-negative integers with only finitely many non-zero λ_i. The positive λ_i are called the *parts* of λ and the number of parts is called the *length* of the partition, denoted by $l(\lambda)$. As usual we identify a partition with its (Young) diagram, and the *conjugate partition* λ' is the partition obtained by reflecting the diagram in the main diagonal. We shall frequently need partitions of rectangular shape. By definition, this is a partition all of whose parts are the same. In order to have a convenient notation, we write (r^n) for the partition $(r,r,\dots,r)$ with n occurrences of r. If λ is a partition of length at most n and largest part at most r, we use the suggestive notation $\lambda\subseteq(r^n)$. Clearly this is equivalent to $\lambda'\subseteq(n^r)$. We say that (i,j) *is a square (in the diagram) of* λ and write $(i,j)\in\lambda$ if and only if $1\leqslant i\leqslant l(\lambda)$ and $1\leqslant j\leqslant\lambda_i$. Following [42], we now define

$$C^-_\lambda(a;q)=\prod_{(i,j)\in\lambda}\big(1-aq^{\lambda_i+\lambda'_j-i-j}\big) \tag{3.1a}$$

$$C^+_\lambda(a;q)=\prod_{(i,j)\in\lambda}\big(1-aq^{\lambda_i-\lambda'_j+j-i+1}\big) \tag{3.1b}$$

$$C^0_\lambda(a;q)=\prod_{(i,j)\in\lambda}\big(1-aq^{j-i}\big). \tag{3.1c}$$

Expressed in terms of ordinary q-binomial coefficients we have

$$C^-_\lambda(a;q)=\prod_{i=1}^n(aq^{n-i};q)_{\lambda_i}\prod_{1\leqslant i<j\leqslant n}\frac{1-aq^{j-i-1}}{1-aq^{\lambda_i-\lambda_j+j-i-1}} \tag{3.2a}$$

$$C^+_\lambda(a;q)=\prod_{i=1}^n\frac{(aq^{2-2i};q)_{2\lambda_i}}{(aq^{2-i-n};q)_{\lambda_i}}\prod_{1\leqslant i<j\leqslant n}\frac{1-aq^{2-i-j}}{1-aq^{\lambda_i+\lambda_j-i-j+2}} \tag{3.2b}$$

$$C^0_\lambda(a;q)=\prod_{i=1}^n(aq^{1-i};q)_{\lambda_i}, \tag{3.2c}$$

where n is an arbitrary integer such that $l(\lambda)\leqslant n$. Since conjugation simply interchanges rows and columns of a partition, it follows readily from (3.1) that

$$C^-_{\lambda'}(a;q)=C^-_\lambda(a;q) \tag{3.3a}$$

$$C^+_{\lambda'}(a;q)=(-aq)^{|\lambda|}q^{3n(\lambda)-3n(\lambda')}C^+_\lambda\big(a^{-1}q^{-2};q\big) \tag{3.3b}$$

$$C^0_{\lambda'}(a;q)=(-a)^{|\lambda|}q^{n(\lambda)-n(\lambda')}C^0_\lambda\big(a^{-1};q\big), \tag{3.3c}$$

where $|\lambda|:=\lambda_1+\lambda_2+\cdots$ and $n(\lambda):=\sum_{i\geqslant1}(i-1)\lambda_i=\sum_{i\geqslant1}\binom{\lambda'_i}{2}$.

3.2. **The Weyl character and dimension formulas.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank r, $\mathfrak{h}$ and $\mathfrak{h}^*$ the Cartan subalgebra and its dual, and Φ the root system spanning $\mathfrak{h}^*$ with basis of simple roots $\{\alpha_1,\dots,\alpha_r\}$, see e.g., [4,17]. Let $\langle\cdot,\cdot\rangle$ denote the usual symmetric bilinear form on $\mathfrak{h}^*$, and assume the standard identification of $\mathfrak{h}$ and $\mathfrak{h}^*$ through the Killing form so that the coroots are given by

$$\alpha^\vee=\frac{2\alpha}{\langle\alpha,\alpha\rangle}=\frac{2\alpha}{\|\alpha\|^2}.$$

Let $\omega_1,\dots,\omega_r$ be the fundamental weights, i.e., $\langle\omega_i,\alpha_j^\vee\rangle=\delta_{ij}$, and denote the root lattice $\mathbb{Z}\alpha_1\oplus\cdots\oplus\mathbb{Z}\alpha_r$ and weight lattice $\mathbb{Z}\omega_1\oplus\cdots\oplus\mathbb{Z}\omega_r$ by Q and P, respectively. Further, let P_+ be the set of dominant (integral) weights,

$$P_+=\big\{\lambda\in P:\ \langle\lambda,\alpha_i^\vee\rangle\geqslant0\text{ for }1\leqslant i\leqslant r\big\},$$

and set

$$Q_+ = \big\{\alpha \in Q :\ \langle \alpha^\vee, \omega_i\rangle \geqslant 0 \text{ for } 1 \leqslant i \leqslant r\big\}.$$

We also denote the set of positive roots by Φ_+, so that $\Phi_+ = Q_+ \cap \Phi$.

The irreducible highest weight modules $V(\lambda)$ of $\mathfrak{g}$ are indexed by dominant weights λ. The characters corresponding to these modules are defined as

$$\operatorname{ch} V(\lambda) := \sum_{\mu\in\mathfrak{h}^*} \dim(V_\mu)\, \mathrm{e}^{\mu},$$

where the V_μ are the weight spaces in the weight-space decomposition of $V(\lambda)$ and e^λ for $\lambda \in P$ is a formal exponential satisfying $\mathrm{e}^\lambda \mathrm{e}^\mu = \mathrm{e}^{\lambda+\mu}$. It is a well-known fact that $\dim(V_\lambda) = 1$ and $\dim(V_\mu) = 0$ if $\lambda - \mu \notin Q_+$. The characters can be computed explicitly using the Weyl character formula

$$\operatorname{ch} V(\lambda) = \frac{\sum_{w\in W} \operatorname{sgn}(w)\, \mathrm{e}^{w(\lambda+\rho)-\rho}}{\prod_{\alpha>0}(1-\mathrm{e}^{-\alpha})}. \tag{3.4}$$

Here, W is the Weyl group of $\mathfrak{g}$, $\alpha > 0$ is shorthand for $\alpha \in \Phi_+$, and $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_{i=1}^r \omega_i$ is the Weyl vector. For $\lambda = 0$, Weyl's formula simplifies to the denominator identity

$$\sum_{w\in W} \operatorname{sgn}(w)\, \mathrm{e}^{w(\rho)-\rho} = \prod_{\alpha>0}(1-\mathrm{e}^{-\alpha}). \tag{3.5}$$

The dimension of the highest weight module $V(\lambda)$ follows from the Weyl character formula by applying the map $\mathrm{e}^\lambda \mapsto 1$. We will require two slightly more general specialisations resulting in q-dimension formulas. Let s be the squared length of the short roots in Φ and define F and $F^\vee$ by

$$\begin{aligned} F &: \mathbb{Z}[\mathrm{e}^{-\alpha_0},\dots,\mathrm{e}^{-\alpha_r}] \to \mathbb{Z}[q^s], & F(\mathrm{e}^{-\alpha_i}) &= q^{\langle \rho,\alpha_i\rangle} \\ F^\vee &: \mathbb{Z}[\mathrm{e}^{-\alpha_0},\dots,\mathrm{e}^{-\alpha_r}] \to \mathbb{Z}[q], & F^\vee(\mathrm{e}^{-\alpha_i}) &= q^{\langle \rho,\alpha_i^\vee\rangle} = q \end{aligned}$$

for all i with $1 \leqslant i \leqslant r$. By defining the q-dimensions by

$$\dim_q V(\lambda) := F\big(\mathrm{e}^{-\lambda} \operatorname{ch} V(\lambda)\big) \quad\text{and}\quad \dim_q^\vee V(\lambda) := F^\vee\big(\mathrm{e}^{-\lambda} \operatorname{ch} V(\lambda)\big),$$

we have the following pair of dimension formulas.

Lemma 3.1. *We have*

$$\dim_q V(\lambda) = \prod_{\alpha>0} \frac{1-q^{\langle \lambda+\rho,\alpha\rangle}}{1-q^{\langle\rho,\alpha\rangle}}, \tag{3.6a}$$

$$\dim_q^\vee V(\lambda) = \prod_{\alpha>0} \frac{1-q^{\langle \lambda+\rho,\alpha^\vee\rangle}}{1-q^{\langle\rho,\alpha^\vee\rangle}}. \tag{3.6b}$$

In the $q \to 1$ limit, (3.6a) implies the Weyl dimension formula

$$\dim V(\lambda) = \prod_{\alpha>0} \frac{\langle \lambda+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}.$$

Proof. Applying F to $\mathrm{e}^{-\lambda}\operatorname{ch}V(\lambda)\in\mathbb{Z}[\mathrm{e}^{-\alpha_1},\dots,\mathrm{e}^{-\alpha_r}]$ and using (3.4), we obtain

$$\dim_q V(\lambda)=\frac{\sum_{w\in W}\operatorname{sgn}(w)q^{-\langle\rho,w(\lambda+\rho)-\lambda-\rho\rangle}}{\prod_{\alpha>0}(1-q^{\langle\rho,\alpha\rangle})}.$$

Since $\langle\rho,w(\lambda{+}\rho)\rangle=\langle w^{-1}(\rho),\lambda{+}\rho\rangle$ and $\operatorname{sgn}(w)=\operatorname{sgn}(w^{-1})$, a change of the summation index from w to w^{-1} results in

$$\dim_q V(\lambda)=\frac{\sum_{w\in W}\operatorname{sgn}(w)q^{-\langle w(\rho)-\rho,\lambda+\rho\rangle}}{\prod_{\alpha>0}(1-q^{\langle\rho,\alpha\rangle})}.$$

The first claim now follows from the denominator formula (3.5) with $\mathrm{e}^{-u}\mapsto q^{-\langle u,\lambda+\rho\rangle}$.

The proof of (3.6b) is nearly identical and is left to the reader. □

In the next four subsections we restrict the Weyl character and dimension formulas to the four classical types and give "dual" forms for the q-dimension formulas needed in our proofs of the discrete Macdonald–Mehta integrals.

3.3. **The Schur functions.** For $x=(x_1,\dots,x_n)$ and λ a partition of length at most n, the *Schur function* $s_\lambda(x)$ is defined by

(3.7) $$s_\lambda(x):=\frac{\det_{1\leqslant i,j\leqslant n}(x_i^{\lambda_j+n-j})}{\det_{1\leqslant i,j\leqslant n}(x_i^{n-j})}.$$

If $\Lambda_n=\mathbb{Z}[x_1,\dots,x_n]^{\mathfrak{S}_n}$ denotes the ring of symmetric functions in n variables, then the Schur functions indexed by partitions of length at most n form a basis of Λ_n. The Schur functions have a simple interpretation in terms of the representation theory of the symmetric group $\mathfrak{S}_n$ and the general linear group $\mathrm{GL}_n(\mathbb{C})$. More precisely, they are exactly the characters of the irreducible (polynomial) representations of $\mathrm{GL}_n(\mathbb{C})$. The representation theory of $\mathrm{SL}_n(\mathbb{C})$ is almost identical to that of $\mathrm{GL}_n(\mathbb{C})$, the only notable difference being that in the former irreducible representations are indexed by partitions of length at most $n-1$, and to interpret such $s_\lambda(x)$ as a character we should impose the restriction $x_1\cdots x_n=1$. Since the Schur function $s_\lambda(x)$ is homogeneous of degree λ and satisfies

$$s_\lambda(x)=(x_1\cdots x_n)^{\lambda_n}s_{(\lambda_1-\lambda_n,\dots,\lambda_{n-1}-\lambda_n,0)}(x),$$

these differences do not affect any of the underlying combinatorics. In particular, if $\mathfrak{g}$ is the Lie algebra $\mathrm{sl}_n(\mathbb{C})$ and ϕ the ring isomorphism

(3.8a) $$\phi:\mathbb{Z}\big[\mathrm{e}^\lambda:\lambda\in P\big]^W\to\mathbb{Z}\big[x_1,\dots,x_{n-1},x_1^{-1}\cdots x_{n-1}^{-1}\big]^{\mathfrak{S}_n}=\Lambda_n'$$

(3.8b) $$\phi(\mathrm{e}^{\omega_i})=x_1\cdots x_i\quad\text{for } 1\leqslant i\leqslant n-1,$$

then

(3.9) $$\phi\big(\operatorname{ch}V(\lambda)\big)=s_\lambda(x)|_{x_n=x_1^{-1}\cdots x_{n-1}^{-1}},$$

where on the left λ is a dominant weight parametrised as

(3.10) $$\lambda=(\lambda_1-\lambda_2)\omega_1+\dots+(\lambda_{n-2}-\lambda_{n-1})\omega_{n-2}+\lambda_{n-1}\omega_{n-1}$$

and on the right λ is the partition $(\lambda_1,\dots,\lambda_{n-1},0)$.

Instead of using the ratio of determinants given in (3.7), we can compute the Schur function in a more combinatorial fashion using semi-standard Young tableaux. Namely,

$$s_\lambda(x) = \sum_T x^T, \tag{3.11}$$

where the sum is over all semi-standard Young tableaux T of shape λ on the alphabet $1 < 2 < \cdots < n$ and $x^T := x_1^{m_1(T)} \cdots x_n^{m_n(T)}$.

From Lemma 3.1 and equation (3.9), it follows that for $l(\lambda) \leqslant n$ we have the principal specialisation formula

$$s_\lambda(1, q, \dots, q^{n-1}) = q^{n(\lambda)} \prod_{1 \leqslant i < j \leqslant n} \frac{1 - q^{\lambda_i - \lambda_j + j - i}}{1 - q^{j-i}}. \tag{3.12}$$

Indeed, since the above only depends on differences between the parts of λ, we may assume without loss of generality that $\lambda_n = 0$. Since the set of positive roots is given by

$$\{\alpha_i + \cdots + \alpha_j : \ 1 \leqslant i \leqslant j \leqslant n-1\},$$

it follows that for $\lambda \in P_+$ parametrised by (3.10) we have

$$\dim_q V(\lambda) = \dim_q^\vee V(\lambda) = \prod_{1 \leqslant i \leqslant j \leqslant n-1} \frac{1 - q^{\lambda_i - \lambda_{j+1} + j - i + 1}}{1 - q^{j-i+1}}. \tag{3.13}$$

Since $F(\mathrm{e}^{-\omega_i}) = q^{i(n-i)/2}$, it follows from (3.8b) that under the induced action of F on Λ_n' we have

$$F(x_i) = q^{i-(n+1)/2} \quad \text{for } 1 \leqslant i \leqslant n-1.$$

We also have $F(\mathrm{e}^{-\lambda}) = q^{(n-1)|\lambda|/2 - n(\lambda)}$, where on the right λ is the partition corresponding to $\lambda \in P_+$ on the left. Hence,

$$\begin{aligned} s_\lambda(1, q, \dots, q^{n-1}) &= q^{(n-1)|\lambda|/2} s_\lambda\big(q^{-(n-1)/2}, q^{-(n-3)/2}, \dots, q^{(n-1)/2}\big) \\ &= q^{(n-1)|\lambda|/2} F\big(s_\lambda(x)\big) \\ &= q^{n(\lambda)} F(\mathrm{e}^{-\lambda} \operatorname{ch} V(\lambda)) = q^{n(\lambda)} \dim_q V(\lambda), \end{aligned}$$

which by (3.13) implies (3.12). All of the above is well-known, although rarely made explicit. Since later we want to refer to analogous results for other groups without spelling out the (less well-known) details, we have included the full details of the Schur function case. We also note that each of the principal specialisation formulas for the classical groups has a dual form obtained by using conjugate partitions. These dual forms will be crucial later.

Lemma 3.2 (Principal specialisation — dual form)**.** *For $\lambda \subseteq (r^n)$, we have*

$$s_\lambda(1, q, \dots, q^{n-1}) = q^{n(\lambda)} \prod_{i=1}^{r} \begin{bmatrix} n+r-1 \\ \lambda_i' + r - i \end{bmatrix} \begin{bmatrix} n+r-1 \\ r-i \end{bmatrix}^{-1} \prod_{1 \leqslant i < j \leqslant r} \frac{1 - q^{\lambda_i' - \lambda_j' + j - i}}{1 - q^{j-i}}.$$

Proof. Perhaps the most elegant proof is to use the *dual Jacobi–Trudi identity* [31, p. 41] and the principal specialisation formula for the *elementary symmetric functions* [31, p. 27], combined with the the determinant evaluation [24, Theorem 26].

In view of the other types yet to be discussed, we will proceed in a slightly different manner. By (3.2), we can write (3.12) as

$$s_\lambda(1,q,\dots,q^{n-1}) = q^{n(\lambda)}\frac{C^0_\lambda(q^n;q)}{C^-_\lambda(q;q)}.$$

According to (3.3), the right-hand side also equals

$$(-q^n)^{|\lambda|}q^{n(\lambda')}\frac{C^0_{\lambda'}(q^{-n};q)}{C^-_{\lambda'}(q;q)},$$

which, by (3.2) with $n\mapsto r$, is

$$(-q^n)^{|\lambda|}q^{n(\lambda')}\prod_{i=1}^{r}\frac{(q^{1-i-n};q)_{\lambda'_i}}{(q^{r-i+1};q)_{\lambda'_i}}\prod_{1\leqslant i<j\leqslant r}\frac{1-q^{\lambda'_i-\lambda'_j+j-i}}{1-q^{j-i}}.$$

By

$$\prod_{i=1}^{r}\frac{(q^{1-i-n};q)_{\lambda'_i}}{(q^{r-i+1};q)_{\lambda'_i}} = (-q^n)^{-|\lambda|}q^{n(\lambda)-n(\lambda')}\prod_{i=1}^{r}\frac{(q;q)_{n+i-1}(q;q)_{r-i}}{(q;q)_{\lambda'_i+r-i}(q;q)_{n+i-\lambda'_i-1}}, \tag{3.14}$$

the lemma follows. □

3.4. **The odd-orthogonal Schur functions.** A sequence $(\lambda_1,\dots,\lambda_n)$ is called a half-partition if $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n>0$ and $\lambda_i\in\mathbb{Z}+1/2$.

For $x=(x_1,\dots,x_n)$ and $\lambda=(\lambda_1,\dots,\lambda_n)$ a partition or half-partition, the *odd-orthogonal Schur functions* are defined as (cf. [13, 29])

$$\mathrm{so}_{2n+1,\lambda}(x) := \frac{\det_{1\leqslant i,j\leqslant n}\bigl(x_i^{\lambda_j+n-j+1/2}-x_i^{-(\lambda_j+n-j+1/2)}\bigr)}{\det_{1\leqslant i,j\leqslant n}\bigl(x_i^{n-j+1/2}-x_i^{-(n-j+1/2)}\bigr)}. \tag{3.15}$$

The $\mathrm{so}_{2n+1,\lambda}(x)$ again arise from (3.4), this time for $\mathfrak{g}=\mathrm{so}_{2n+1}(\mathbb{C})$. Defining ϕ by

$$\phi:\mathbb{Z}\bigl[\,\mathrm{e}^\lambda:\lambda\in P\bigr]^W\to\mathbb{Z}\bigl[x_1^{\pm1/2},\dots,x_n^{\pm1/2}\bigr]^{B_n}$$
$$\phi(\mathrm{e}^{-\omega_i}) = \begin{cases} x_1\cdots x_i, & \text{for } 1\leqslant i\leqslant n-1,\\ (x_1\cdots x_n)^{1/2}, & \text{for } i=n,\end{cases}$$

where B_n is the *hyperoctahedral group* acting on the x_i by permuting them and by sending x_i to x_i^{-1} for some i, we have

$$\phi\bigl(\,\mathrm{ch}\,V(\lambda)\bigr) = \mathrm{so}_{2n+1,\lambda}(x), \tag{3.16}$$

where on the left λ is a dominant weight parametrised as

$$\lambda = (\lambda_1-\lambda_2)\omega_1+\cdots+(\lambda_{n-1}-\lambda_n)\omega_{n-1}+2\lambda_n\omega_n,$$

and on the right λ is the partition or half-partition $(\lambda_1,\dots,\lambda_n)$.

For later use, we will also define the companion

$$\mathrm{so}^+_{2n+1,\lambda}(x) := \frac{\det_{1\leqslant i,j\leqslant n}\bigl(x_i^{\lambda_j+n-j+1/2}+x_i^{-(\lambda_j+n-j+1/2)}\bigr)}{\det_{1\leqslant i,j\leqslant n}\bigl(x_i^{n-j+1/2}+x_i^{-(n-j+1/2)}\bigr)}. \tag{3.17}$$

If λ is a partition, it readily follows that

$$\mathrm{so}^+_{2n+1,\lambda}(x) = (-1)^{|\lambda|}\,\mathrm{so}_{2n+1,\lambda}(-x). \tag{3.18}$$

For half-partitions, however, $\mathrm{so}^+_{2n+1,\lambda}(x)$ is a rational function such that

$$\mathrm{so}^+_{2n+1,\lambda}(x)D(x) \in \mathbb{Z}[x^{\pm}]^{B_n}, \qquad D(x) := \prod_{i=1}^{n}(x_i^{1/2} + x_i^{-1/2}).$$

Since for half-partitions $\mathrm{so}_{2n+1,\lambda}(x)D(x) \in \mathbb{Z}[x^{\pm}]^{B_n}$, it follows that, regardless of the type of λ, we have

$$\mathrm{so}_{2n+1,\lambda}(x)\,\mathrm{so}^+_{2n+1,\lambda}(x) \in \mathbb{Z}[x^{\pm}]^{B_n}.$$

In terms of the Sundaram tableaux introduced on page 4, for λ a partition we have

$$\mathrm{so}_{2n+1,\lambda}(x) = \sum_T x^T,$$

where the sum is over all Sundaram tableaux of shape λ and

$$x^T := \prod_{k=1}^{n} x_k^{m_k(T)-m_{\bar{k}}(T)}. \tag{3.19}$$

Lemma 3.3 (Principal specialisation — dual form)**.** *For $\lambda \subseteq (r^n)$ a partition, we have*

$$\begin{aligned}\mathrm{so}_{2n+1,\lambda}(q,q^2,\dots,q^n) = q^{n(\lambda)-n|\lambda|}\prod_{i=1}^{r}\begin{bmatrix}2n+2r-1\\ \lambda'_i+r-i\end{bmatrix}\begin{bmatrix}2n+2r-1\\ r-i\end{bmatrix}^{-1}\\ \times\prod_{1\leqslant i<j\leqslant r}\frac{1-q^{\lambda'_i-\lambda'_j+j-i}}{1-q^{j-i}}\cdot\frac{1-q^{2n-\lambda'_i-\lambda'_j+i+j-1}}{1-q^{2n+i+j-1}}\end{aligned} \tag{3.20a}$$

and

$$\begin{aligned}\mathrm{so}_{2n+1,\lambda}(q^{1/2},q^{3/2},\dots,q^{n-1/2}) = q^{n(\lambda)-(n-1/2)|\lambda|}\\ \times\prod_{i=1}^{r}\frac{1+q^{n-\lambda'_i+i-1/2}}{1+q^{n+i-1/2}}\begin{bmatrix}2n+2r-1\\ \lambda'_i+r-i\end{bmatrix}\begin{bmatrix}2n+2r-1\\ r-i\end{bmatrix}^{-1}\\ \times\prod_{1\leqslant i<j\leqslant r}\frac{1-q^{\lambda'_i-\lambda'_j+j-i}}{1-q^{j-i}}\cdot\frac{1-q^{2n-\lambda'_i-\lambda'_j+i+j-1}}{1-q^{2n+i+j-1}}.\end{aligned} \tag{3.20b}$$

Proof. Let $\epsilon_1,\dots,\epsilon_n$ be the standard unit vectors in $\mathbb{R}^n$. Assuming the realisation $\{\alpha_1,\dots,\alpha_n\} = \{\epsilon_1-\epsilon_2,\dots,\epsilon_{n-1}-\epsilon_n,\epsilon_n\}$ for the simple roots of $\mathrm{so}_{2n+1}(\mathbb{C})$ (see [17]), the fundamental weights and positive roots are given by

$$\begin{aligned}\{\omega_1,\dots,\omega_n\} &= \{\epsilon_1,\epsilon_1+\epsilon_2,\dots,\epsilon_1+\dots+\epsilon_{n-1},\tfrac{1}{2}(\epsilon_1+\dots+\epsilon_n)\},\\ \{\alpha\in\Phi:\alpha>0\} &= \{\epsilon_i : 1\leqslant i\leqslant n\}\cup\{\epsilon_i\pm\epsilon_j : 1\leqslant i<j\leqslant n\}.\end{aligned}$$

Hence, by (3.6b), (3.16) and $F(x_i)=q^{n-i+1}$, we have

$$\begin{aligned}\mathrm{so}_{2n+1,\lambda}(q,q^2,\dots,q^n)&=q^{n(\lambda)-n|\lambda|}\dim_q^{\vee}V(\Lambda)\\&=q^{n(\lambda)-n|\lambda|}\prod_{i=1}^{n}\frac{1-q^{2\lambda_i+2n-2i+1}}{1-q^{2n-2i+1}}\\&\quad\times\prod_{1\leqslant i<j\leqslant n}\frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}\cdot\frac{1-q^{\lambda_i+\lambda_j+2n-i-j+1}}{1-q^{2n-i-j+1}}.\end{aligned}\tag{3.21}$$

It follows from (3.2) that the right-hand side can be expressed in terms of the generalised q-shifted factorials as

$$q^{n(\lambda)-n|\lambda|}\,\frac{C^0_\lambda(q^n,-q^n,q^{n+1/2},-q^{n+1/2};q)}{C^-_\lambda(q;q)C^+_\lambda(q^{2n-1};q)},$$

where $C^0_\lambda(a_1,\dots,a_k;q)=C^0_\lambda(a_1;q)\cdots C^0_\lambda(a_k;q)$. By (3.3), this is also

$$(-q^{n+1})^{|\lambda|}q^{n(\lambda')}\,\frac{C^0_{\lambda'}(q^{-n},-q^{-n},q^{-n-1/2},-q^{-n-1/2};q)}{C^-_{\lambda'}(q;q)C^+_{\lambda'}(q^{-2n-1};q)}.$$

Again using (3.2), but now with n replaced by r, this is

$$(-q^{n+r})^{|\lambda|}q^{n(\lambda')}\prod_{i=1}^{r}\frac{(q^{1-i-2n-r};q)_{\lambda'_i}}{(q^{r-i+1};q)_{\lambda'_i}}\prod_{1\leqslant i<j\leqslant r}\frac{1-q^{\lambda'_i-\lambda'_j+j-i}}{1-q^{j-i}}\cdot\frac{1-q^{2n-\lambda_i-\lambda_j+i+j-1}}{1-q^{2n+i+j-1}}.\tag{3.22}$$

By (3.14) with $n\mapsto 2n+r$, the first claim follows.

The second specialisation (3.20b) follows in much the same way by applying (3.2) and (3.3) to

$$\begin{aligned}&\mathrm{so}_{2n+1,\lambda}(q^{1/2},q^{3/2},\dots,q^{n-1/2})\\&\quad=q^{n(\lambda)-(n-1/2)|\lambda|}\dim_q V(\Lambda)\\&\quad=q^{n(\lambda)-(n-1/2)|\lambda|}\prod_{i=1}^{n}\frac{1-q^{\lambda_i+n-i+1/2}}{1-q^{n-i+1/2}}\\&\qquad\times\prod_{1\leqslant i<j\leqslant n}\frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}\cdot\frac{1-q^{\lambda_i+\lambda_j+2n-i-j+1}}{1-q^{2n-i-j+1}}.\end{aligned}\tag{3.23}$$

□

For later reference we also state the principal specialisation of $\mathrm{so}^+_{2n+1,\lambda}(x)$.

Lemma 3.4. *For $\lambda=(\lambda_1,\dots,\lambda_n)$ a partition or half-partition, we have*

$$\begin{aligned}\mathrm{so}^+_{2n+1,\lambda}(q^{1/2},\dots,q^{n-1/2})&=q^{n(\lambda)-(n-1/2)|\lambda|}\prod_{i=1}^{n}\frac{1+q^{\lambda_i+n-i+1/2}}{1+q^{n-i+1/2}}\\&\quad\times\prod_{1\leqslant i<j\leqslant n}\frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}\cdot\frac{1-q^{\lambda_i+\lambda_j+2n-i-j+1}}{1-q^{2n-i-j+1}}.\end{aligned}\tag{3.24}$$

Proof. According to (3.5), the denominator identity for B_n (or $\mathrm{so}_{2n+1,\lambda}(\mathbb{C})$) is given by (see also [24, Equation (2.4)])

$$\det_{1\leqslant i,j\leqslant n}\Big(x_i^{n-j+1/2}-x_i^{-(n-j+1/2)}\Big) = (-1)^{\binom{n+1}{2}}\prod_{i=1}^{n}x_i^{1/2-n}(1-x_i)\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(1-x_ix_j). \tag{3.25}$$

Replacing x_i by $-x_i$ (readers worried about a choice of branch-cut should first multiply both sides by $\prod_i x_i^{-1/2}$ and later divide by this factor) and taking the transpose of the determinant, we obtain (see also [24, Equation (2.6)])

$$\det_{1\leqslant i,j\leqslant n}\Big(x_j^{n-i+1/2}+x_j^{-(n-i+1/2)}\Big)=\prod_{i=1}^{n}x_i^{1/2-n}(1+x_i)\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(1-x_ix_j). \tag{3.26}$$

If we specialise $x_i=q^{n-i+1/2}$ $(1\leqslant i\leqslant n)$ in (3.17), then we get

$$\mathrm{so}^{+}_{2n+1,\lambda}(q^{1/2},\dots,q^{n-1/2})=\frac{\det_{1\leqslant i,j\leqslant n}\Big(q^{(\lambda_j+n-j+1/2)(n-i+1/2)}+q^{-(\lambda_j+n-j+1/2)(n-i+1/2)}\Big)}{\det_{1\leqslant i,j\leqslant n}\Big(q^{(n-j+1/2)(n-i+/2)}+q^{-(n-j+1/2)(n-i+1/2)}\Big)}. \tag{3.27}$$

By (3.26) with $x_j=q^{\lambda_j+n-j+1/2}$ or $x_j=q^{n-j+1/2}$, both determinants on the right-hand side can be expressed in product form, resulting in (3.24). ☐

3.5. **The symplectic Schur functions.** For $x=(x_1,\dots,x_n)$ and λ a partition of length at most n, the *symplectic Schur functions* are defined as

$$\mathrm{sp}_{2n,\lambda}(x):=\frac{\det_{1\leqslant i,j\leqslant n}\Big(x_i^{\lambda_j+n-j+1}-x_i^{-(\lambda_j+n-j+1)}\Big)}{\det_{1\leqslant i,j\leqslant n}\Big(x_i^{n-j+1}-x_i^{-(n-j+1)}\Big)}. \tag{3.28}$$

If $\mathfrak{g}=\mathrm{sp}_{2n}(\mathbb{C})$, then

$$\phi\big(\operatorname{ch}V(\lambda)\big)=\mathrm{sp}_{2n,\lambda}(x),$$

where $\phi(\mathrm{e}^{-\omega_i})=x_1\cdots x_i$ $(1\leqslant i\leqslant n)$ and

$$P_+\ni\lambda=(\lambda_1-\lambda_2)\omega_1+\cdots+(\lambda_{n-1}-\lambda_n)\omega_{n-1}+\lambda_n\omega_n.$$

To express this combinatorially, we need the symplectic tableaux of King and El-Sharkaway [20, 21]. These are semi-standard Young tableaux on $1<\bar{1}<2<\bar{2}<\cdots<n<\bar{n}$ such that all entries in row k are at least k. For example,

$$\begin{array}{|c|c|c|c|c|}\hline 1&\bar{1}&2&3&\bar{5}\\\hline 2&\bar{2}&3&4\\\cline{1-4} \bar{4}&\bar{4}&5\\\cline{1-3}\end{array}$$

is a symplectic tableau for $n\geqslant 5$. The symplectic analogue of (3.11) then is

$$\mathrm{sp}_{2n,\lambda}(x)=\sum_T x^T,$$

where the sum is over all symplectic tableaux of shape λ and x^T is again given by (3.19).

Lemma 3.5 (Principal specialisation — dual form)**.** *For $\lambda \in (r^n)$, we have*

$$\operatorname{sp}_{2n,\lambda}(q,q^2,\dots,q^n) = q^{n(\lambda)-n|\lambda|} \prod_{i=1}^{r} \frac{1-q^{n-\lambda'_i+i}}{1-q^{n+i}} \begin{bmatrix} 2n+2r \\ \lambda'_i+r-i \end{bmatrix} \begin{bmatrix} 2n+2r \\ r-i \end{bmatrix}^{-1} \times \prod_{1\leqslant i<j\leqslant r} \frac{1-q^{\lambda'_i-\lambda'_j+j-i}}{1-q^{j-i}} \cdot \frac{1-q^{2n-\lambda'_i-\lambda'_j+i+j}}{1-q^{2n+i+j}} \tag{3.29a}$$

and

$$\begin{aligned}\operatorname{sp}_{2n,\lambda}&(q^{1/2},q^{3/2},\dots,q^{n-1/2}) \\ &= q^{n(\lambda)-(n-1/2)|\lambda|} \prod_{i=1}^{r} \frac{1-q^{2(n-\lambda'_i+i)}}{1-q^{2(n+i)}} \begin{bmatrix} 2n+2r \\ \lambda'_i+r-i \end{bmatrix} \begin{bmatrix} 2n+2r \\ r-i \end{bmatrix}^{-1} \\ &\quad \times \prod_{1\leqslant i<j\leqslant r} \frac{1-q^{\lambda'_i-\lambda'_j+j-i}}{1-q^{j-i}} \cdot \frac{1-q^{2n-\lambda'_i-\lambda'_j+i+j}}{1-q^{2n+i+j}}.\end{aligned} \tag{3.29b}$$

Proof. If we take the simple roots to be $\{\alpha_1,\dots,\alpha_n\} = \{\epsilon_1-\epsilon_2,\dots,\epsilon_{n-1}-\epsilon_n,2\epsilon_n\}$ (see [17]), then

$$\{\omega_1,\dots,\omega_n\} = \{\epsilon_1,\epsilon_1+\epsilon_2,\dots,\epsilon_1+\dots+\epsilon_n\},$$
$$\{\alpha\in\Phi:\alpha>0\} = \{2\epsilon_i : 1\leqslant i\leqslant n\} \cup \{\epsilon_i\pm\epsilon_j : 1\leqslant i<j\leqslant n\}.$$

From Lemma 3.1, it then follows that

$$\begin{aligned}\operatorname{sp}_{2n,\lambda}(q,q^2,\dots,q^n) &= q^{n(\lambda)-n|\lambda|}\dim_q V(\lambda) \\ &= q^{n(\lambda)-n|\lambda|}\prod_{i=1}^{n}\frac{1-q^{2(\lambda_i+n-i+1)}}{1-q^{2(n-i+1)}} \\ &\quad\times\prod_{1\leqslant i<j\leqslant n}\frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}\cdot\frac{1-q^{\lambda_i+\lambda_j+2n-i-j+2}}{1-q^{2n-i-j+2}}\end{aligned} \tag{3.30a}$$

and

$$\begin{aligned}\operatorname{sp}_{2n,\lambda}(q^{1/2},q^{3/2},\dots,q^{n-1/2}) &= q^{n(\lambda)-(n-1/2)|\lambda|}\dim_q^{\vee} V(\Lambda) \\ &= q^{n(\lambda)-(n-1/2)|\lambda|}\prod_{i=1}^{n}\frac{1-q^{\lambda_i+n-i+1}}{1-q^{n-i+1}} \\ &\quad\times\prod_{1\leqslant i<j\leqslant n}\frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}\cdot\frac{1-q^{\lambda_i+\lambda_j+2n-i-j+2}}{1-q^{2n-i-j+2}}.\end{aligned} \tag{3.30b}$$

The rest of the proof is analogous to that of Lemma 3.3; we omit the details. □

3.6. **The even-orthogonal Schur functions.** Let a D_n partition be a weakly decreasing sequence $(\lambda_1,\dots,\lambda_n)$ such that each $\lambda_i\in\mathbb{Z}$ or each $\lambda_i\in\mathbb{Z}+1/2$, and such that $\lambda_{n-1}\geqslant|\lambda_n|$. If λ is a D_n partition then so is $\bar\lambda := (\lambda_1,\dots,\lambda_{n-1},-\lambda_n)$.

For $x = (x_1, \dots, x_n)$ and λ a D_n partition, the *even-orthogonal Schur functions* are defined by

$$\mathrm{so}_{2n,\lambda}(x) := \sum_{\sigma \in \{\pm 1\}} \frac{\det_{1 \leqslant i,j \leqslant n}\big(\sigma x_i^{\lambda_j+n-j} + x_i^{-(\lambda_j+n-j)}\big)}{\det_{1 \leqslant i,j \leqslant n}\big(x_i^{n-j} + x_i^{-(n-j)}\big)}. \tag{3.31}$$

We note that $\mathrm{so}_{2n,\bar{\lambda}}(x) = \mathrm{so}_{2n,\lambda}(\bar{x})$, where $\bar{x} := (x_1, \dots, x_{n-1}, x_n^{-1})$. Assuming $\mathfrak{g} = \mathrm{so}_{2n}(\mathbb{C})$, we have

$$\phi\big(\operatorname{ch} V(\lambda)\big) = \mathrm{so}_{2n,\lambda}(x),$$

where

$$\phi(\mathrm{e}^{-\omega_i}) = \begin{cases} x_1 \cdots x_i, & \text{for } 1 \leqslant i \leqslant n-2, \\ (x_1 \cdots x_{n-1} x_n^{-1})^{1/2}, & \text{for } i = n-1, \\ (x_1 \cdots x_n)^{1/2}, & \text{for } i = n, \end{cases}$$

and

$$P_+ \ni \lambda = (\lambda_1 - \lambda_2)\omega_1 + \cdots + (\lambda_{n-1} - \lambda_n)\omega_{n-1} + (\lambda_{n-1} + \lambda_n)\omega_n.$$

For our purposes it is not enough to consider $\mathrm{so}_{2n,\lambda}(x)$; we also need the closely related *even-orthogonal characters* (cf. [22])

$$\mathrm{o}_{2n,\lambda}(x) = u_\lambda \frac{\det_{1 \leqslant i,j \leqslant n}\big(x_i^{\lambda_j+n-j} + x_i^{-(\lambda_j+n-j)}\big)}{\det_{1 \leqslant i,j \leqslant n}\big(x_i^{n-j} + x_i^{-(n-j)}\big)}, \tag{3.32}$$

where λ is a partition or half-partition and $u_\lambda = 1$ if $l(\lambda) < n$ and $u_\lambda = 2$ if $l(\lambda) = n$. Note that

$$\mathrm{o}_{2n,\lambda}(x) = \begin{cases} \mathrm{so}_{2n,\lambda}(x), & \text{if } l(\lambda) < n, \\ \mathrm{so}_{2n,\lambda}(x) + \mathrm{so}_{2n,\bar{\lambda}}(x), & \text{if } l(\lambda) = n. \end{cases} \tag{3.33}$$

Also the even-orthogonal characters can be expressed in terms of a tableau sum, see, e.g., [12, 41]. We will however not define these tableaux here and instead restrict our attention to the simpler "even Sundaram tableaux" of [12]. An *even Sundaram tableau* is a semi-standard Young tableau on the alphabet $1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n} < \infty$ such that all entries in row k are at least $\bar{k}$, with the exception that ∞ may occur multiple times in each column but at most once in each row. Note that the only difference with the earlier definition of Sundaram tableaux is that entries in row k have to be at least $\bar{k}$ instead of k. This implies that 1 cannot actually occur in an even Sundaram tableaux. Due to the absence of the letter 1, it is not known how to assign monomials to even Sundaram tableaux so that they generate $\mathrm{o}_{2n,\lambda}(x)$. It is however shown in [12] that $\mathrm{o}_{2n,\lambda}(1^n)$ correctly counts the number of even Sundaram tableaux of shape λ.

Lemma 3.6. *For λ a partition contained in (r^n), we have*

$$
\begin{aligned}
(3.34)\quad & \mathrm{o}_{2n,\lambda}(q^{1/2},q^{3/2},\dots,q^{n-1/2}) \\
&= q^{n(\lambda)-(n-1/2)|\lambda|}\prod_{i=1}^{r}\begin{bmatrix}2n+2r-2\\ \lambda_i'+r-i\end{bmatrix}\begin{bmatrix}2n+2r-2\\ r-i\end{bmatrix}^{-1} \\
&\quad\times\prod_{1\leqslant i<j\leqslant r}\frac{1-q^{\lambda_i'-\lambda_j'+j-i}}{1-q^{j-i}}\cdot\frac{1-q^{2n-\lambda_i'-\lambda_j'+i+j-2}}{1-q^{2n+i+j-2}}.
\end{aligned}
$$

There is a similar result for $\mathrm{o}_{2n,\lambda}(1,q,\dots,q^{n-1})$, but this is not needed.

Proof. If we specialise $x_i=q^{n-i+1/2}$ in (3.32), with $1\leqslant i\leqslant n$, and then use the determinant evaluation (3.26) with $x_j=q^{\lambda_j+n-j}$ or $x_j=q^{n-j}$, we obtain

$$
\begin{aligned}
(3.35)\quad \mathrm{o}_{2n,\lambda}(q^{1/2},q^{3/2},\dots,q^{n-1/2}) &= u_\lambda\, q^{n(\lambda)-(n-1/2)|\lambda|}\prod_{i=1}^{n}\frac{1+q^{\lambda_i+n-i}}{1+q^{n-i}} \\
&\quad\times\prod_{1\leqslant i<j\leqslant n}\frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}\cdot\frac{1-q^{\lambda_i+\lambda_j+2n-i-j}}{1-q^{2n-i-j}}.
\end{aligned}
$$

The rest of the proof follows that of Lemma 3.3. □

For later reference we note that it follows in much the same way from (3.25) and (3.26) that

$$
\begin{aligned}
(3.36)\quad & \mathrm{so}_{2n,\lambda}(q^{1/2},q^{3/2},\dots,q^{n-1/2}) \\
&= q^{n(\lambda)-(n-1/2)|\lambda|}\left(\prod_{i=1}^{n}\frac{1+q^{\lambda_i+n-i}}{1+q^{n-i}}+\prod_{i=1}^{n}\frac{1-q^{\lambda_i+n-i}}{1+q^{n-i}}\right) \\
&\quad\times\prod_{1\leqslant i<j\leqslant n}\frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}\cdot\frac{1-q^{\lambda_i+\lambda_j+2n-i-j}}{1-q^{2n-i-j}}.
\end{aligned}
$$

4. Okada-type formulas

With the exception of type A_{r-1}, our proofs of the discrete analogues of Macdonald–Mehta integrals for $\gamma=1/2$ given in the next section rely on formulas for the multiplication of Schur functions of type $\mathfrak{g}$ indexed by partitions of rectangular shape. Such formulas have been given by Okada in [36]. We use several of his formulas, but we also require additional ones. In the subsection below, we list all these results, and we present the (principal) specialisations of these formulas that we actually need. Subsection 4.2 provides the proofs of the new results not contained in [36]. These proofs heavily rely on "preparatory results" from [36].

4.1. **Main results.** Our first result applies to $\mathfrak{g}=\mathrm{so}_{2n+1}(\mathbb{C})$. Let $\mathrm{so}^{-}_{2n+1,\lambda}(x):=\mathrm{so}_{2n+1,\lambda}(x)$.

Theorem 4.1. *Let r be a non-negative integer, $\varepsilon\in\{-1,1\}$ and $s:=\frac{1}{2}r$. Then*

$$
(4.1)\qquad \sum_{\lambda\subseteq(r^n)}\varepsilon^{|\lambda|}\,\mathrm{so}_{2n+1,\lambda}(\varepsilon x)=\mathrm{so}_{2n+1,(s^n)}(x)\,\mathrm{so}^{\sigma}_{2n+1,(s^n)}(x),
$$

where the sum on the left is over partitions, and $\sigma = -$ if $\varepsilon = 1$ and $\sigma = +$ if $\varepsilon = -1$.

For $\varepsilon = 1$ this is (a special case of) Okada's [36, Theorem 2.5(1)].

Later we require (4.1) in principally specialised form as follows from (3.21), (3.23) and (3.24) for $\lambda = (s^n)$.

Corollary 4.2. *For r a non-negative integer and $\varepsilon \in \{-1, 1\}$, we have*

$$\sum_{\lambda \subseteq (r^n)} \mathrm{so}_{2n+1,\lambda}(q, q^2, \dots, q^n) = q^{-r\binom{n+1}{2}} \frac{(q^{r+1}; q^2)_n}{(q; q^2)_n} \prod_{i,j=1}^{n} \frac{1 - q^{i+j+r-1}}{1 - q^{i+j-1}} \tag{4.2a}$$

and

$$\sum_{\lambda \subseteq (r^n)} \varepsilon^{|\lambda|} \mathrm{so}_{2n+1,\lambda}(\varepsilon q^{1/2}, \varepsilon q^{3/2}, \dots, \varepsilon q^{n-1/2}) \\ = q^{-rn^2/2} \frac{(q^{(r+1)/2}; q)_n (\varepsilon q^{(r+1)/2}; q)_n}{(q^{1/2}; q)_n (\varepsilon q^{1/2}; q)_n} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{1 - q^{i+j+r-1}}{1 - q^{i+j-1}}, \tag{4.2b}$$

where λ is summed over partitions.

Letting q tend to 1 in (4.2a) (or the $\varepsilon = 1$ case of (4.2b)) yields the unweighted enumeration of Sundaram tableaux given in Theorem 1.2. Taking $\varepsilon = -1$ in (4.2b), then using

$$\frac{(q^{(r+1)/2}; q)_n (-q^{(r+1)/2}; q)_n}{(q^{1/2}; q)_n (-q^{1/2}; q)_n} = \frac{(q^{r+1}; q^2)_n}{(q; q^2)_n},$$

and finally letting $q^{1/2}$ tend to ± 1 gives

$$\sum_{T \subseteq (r^n)} (-1)^{|T|} (\mp 1)^{\sum_{k=1}^{n} (m_k(T) + m_{\bar{k}}(T))} = (\pm 1)^{rn} \prod_{i,j=1}^{n} \frac{i + j + r - 1}{i + j - 1}.$$

Since

$$|T| = m_\infty(T) + \sum_{k=1}^{n} \big(m_k(T) + m_{\bar{k}}(T)\big),$$

this results in the two weighted enumerations of that theorem.

Next we consider $\mathfrak{g} = \mathrm{sp}_{2n}(\mathbb{C})$.

Theorem 4.3. *Let r be a non-negative integer and $s := \lfloor \frac{1}{2} r \rfloor$, $t := \lceil \frac{1}{2} r \rceil$. Then*

$$\sum_{\lambda \subseteq (r^n)} \mathrm{sp}_{2n,\lambda}(x) = \mathrm{sp}_{2n,(s^n)}(x)\, \mathrm{so}_{2n+1,(t^n)}(x). \tag{4.3}$$

This identity follows from [36, Theorem 2.5(1)] by observing that (see e.g. [41, Proposition A2.1(c)])

$$\mathrm{so}_{2n+1,\lambda+1/2}(x) = \mathrm{sp}_{2n,\lambda}(x) \prod_{i=1}^{n} (x_i^{1/2} + x_i^{-1/2}),$$

where $\lambda + 1/2$ stands for $(\lambda_1 + 1/2, \dots, \lambda_n + 1/2)$. It is interesting to note that Proctor [39, Lemma 4, equation for $A_{2n}(m\omega_r)$, case $r = n$] obtained this same sum from a specialised Schur function. (In representation-theoretic terms: the restriction of an

$\mathrm{SL}_{2n+1}(\mathbb{C})$-character indexed by a rectangular shape to $\mathrm{Sp}_{2n}(\mathbb{C})$ decomposes into the sum of symplectic characters indexed by all shapes contained in that rectangle; see also [23, Equation (3.4)].) He used his result to prove the (at the time conjectured) formula for the number of symmetric self-complementary plane partitions contained in a given box.

Once again, use of (3.21) as well as (3.30) yields our second corollary.

Corollary 4.4. *For r a non-negative integer, we have*

$$\sum_{\lambda\subseteq(r^n)} \mathrm{sp}_{2n,\lambda}(q,q^2,\dots,q^n) = q^{-r\binom{n+1}{2}} \prod_{i=1}^{n+1}\prod_{j=1}^{n} \frac{1-q^{i+j+r-1}}{1-q^{i+j-1}} \tag{4.4a}$$

and

$$\sum_{\lambda\subseteq(r^n)} \mathrm{sp}_{2n,\lambda}(q^{1/2},q^{3/2},\dots,q^{n-1/2}) = q^{-rn^2/2} \prod_{i=1}^{2n} \frac{1-q^{(i+r)/2}}{1-q^{i/2}} \prod_{i=1}^{n}\prod_{j=1}^{n-1} \frac{1-q^{i+j+r}}{1-q^{i+j}}. \tag{4.4b}$$

Letting $q^{1/2}$ tend to ±1 in (4.4b) implies two counting formulas for symplectic tableaux.

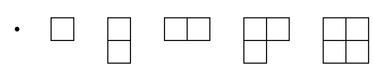
Theorem 4.5. *The number of symplectic tableaux of height at most n and width at most r is given by*

$$\prod_{i=1}^{n+1}\prod_{j=1}^{n} \frac{i+j+r-1}{i+j-1} \tag{4.5}$$

and the number of such tableaux weighted by $(-1)^{|T|}$ is

$$(-1)^{rn} \prod_{i=1}^{n} \frac{i+\lfloor r/2\rfloor}{i} \prod_{i=1}^{n}\prod_{j=1}^{n-1} \frac{i+j+r}{i+j}.$$

For example, when $r=n=2$ there are $(3\cdot 4^2\cdot 5^2\cdot 6)/(1\cdot 2^2\cdot 3^2\cdot 4)=50$ symplectic tableaux, with the following break-down according to shape



1 4 5 10 16 14

so that the signed enumeration is $1-4+5+10-16+14=10=(2\cdot3\cdot4\cdot5)/(1\cdot2^2\cdot3)$.

We remark that (4.5) is not actually new, and it is implicit in [39] that the number of symplectic tableaux contained in (r^m) $(0\leqslant m\leqslant n)$ is given by

$$\prod_{i=1}^{2n-m+1}\prod_{j=1}^{m} \frac{i+j+r-1}{i+j-1}.$$

See also [26, Theorem 7] for an equivalent statement in terms of vicious walkers (non-intersecting lattice paths).

Our final Okada-type formula involves the even-orthogonal as well as orthogonal characters.

Theorem 4.6. *Let r be a positive integer. Then*

$$\sum_{\lambda\subseteq(r^n)} \mathrm{so}_{2n,\lambda}(x)=\mathrm{so}_{2n,(s^n)}(x)\,\mathrm{so}_{2n+1,(s^n)}(x), \tag{4.6a}$$

where $s:=\frac{1}{2}r$, and

$$\sum_{\substack{\lambda\subseteq(r^n)\\ l(\lambda)=n}} \mathrm{o}_{2n,\lambda}(x)=\mathrm{o}_{2n,(s^n)}(x)\,\mathrm{so}_{2n+1,(t^n)}(x), \tag{4.6b}$$

where $s:=\frac{1}{2}(r+1)$ and $t:=\frac{1}{2}(r-1)$.

We remark that (4.6b) also holds when the orthogonal characters are replaced by even-orthogonal Schur functions, but in some sense this is a weakening of the result. In the other direction, the analogous result does not hold for (4.6a) in that we cannot replace the even-orthogonal Schur functions by orthogonal characters.

By (3.23), (3.35) and (3.36), the above two identities result in the final corollary of this section.

Corollary 4.7. *For r a positive integer, we have*

$$\begin{aligned}\sum_{\lambda\subseteq(r^n)} &\mathrm{so}_{2n,\lambda}(q^{1/2},q^{3/2},\dots,q^{n-1/2}) \\ &=q^{-rn^2/2}\frac{(q^{r/2+1/2};q)_n}{(q^{1/2};q)_n}\bigg(\frac{(-q^{r/2};q)_n}{(-1;q)_n}+\frac{(q^{r/2};q)_n}{(-1;q)_n}\bigg)\prod_{i=1}^{n}\prod_{j=1}^{n-1}\frac{1-q^{i+j+r-1}}{1-q^{i+j-1}}\end{aligned} \tag{4.7a}$$

and

$$\begin{aligned}\sum_{\substack{\lambda\subseteq(r^n)\\ l(\lambda)=n}} &\mathrm{o}_{2n,\lambda}(q^{1/2},q^{3/2},\dots,q^{n-1/2}) \\ &=2q^{-rn^2/2}\frac{(q^{r/2};q)_n}{(q^{1/2};q)_n}\cdot\frac{(-q^{r/2+1/2};q)_n}{(-1;q)_n}\prod_{i=1}^{n}\prod_{j=1}^{n-1}\frac{1-q^{i+j+r-1}}{1-q^{i+j-1}}.\end{aligned} \tag{4.7b}$$

If we let $q\to 1$ in (4.7b), we obtain a closed-form expression for the number of even Sundaram tableaux of height exactly n and width at most r. From (3.33) and $\mathrm{so}_{2n,\bar{\lambda}}(x)=\mathrm{so}_{2n,\lambda}(\bar{x})$, it follows that

$$\mathrm{o}_{2n,\lambda}(1^n)=u_\lambda\,\mathrm{so}_{2n,\lambda}(1^n). \tag{4.8}$$

Hence we can combine (4.7a) and (4.7b) to also obtain the enumeration of such tableaux contained in (r^n).

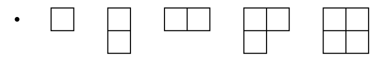
Theorem 4.8. *The number of even Sundaram tableaux of height at most n and width at most r is given by*

$$2^{2n-1}\frac{(\frac{1}{2}r+\frac{1}{2})_n+(\frac{1}{2}r)_n}{n!}\prod_{i=1}^{n}\prod_{j=1}^{n-1}\frac{i+j+r-1}{i+j},$$

and the number of such tableaux of height exactly n is

$$2^{2n}\frac{(\frac{1}{2}r)_n}{n!}\prod_{i=1}^{n}\prod_{j=1}^{n-1}\frac{i+j+r-1}{i+j}.$$

For example, when $r = n = 2$ there are 46 even Sundaram tableaux, with the following break-down according to shape



1 4 6 9 16 10

so that exactly 32 of these have height 2.

4.2. **Proof of Theorems 4.1 and 4.6.** Our proofs closely follow Okada's Pfaffian-based approach, which relies on two key results: the Ishikawa–Wakayama minor summation formula [19, Theorem 2] (see also [35, Theorem 3]) and Okada's Pfaffian evaluation [36, Theorem 4.4].

Recall that the Pfaffian of a $2m \times 2m$ skew-symmetric matrix M is defined as

$$\mathrm{Pf}(M) = \sum_{\pi}(-1)^{c(\pi)}\prod_{(i,j)\in\pi} M_{ij},$$

where the sum is over perfect matchings π (or 1-factorisations) of the complete graph on $2m$ vertices (labelled $1, 2, \dots, 2m$), and the product is over all edges (i, j) in the matching, $1 \le i < j \le 2m$. The crossing number $c(\pi)$ of a perfect matching π is the number of pairs of edges (i, j) and (k, l) of π such that $i < k < j < l$.

Theorem 4.9 (Minor summation formula)**.** *Let n and r be positive integers such that n is even and $n \leqslant r$, and let M be an arbitrary $n \times r$ matrix. Then*

$$\sum_{\substack{J\subseteq\{1,\dots,r\}\\|J|=n}}\det_{\substack{1\leqslant i\leqslant n\\ j\in J}}\big(M_{ij}\big) = \mathrm{Pf}(B), \tag{4.9}$$

where B is the $n \times n$ skew-symmetric matrix

$$B = MAM^t, \tag{4.10}$$

with A the $r \times r$ skew-symmetric matrix with entries $A_{ij} = 1$ for $j > i$.

Here it should be understood that J is viewed as an ordered n-subset of $\{1, \dots, r\}$, i.e., $J = \{j_1 < j_2 < \cdots < j_n\}$.

Theorem 4.10 (Okada's Pfaffian evaluation)**.** *Let $x = (x_1, \dots, x_n)$ where n is even. Let $Q(x; a, b)$ be the $n \times n$ skew-symmetric matrix with entries*

$$Q_{ij}(x; a, b) = \frac{q(x_i, x_j, a_i, a_j)q(x_i, x_j, b_i, b_j)}{(x_i - x_j)(1 - x_i x_j)}, \tag{4.11}$$

where $q(\alpha, \beta, \gamma, \delta) := (\alpha - \beta)(1 - \gamma\delta) - (1 - \alpha\beta)(\gamma - \delta)$, and let $W(x; a)$ be the $n \times n$ matrix with entries

$$W_{ij}(x; a) = a_i x_i^{n-j} - x_i^{j-1}. \tag{4.12}$$

Then

$$\operatorname{Pf}\big(Q(x;a,b)\big)=\frac{\det\big(W(x;a)\big)\det\big(W(x;b)\big)}{\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(1-x_ix_j)}. \tag{4.13}$$

Combining these two theorems we readily obtain the following result.

Corollary 4.11. *Let n,r be positive integers such that n is even, $\varepsilon\in\{\pm1\}$, and $M=M(a,\varepsilon)$ is the $n\times r$ matrix with entries*

$$M_{ij}=x_i^{j-a}-\varepsilon x_i^{-j+a}.$$

Then

$$\sum_{\substack{J\subseteq\{1,\dots,r\}\\|J|=n}}\det_{\substack{1\leqslant i\leqslant n\\ j\in J}}\big(M_{ij}\big)=(-1)^{n/2}\det_{1\leqslant i,j\leqslant n}\Big(x_i^{r/2+n/2-j-a+1}-\varepsilon x_i^{-(r/2+n/2-a-j+1)}\Big)$$
$$\times\frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{r/2+n/2-j+1/2}-x_i^{-(r/2+n/2-j+1/2)}\big)}{\det_{1\leqslant i,j\leqslant n}\big(x_i^{n-j+1/2}-x_i^{-(n-j+1/2)}\big)}.$$

Proof. A routine calculation using the summation of geometric series shows that for the above choice of M, the matrix B in (4.10) is given by

$$B_{ij}=\sum_{1\leqslant k<l\leqslant r}\big(M_{ik}M_{jl}-M_{il}M_{jk}\big)=\frac{(x_ix_j)^{a-r}}{(1-x_i)(1-x_j)}\,Q_{ij}\big(x;x^r,\varepsilon x^{r-2a+1}\big),$$

where x^a is shorthand for $(x_1^a,\dots,x_n^a)$. Since $\operatorname{Pf}(u_iu_jv_{ij})=\big(\prod_i u_i\big)\operatorname{Pf}(v_{ij})$, we obtain

$$\begin{aligned}\operatorname{Pf}(B)&=\bigg(\prod_{i=1}^n\frac{x_i^{a-r}}{1-x_i}\bigg)\operatorname{Pf}\Big(Q\big(x;x^r,\varepsilon x^{r-2a+1}\big)\Big)\\&=\frac{\det\big(W(x;x^r)\big)\det\big(W(x;\varepsilon x^{r-2a+1})\big)}{\prod_{i=1}^n x_i^{r-a}(1-x_i)\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(1-x_ix_j)}.\end{aligned}$$

Use of (4.12), the B_n Vandermonde determinant (3.25), and the fact that n is even and $\varepsilon^2=1$ completes the proof. □

Proof of Theorem 4.1. From (see, e.g., [36, Lemma 5.3(2)])

$$\lim_{x_n\to0}x_n^r\operatorname{so}^{\sigma}_{2n+1,\lambda}(x_1,\dots,x_n)=\begin{cases}\operatorname{so}^{\sigma}_{2n-1,\mu}(x_1,\dots,x_{n-1}), & \text{if } r=\lambda_1,\\ 0, & \text{if } r>\lambda_1,\end{cases} \tag{4.14}$$

for $\mu:=(\lambda_2,\dots,\lambda_{n-1})$, it follows that, if we multiply both sides of (4.1) by x_n^r and let x_n tend to zero, we obtain (4.1) with n replaced by $n-1$. Hence it suffices to prove the claim for even values of n.

Let S_r denote the left-hand side of (4.1). From (3.18), it follows that

$$S_r=\sum_{\lambda\subseteq(r^n)}\operatorname{so}^{\sigma}_{2n+1,\lambda}(x).$$

By (3.15) and (3.17), this can also be written as

$$S_r=\frac{\sum_{\lambda\subseteq(r^n)}\det_{1\leqslant i,j\leqslant n}\big(x_i^{\lambda_j+n-j+1/2}-\varepsilon x_i^{-(\lambda_j+n-j+1/2)}\big)}{\det_{1\leqslant i,j\leqslant n}\big(x_i^{n-j+1/2}-\varepsilon x_i^{-(n-j+1/2)}\big)}.$$

If we replace the sum over λ by a sum over $k_1,\dots,k_n$ via the substitution

$$\lambda_j = k_{n-j+1} - \rho_j - 1/2, \quad \text{for } 1 \leqslant j \leqslant n,$$

and reverse the order of the columns in the determinant, this leads to

$$S_r = (-1)^{\binom{n}{2}} \sum_{1\leqslant k_1<k_2<\cdots<k_n\leqslant r+n} \frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{k_j-1/2} - \varepsilon x_i^{-(k_j-1/2)}\big)}{\det_{1\leqslant i,j\leqslant n}\big(x_i^{n-j+1/2} - \varepsilon x_i^{-(n-j+1/2)}\big)}.$$

Now assume that n is even. We can then apply Corollary 4.11 with $r \mapsto r+n$ and $a = 1/2$ to find

$$S_r = \frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{r/2+n-j+1/2} - \varepsilon x_i^{-(r/2+n-j+1/2)}\big)}{\det_{1\leqslant i,j\leqslant n}\big(x_i^{n-j+1/2} - \varepsilon x_i^{-(n-j+1/2)}\big)} \times \frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{r/2+n-j+1/2} - x_i^{-(r/2+n-j+1/2)}\big)}{\det_{1\leqslant i,j\leqslant n}\big(x_i^{n-j+1/2} - x_i^{-(n-j+1/2)}\big)}.$$

Finally, recalling (3.15) and (3.17), we obtain

$$S_r = \mathrm{so}^{\sigma}_{2n+1,(s^n)}(x)\,\mathrm{so}_{2n+1,(s^n)}(x),$$

with $s = \frac{1}{2}r$. □

Proof of Theorem 4.6. Equation 4.14 once again holds when $\mathrm{so}^{\sigma}_{2n+1,\lambda}$ is replaced by $\mathrm{so}_{2n,\lambda}$ or $\mathrm{o}_{2n,\lambda}$, so that we may again take n to be even.

Let S_r and S'_r denote the left-hand sides of (4.6a) and (4.6b), respectively. Using (3.31) and (3.32) and making the substitutions

$$\begin{cases} S_r: & \lambda_j = k_{n-j+1} - n + j - 1, \\ S'_r: & \lambda_j = k_{n-j+1} - n + j, \end{cases} \qquad \text{for } 1 \leqslant j \leqslant n,$$

we get

$$S_r = (-1)^{\binom{n}{2}} \sum_{\sigma\in\{\pm1\}} \sum_{1\leqslant k_1<k_2<\cdots<k_n\leqslant r+n} \frac{\det_{1\leq i,j\leq n}\big(\sigma x_i^{k_j-1} + x_i^{-(k_j-1)}\big)}{\det_{1\leq i,j\leq n}\big(x_i^{n-j} + x_i^{-(n-j)}\big)}$$

and

$$S'_r = 2(-1)^{\binom{n}{2}} \sum_{1\leqslant k_1<k_2<\cdots<k_n\leqslant r+n-1} \frac{\det_{1\leq i,j\leq n}\big(x_i^{k_j} + x_i^{-k_j}\big)}{\det_{1\leq i,j\leq n}\big(x_i^{n-j} + x_i^{-(n-j)}\big)}.$$

By Corollary 4.11 with $r \mapsto r+n$, $a = 1$, $\varepsilon = -\sigma$ and $r \mapsto r+n-1$, $a = 0$, $\varepsilon = -1$, respectively, this yields

$$S_r = \frac{\sum_{\sigma\in\{\pm1\}}\det_{1\leqslant i,j\leqslant n}\big(\sigma x_i^{r/2+n-j} + x_i^{-(r/2+n-j)}\big)}{\det_{1\leq i,j\leq n}\big(x_i^{n-j} + x_i^{-(n-j)}\big)} \times \frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{r/2+n/2-j+1/2} - x_i^{-(r/2+n/2-j+1/2)}\big)}{\det_{1\leqslant i,j\leqslant n}\big(x_i^{n-j+1/2} - x_i^{-(n-j+1/2)}\big)}$$

and

$$S'_r = 2\,\frac{\det_{1\leqslant i,j\leqslant n}\left(x_i^{r/2+n-j+1/2}+x_i^{-(r/2+n-j+1/2)}\right)}{\det_{1\leq i,j\leq n}\left(x_i^{n-j}+x_i^{-(n-j)}\right)} \times \frac{\det_{1\leqslant i,j\leqslant n}\left(x_i^{r/2+n-j}-x_i^{-(r/2+n-j)}\right)}{\det_{1\leqslant i,j\leqslant n}\left(x_i^{n-j+1/2}-x_i^{-(n-j+1/2)}\right)},$$

where we have used that n is even. From (3.15) and (3.31), we see that the expression for S_r is exactly

$$\mathrm{so}_{2n,(s^n)}(x)\,\mathrm{so}_{2n+1,(s^n)}(x), \qquad s:=\tfrac{1}{2}r,$$

and that for S'_r

$$\mathrm{o}_{2n,(s^n)}(x)\,\mathrm{so}_{2n+1,(t^n)}(x), \qquad s:=\tfrac{1}{2}(r+1),\ t:=\tfrac{1}{2}(r-1). \qquad \square$$

5. Discrete Macdonald–Mehta integrals for $\gamma = 1/2$

We will slightly extend our earlier definition (1.7) by considering $\mathcal{S}_{r,n}(\alpha,\gamma,\delta)$ for n a non-negative integer or half-integer. In the latter case, the sum over $k_1,\dots,k_r$ is assumed to range over half-integers, so that in both cases the k_i are summed over $\{-n,-n+1,\dots,n\}$.

5.1. **The evaluation of $\mathcal{S}_{r,n}(1,\frac{1}{2},0)$.** Instead of computing this sum directly, we first consider a q-analogue.

Proposition 5.1 (A_{r-1} summation)**.** *Let $0<q<1$, r a positive integer and n an integer or half-integer such that $n\geqslant (r-1)/2$. Then*

$$\sum_{k_1,\dots,k_r=-n}^{n}\ \prod_{1\leqslant i<j\leqslant r}\bigl|[k_i-k_j]_q\bigr|\prod_{i=1}^{r} q^{(k_i+n-r+i)^2/2}\begin{bmatrix}2n\\ n+k_i\end{bmatrix} = \frac{r!}{[r]_{q^{1/2}}!}\prod_{i=1}^{r}(-q^{1/2};q^{1/2})_i(-q^{i/2+1};q)_{2n-r} \times\prod_{i=1}^{r}\frac{\Gamma_q(1+\frac{1}{2}i)}{\Gamma_q(\frac{3}{2})}\cdot\frac{\Gamma_q(2n+1)\,\Gamma_q(2n-i+\frac{5}{2})}{\Gamma_q(2n-i+2)\,\Gamma_q(2n-\frac{1}{2}i+2)}. \tag{5.1}$$

Taking the $q\to 1$ limit, we arrive at (cf. (2.2))

$$\mathcal{S}_{r,n}(1,\tfrac{1}{2},0)=\sum_{k_1,\dots,k_r=-n}^{n}\ \prod_{1\leqslant i<j\leqslant r}|k_i-k_j|\prod_{i=1}^{r}\binom{2n}{n+k_i} = 2^{2rn-\binom{r}{2}}\prod_{i=1}^{r}\frac{\Gamma(1+\frac{1}{2}i)}{\Gamma(\frac{3}{2})}\cdot\frac{\Gamma(2n+1)\,\Gamma(2n-i+\frac{5}{2})}{\Gamma(2n-i+2)\,\Gamma(2n-\frac{1}{2}i+2)}. \tag{5.2}$$

The evaluation of $S_{1,1}(n)$ in [6, Equation (5.6)] is the special case $r=2$ of this identity.

Proof of Proposition 5.1. Denote the sum on the left of (5.1) by $f_{n,r}$. Since

$$q^{\sum_{i=1}^r (k_i+n-r+i)^2/2} \prod_{1\leqslant i<j\leqslant r} \bigl|1-q^{k_i-k_j}\bigr| \\ = q^{(n-1/2)\binom{r}{2}-2\binom{r}{3}+\sum_{i=1}^r (k_i+n-r+1)^2/2} \prod_{1\leqslant i<j\leqslant r} \bigl|q^{k_i}-q^{k_j}\bigr|,$$

the summand of $f_{n,r}$ is a symmetric function which vanishes unless all k_i are pairwise distinct. Anti-symmetrisation thus yields

$$f_{n,r} = \frac{r!}{(1-q)^{\binom{r}{2}}} \sum_{n\geqslant k_1>\cdots>k_r\geqslant -n} \prod_{1\leqslant i<j\leqslant r} \bigl(1-q^{k_i-k_j}\bigr) \prod_{i=1}^r q^{(k_i+n-r+i)^2/2} \begin{bmatrix} 2n \\ n+k_i \end{bmatrix}.$$

We write this as a sum over partitions $\lambda\subseteq (r^{2n-r+1})$ via

$$k_i = \lambda'_i - n + r - i, \qquad 1\leqslant i\leqslant r.$$

Then

$$f_{n,r} = \frac{r!}{(1-q)^{\binom{r}{2}}} \sum_{\lambda\subseteq (r^{2n-r+1})} q^{n(\lambda)+|\lambda|/2} \prod_{1\leqslant i<j\leqslant r} \bigl(1-q^{\lambda'_i-\lambda'_j+j-i}\bigr) \prod_{i=1}^r \begin{bmatrix} 2n \\ \lambda'_i+r-i \end{bmatrix}.$$

By Lemma 3.2 and the fact that s_λ is homogeneous of degree $|\lambda|$, this can be written as a sum over principally specialised Schur functions. Performing in addition the replacement $n\mapsto (n+r-1)/2$, we arrive at

$$f_{(n+r-1)/2,r} = \frac{r!}{(1-q)^{\binom{r}{2}}} \prod_{1\leqslant i<j\leqslant r} \bigl(1-q^{j-i}\bigr) \prod_{i=1}^r \begin{bmatrix} n+r-1 \\ r-i \end{bmatrix} \\ \times \sum_{\lambda\subseteq (r^n)} s_\lambda\bigl(q^{1/2},q^{3/2},\dots,q^{n-1/2}\bigr),$$

for n a non-negative integer. (When $n=0$, the sum on the right should be interpreted as 1.) The sum can be computed by [31, p. 85][2]

$$\sum_{\lambda\subseteq (r^n)} s_\lambda\bigl(q^{1/2},q^{3/2},\dots,q^{n-1/2}\bigr) = \prod_{i=1}^n \frac{1-q^{i+(r-1)/2}}{1-q^{i-1/2}} \prod_{1\leqslant i<j\leqslant n} \frac{1-q^{r+i+j-1}}{1-q^{i+j-1}}.$$

Some elementary simplifications of the q-products and the subsequent replacement $n\mapsto 2n-r+1$ result in

$$f_{n,r} = \frac{r!}{(1-q)^{\binom{r}{2}}} \frac{(q^{(r+1)/2};q)_{2n-r+1}}{(q^{1/2};q)_{2n-r+1}} \prod_{i=1}^r \frac{(q;q)_{2n}(q;q)_{i-1}(q^i;q^2)_{2n-r+1}}{(q;q)_{2n-i+1}^2}.$$

To transform this into the claimed product over q-gamma functions is somewhat delicate. First we use $(a^2;q^2)_n=(a;q)_n(-a;q)_n$ to write

$$\frac{(q^{(r+1)/2};q)_{2n-r+1}}{(q^{1/2};q)_{2n-r+1}} \prod_{i=1}^r \frac{(q^i;q^2)_{2n-r+1}}{(q;q)_{2n-i+1}} = \prod_{i=1}^r \frac{(-q^{i/2};q)_{2n-r+1}(q^{(i+1)/2};q)_{2n-r+1}}{(q;q)_{2n-i+1}}.$$

---

[2] This is equivalent to MacMahon's formula [32] for the generating function of symmetric plane partitions that fit in a box of size $n\times n\times r$, proved by Andrews [1] and Macdonald [31].

The first term in the numerator is wanted, but we further need to transform the other two terms as follows:

$$\prod_{i=1}^{r} \frac{(q^{(i+1)/2};q)_{2n-r+1}}{(q;q)_{2n-i+1}} = \prod_{i=1}^{r} \frac{(-q^{1/2};q^{1/2})_{i-1}(q^{3/2};q)_{2n-i+1}}{(q;q)_{i-1}(q^{i/2+1};q)_{2n-i+1}} \cdot \frac{1-q^{1/2}}{1-q^{i/2}}.$$

Putting all this together, we get

$$f_{n,r} = \frac{r!}{(1-q)^{\binom{r}{2}}\,[r]_{q^{1/2}}!} \prod_{i=1}^{r} (-q^{1/2};q^{1/2})_i (-q^{i/2+1};q)_{2n-r} \cdot \frac{(q;q)_{2n}(q^{3/2};q)_{2n-i+1}}{(q;q)_{2n-i+1}(q^{i/2+1};q)_{2n-i+1}}.$$

By the definition of the q-gamma function, the result now follows. □

5.2. **The evaluation of $\mathcal{S}_{r,n}(2,\frac{1}{2},1)$.** Again we first consider a q-analogue.

Proposition 5.2 (B_r summation)**.** *Let $0<q<1$, r a positive integer and n an integer or half-integer such that $n \geqslant r-1/2$. Then*

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} \big|[k_i-k_j]_q\,[k_i+k_j]_q\big| \prod_{i=1}^{r} \big|[k_i]_q\big|\, q^{\binom{k_i-r+i}{2}-\binom{\lceil n\rceil-n}{2}} \begin{bmatrix} 2n \\ n+k_i \end{bmatrix} \tag{5.3}$$

$$= 2^r r! \prod_{i=1}^{r} (-q;q^{1/2})_{2n-2i} \frac{\Gamma_q(i)}{\Gamma_q(\frac{3}{2})} \cdot \frac{\Gamma_q(2n+1)\,\Gamma_q(\lceil n\rceil - i + \frac{3}{2})}{\Gamma_q(2n-i+2)\,\Gamma_q(\lceil n\rceil-i+1)}.$$

Taking the $q\to 1$ limit and using $r!\prod_{i=1}^{r}\Gamma(i)=\prod_{i=1}^{r}\Gamma(i+1)$, we obtain (cf. (2.1))

$$\mathcal{S}_{r,n}(2,\tfrac{1}{2},1) := \sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} \big|k_i^2-k_j^2\big| \prod_{i=1}^{r} |k_i| \binom{2n}{n+k_i} \tag{5.4}$$

$$= 2^{(2n+1)r-r(r+1)} \prod_{i=1}^{r} \frac{\Gamma(1+i)}{\Gamma(\frac{3}{2})} \cdot \frac{\Gamma(2n+1)\,\Gamma(n-i+\frac{3}{2})}{\Gamma(2n-i+2)\,\Gamma(n-i+1)}.$$

Equation (5.12) in [6] is the special case $r=2$ of this identity.

Proof. Once again, the sum will be denoted by $f_{n,r}$. This time the summand is symmetric under signed permutations of the k_i. Exploiting this hyperoctahedral symmetry, we obtain

$$f_{n,r} = \frac{2^r r!}{(1-q)^{r^2}} \sum_{n\geqslant k_1>\dots>k_r>0} \prod_{1\leqslant i<j\leqslant r} (1-q^{k_i-k_j})(1-q^{k_i+k_j})$$

$$\times \prod_{i=1}^{r} (1-q^{k_i})\, q^{\binom{k_i-r+i}{2}-\binom{\lceil n\rceil-n}{2}} \begin{bmatrix} 2n \\ n+k_i \end{bmatrix}.$$

We now set

$$k_i = n-i-\lambda'_{r-i+1}+1, \qquad 1\leqslant i\leqslant r, \tag{5.5}$$

where λ is a partition contained in $(r^{\lceil n\rceil-r})$. If we then replace $n\mapsto \lfloor n\rfloor + r$ and use the dual C_n specialisation formula (3.29a) in the integer-n case or the dual B_n specialisation formula (3.20b) with $q^{1/2}\mapsto -q^{1/2}$ in the half-integer case, we get

$$f_{n+r,r}=\frac{2^r r!\,q^{r\binom{n+1}{2}}}{(1-q)^{r^2}}\,\frac{(q;q)_{n+r}}{(q;q)_n}\prod_{i=1}^{r}\frac{(q;q)_{2n+2r}}{(q;q)_{2n+2r-2i+2}}\sum_{\lambda\subseteq(r^n)}\mathrm{sp}_{2n,\lambda}(q,q^2,\dots,q^n)$$

and

$$\begin{aligned}f_{n+r-1/2,r}=\frac{2^r r!\,q^{rn^2/2}}{(1-q)^{r^2}}\,\frac{(q^{1/2};q)_{n+r}}{(q^{1/2};q)_n}\prod_{i=1}^{r}\frac{(q;q)_{2n+2r-1}}{(q;q)_{2n+2r-2i+1}}&\\ \times\sum_{\lambda\subseteq(r^n)}(-1)^{|\lambda|}\,\mathrm{so}_{2n+1,\lambda}\big(-q^{1/2},-q^{3/2},\dots,-q^{n-1/2}\big)&,\end{aligned}$$

where n is a non-negative integer. (The two sums on the right are again to be interpreted as 1 when $n=0$.) By Corollaries 4.2 and 4.4, we can carry out the summations, resulting in

$$f_{n+r,r}=\frac{2^r r!}{(1-q)^{r^2}}\prod_{i=1}^{r}\frac{(q;q)_{2n+2r}(q;q)_{2n+i}(q;q)_{i-1}}{(q;q)_{2n+2r-2i+2}(q;q)_{n+r-i}^2}$$

and

$$f_{n+r-1/2,r}=\frac{2^r r!}{(1-q)^{r^2}}\,\frac{(q^{1/2};q)_{n+r}}{(q^{1/2};q)_n}\prod_{i=1}^{r}\frac{(q;q)_{2n+2r-1}(q;q)_{2n+i-1}(q;q)_{i-1}}{(q;q)_{2n+2r-2i+1}(q;q)_{n+r-i}^2},$$

respectively. The replacement $n\mapsto n-r$ or $n\mapsto n-r+1/2$ and some elementary manipulations lead to

$$f_{n,r}=\frac{2^r r!}{(1-q)^{r^2}}\prod_{i=1}^{r}\frac{(-q^{1/2};q^{1/2})_{2n-2i+1}(q^{1/2};q)_{\lceil n\rceil-i+1}(q;q)_{2n}(q;q)_{i-1}}{(q;q)_{2n-i+1}(q;q)_{\lceil n\rceil-i}}.$$

The proof is completed by writing this in terms of q-gamma functions. ☐

5.3. **The evaluation of $\mathcal{S}_{r,n}(2,\frac12,0)$.** We first restate Proposition 1.1, now including the half-integral case (cf. (2.1)).

Proposition 5.3 (D_r summation)**.** *Let r be a positive integer and n an integer or half-integer such that $n\geqslant r-1$. Then*

$$\begin{aligned}\mathcal{S}_{r,n}(2,\tfrac12,0)&=\sum_{k_1,\dots,k_r=-n}^{n}\prod_{1\leqslant i<j\leqslant r}\big|k_i^2-k_j^2\big|\prod_{i=1}^{r}\binom{2n}{n+k_i}\\&=2^{2rn-r(r-1)}\,\frac{\Gamma(1+\frac12 r)}{\Gamma(\frac32)}\cdot\frac{\Gamma(\lfloor n\rfloor-\frac12 r+\frac32)}{\Gamma(\lfloor n\rfloor+1)}\\&\quad\times\prod_{i=1}^{r-1}\frac{\Gamma(i+1)}{\Gamma(\frac32)}\cdot\frac{\Gamma(2n+1)\,\Gamma(\lfloor n\rfloor-i+\frac32)}{\Gamma(2n-i+1)\,\Gamma(\lfloor n\rfloor-i+1)}.\end{aligned}\tag{5.6}$$

As already pointed out in the introduction, the special cases $r=2,3,4$ cover [7, Theorem 1], and Theorem 4.1 and Conjecture 4.1 in [6], respectively.

When n is a half-integer, the identity (5.6) admits a q-analogue:

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} \big|[k_i-k_j]_q\,[k_i+k_j]_q\big| \prod_{i=1}^{r} q^{\binom{k_i-r+i+1/2}{2}} \begin{bmatrix} 2n \\ n+k_i \end{bmatrix} \tag{5.7}$$
$$= 2^r\,[2]_q^n\,\frac{r!}{[r]_q!}\cdot\frac{1}{(-q;q)_{n-r}}\prod_{i=1}^{r}(-q;q^{1/2})_{2n-2i}\frac{\Gamma_{q^2}(1+\frac{1}{2}r)}{\Gamma_q(\frac{3}{2})}\prod_{i=1}^{r-1}\frac{\Gamma_q(i+1)}{\Gamma_q(\frac{3}{2})}$$
$$\times\frac{\Gamma_{q^2}(n-\frac{1}{2}r+1)}{\Gamma_q(n+\frac{1}{2})}\prod_{i=1}^{r-1}\frac{\Gamma_q(2n+1)\,\Gamma_q(n-i+1)}{\Gamma_q(2n-i+1)\,\Gamma_q(n-i+\frac{1}{2})}.$$

Proof. As usual, we denote the sum on the left by $f_{n,r}$. Due to the hyperoctahedral symmetry of the summand, we have

$$f_{n,r} = r!\cdot 2^r \sum_{n\geqslant k_1>\dots>k_r\geqslant 0}(1-\tfrac{1}{2}\delta_{k_r,0})\prod_{1\leqslant i<j\leqslant r}(k_i^2-k_j^2)\prod_{i=1}^{r}\binom{2n}{n+k_i}.$$

Since $n+k_r$ must be an integer, the effective lower bound is $1/2$ when n is a half-integer. In this case $(1-\frac{1}{2}\delta_{k_r,0})=1$.

Again we make the variable change (5.5). Due to the different lower bound compared to the B_r summation in Proposition 5.2, this means that we will now be summing over partitions λ contained in $r^{\lfloor n\rfloor-r+1}$. We also note that in the integer-n case $(1-\frac{1}{2}\delta_{k_r,0})$ transforms into

$$(1-\tfrac{1}{2}\delta_{\lambda'_1,n-r+1}) = (1-\tfrac{1}{2}\delta_{l(\lambda),n-r+1}). \tag{5.8}$$

Next we replace $n\mapsto\lceil n\rceil+r-1$. Note that this turns (5.8) into

$$(1-\tfrac{1}{2}\delta_{l(\lambda),n}) = u_\lambda^{-1},$$

with u_λ as in (3.32). In the integer-n case, we can then use (3.34) for $q=1$ combined with (4.8) to find

$$f_{n+r-1,r} = 2^r r!\prod_{i=1}^{r}\frac{(2n+2r-2)!}{(2n+2r-2i)!}\sum_{\lambda\subseteq(r^n)}\mathrm{so}_{2n,\lambda}(1^n).$$

In the half-integer case, we can use the $q=1$ instance of (3.20a). This results in

$$f_{n+r-1/2,r} = 2^r r!\prod_{i=1}^{r}\frac{(2n+2r-1)!}{(2n+2r-2i+1)!}\sum_{\lambda\subseteq(r^n)}\mathrm{so}_{2n+1,\lambda}(1^n). \tag{5.9}$$

As before, the sums on the right are 1 when $n=0$. Evaluation of these sums for general n by the $q=1$ cases of (4.7a) and (4.2b) (with $\varepsilon=1$), respectively, gives

$$f_{n+r-1,r} = 2^{2n+r-1}\frac{(\frac{1}{2}r+\frac{1}{2})_n}{n!}\prod_{i=1}^{r}\frac{(2n+2r-2)!}{(2n+2r-2i)!}\prod_{i=1}^{r-1}\frac{(2n+i-1)!\,(i+1)!}{(n+r-i)!\,(n+r-i-1)!}$$

and

$$f_{n+r-1/2,r}(q) = 2^r\,\frac{(\frac{1}{2}r+\frac{1}{2})_n}{(\frac{1}{2})_n}\prod_{i=1}^{r}\frac{(2n+2r-1)!\,(2n+i-1)!\,i!}{(2n+2r-2i+1)!\,(n+r-i)!^2}. \tag{5.10}$$

Replacing $n \mapsto n-r+1$ or $n \mapsto n-r+1/2$ and then expressing $f_{n,r}$ in terms of gamma functions, we arrive at the right-hand side of (5.6).

The proof of the q-case for half-integer n proceeds along exactly the same lines, with (5.9) replaced by

$$f_{n+r-1/2,r}(q) = \frac{2^r r!}{(1-q)^{r^2-r}}\, q^{r\binom{n+1}{2}} \prod_{i=1}^{r} \frac{(q;q)_{2n+2r-1}}{(q;q)_{2n+2r-2i+1}} \sum_{\lambda \subseteq r^n} \mathrm{so}_{2n+1,\lambda}(q, q^2, \dots, q^n).$$

and (5.10) by

$$f_{n+r-1/2,r}(q) = \frac{2^r r!}{(1-q)^{r^2-r}} \frac{(q^{r+1};q^2)_n}{(q;q^2)_n} \prod_{i=1}^{r} \frac{(q;q)_{2n+2r-1}(q;q)_{2n+i-1}(q;q)_{i-1}}{(q;q)_{2n+2r-2i+1}(q;q)^2_{n+r-i}}. \qquad \square$$

5.4. **The evaluation of $\mathcal{S}_{r,n}(2,\frac{1}{2},2)$.** This is the $(\alpha,\gamma) = (2,1/2)$ case in Table 1. It has no interpretation in terms of finite reflection groups. It is also the only case that apparently does not admit a simple closed-form product formula for half-integer n.

Proposition 5.4. *Let $0 < q < 1$, r a positive integer and n an integer such that $n \geqslant r$. Then*

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} \big|[k_i-k_j]_q\,[k_i+k_j]_q\big| \prod_{i=1}^{r} [k_i]_q^2\, q^{(k_i-r+i-1)^2/2} \begin{bmatrix} 2n \\ n+k_i \end{bmatrix} \tag{5.11}$$
$$= 2^r \frac{r!}{[r]!_q} \cdot \frac{(-1;q^{1/2})_{r+1}(-q^{r/2+1};q)_{n-r}}{(-1;q)_{n-r}} \prod_{i=1}^{r+1} \frac{(-q^{1/2};q^{1/2})_{i-1}(-q^{1/2};q^{1/2})_{2n-i-r}}{(-q^{1/2};q^{1/2})_{2n-2i+2}}$$
$$\times \prod_{i=1}^{r} \frac{\Gamma_q^2(1+\frac{1}{2}i)}{\Gamma_q(\frac{1}{2})} \cdot \frac{\Gamma_q(2n+1)\,\Gamma_q(n-i+\frac{3}{2})}{\Gamma_q(n-i+1)\,\Gamma_q^2(n-\frac{1}{2}i+1)}.$$

In the $q \to 1$ limit, this becomes (cf. (2.1))

$$\mathcal{S}_{r,n}(2,\tfrac{1}{2},2) = \sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} \big|k_i^2-k_j^2\big| \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i} \tag{5.12}$$
$$= 2^r \prod_{i=1}^{r} \frac{\Gamma^2(1+\frac{1}{2}i)}{\Gamma(\frac{1}{2})} \cdot \frac{\Gamma(2n+1)\,\Gamma(n-i+\frac{3}{2})}{\Gamma(n-i+1)\,\Gamma^2(n-\frac{1}{2}i+1)}. \tag{5.13}$$

Equation (5.13) in [6] is the special case $r=2$ of this identity.

Proof. If we denote the sum on the left by $f_{n,r}$ and define $k_{r+1} := 0$, then the summand of $f_{n,r}$ can be rewritten as

$$\begin{bmatrix} 2n \\ n \end{bmatrix}^{-1} \prod_{1\leqslant i<j\leqslant r+1} \big|[k_i-k_j]_q\,[k_i+k_j]_q\big| \prod_{i=1}^{r+1} q^{(k_i-r+i-1)^2/2} \begin{bmatrix} 2n \\ n+k_i \end{bmatrix}.$$

Hence, after anti-symmetrisation and the variable change

$$k_i = n-i-\lambda'_{r-i+2}+1, \qquad 1 \leqslant i \leqslant r+1$$

(so that $\lambda_1' := n-r$), we obtain

$$f_{n+r-1,r-1} = \frac{2^{r-1}(r-1)!\, q^{rn^2/2}}{(1-q)^{r^2+r}} \begin{bmatrix} 2n+2r-2 \\ n+r-1 \end{bmatrix}^{-1} \prod_{i=1}^{r} \frac{(q;q)_{2n+2r-2}}{(q;q)_{2n+2i-2}} \times \sum_{\substack{\lambda\subseteq(r^n)\\ l(\lambda)=n}} \mathrm{o}_{2n,\lambda}(q^{1/2},q^{3/2},\dots,q^{n-1/2}),$$

where we have also used (3.34). Next we apply (4.7b) so that

$$f_{n+r-1,r-1} = 2^r(r-1)!\, \frac{(-q^{1/2};q^{1/2})_{2n-1}(-q^{(r+1)/2};q)_n}{(-1;q)_n} \times \frac{(q^{r/2};q)_n}{(q;q)_n} \begin{bmatrix} 2n+2r-2 \\ n+r-1 \end{bmatrix}^{-1} \prod_{i=1}^{r-1} \frac{(q;q)_{2n+2r-2}(q;q)_{2n+i-1}(q;q)_i}{(q;q)_{2n+2r-2i-2}(q;q)_{n+r-i}(q;q)_{n+r-i-1}}.$$

The rest follows as in earlier cases. □

6. Discrete Macdonald–Mehta integrals for $\gamma=1$ and $\alpha=2$

In this section, we present our results concerning evaluations of $S_{r,n}(\alpha,\gamma,\delta)$ for $\gamma=1$ and $\alpha=2$. In contrast to the previous section, where identities for classical group characters played a key role, here our starting point is a transformation formula for elliptic hypergeometric series. Along the lines of Section 5, in each case we shall start with a q-analogue, from which the evaluations of $S_{r,n}(2,1,\delta)$ for $\delta=0,1,2,3$ follow by a straightforward $q\to1$ limit. An additional feature is that the identities in this section typically contain an additional parameter.

We start with the $p=0$, $x=q$ special case of a transformation formula originally conjectured by the third author [48, Conjecture 6.1] and proven independently by Rains [43, Theorem 4.9] and by Coskun and Gustafson [9].

Theorem 6.1. *Let a,b,c,d,e,f be indeterminates, m a non-negative integer, and $r\geq 1$. Then*

$$\sum_{0\leqslant k_1<k_2<\cdots<k_r\leqslant m} q^{\sum_{i=1}^r(2i-1)k_i} \prod_{1\leqslant i<j\leqslant r}(1-q^{k_i-k_j})^2\,(1-aq^{k_i+k_j})^2 \times \prod_{i=1}^{r} \frac{(1-aq^{2k_i})\,(a,b,c,d,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q)_{k_i}}{(1-a)\,(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{r-1-m}/\lambda,aq^{1+m};q)_{k_i}}$$
$$= \prod_{i=1}^{r} \frac{(b,c,d,ef/a;q)_{i-1}}{(\lambda b/a,\lambda c/a,\lambda d/a,ef/\lambda;q)_{i-1}} \times \prod_{i=1}^{r} \frac{(aq;q)_m\,(aq/ef;q)_{m-r+1}\,(\lambda q/e,\lambda q/f;q)_{m-i+1}}{(\lambda q;q)_m\,(\lambda q/ef;q)_{m-r+1}\,(aq/e,aq/f;q)_{m-i+1}}$$
$$\times \sum_{0\leqslant k_1<k_2<\cdots<k_r\leqslant m} q^{\sum_{i=1}^r(2i-1)k_i} \prod_{1\leqslant i<j\leqslant r}(1-q^{k_i-k_j})^2\,(1-\lambda q^{k_i+k_j})^2 \times \prod_{i=1}^{r} \frac{(1-\lambda q^{2k_i})\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q)_{k_i}}{(1-\lambda)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{r-1-m}/a,\lambda q^{1+m};q)_{k_i}}, \tag{6.1}$$

where $\lambda = a^2q^{2-r}/bcd$.

In the above formula, we let $m \to \infty$ to obtain

$$\begin{aligned}(6.2)\qquad &\sum_{0\leqslant k_1<k_2<\cdots<k_r} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\leqslant i<j\leqslant r} (1-q^{k_i-k_j})^2\,(1-aq^{k_i+k_j})^2\\
&\quad\times\prod_{i=1}^r\Big(\frac{a^2}{q^{2r-3}bcdef}\Big)^{k_i}\frac{(1-aq^{2k_i})\,(a,b,c,d,e,f;q)_{k_i}}{(1-a)\,(q,aq/b,aq/c,aq/d,aq/e,aq/f;q)_{k_i}}\\
&\quad=\prod_{i=1}^r\frac{(b,c,d,ef/a;q)_{i-1}}{(\lambda b/a,\lambda c/a,\lambda d/a,ef/\lambda;q)_{i-1}}\frac{(aq,aq/ef,\lambda q/e,\lambda q/f;q)_\infty}{(\lambda q,\lambda q/ef,aq/e,aq/f;q)_\infty}\\
&\quad\times\sum_{0\leqslant k_1<k_2<\cdots<k_r} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\leqslant i<j\leqslant r} (1-q^{k_i-k_j})^2\,(1-\lambda q^{k_i+k_j})^2\\
&\qquad\times\prod_{i=1}^r\Big(\frac{a}{q^{r-1}ef}\Big)^{k_i}\frac{(1-\lambda q^{2k_i})\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f;q)_{k_i}}{(1-\lambda)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f;q)_{k_i}}.\end{aligned}$$

The two specialisations which are relevant for us are $b = aq/c$ and $b = aq^2/c$. The case $b = aq/c$ (which has the effect of generating terms $(\lambda d/a;q)_{k_i} = (q^{1-r};q)_{k_i}$ in the right-hand side sum of (6.2), in turn implying that the only choice for the summation indices k_i to produce a non-vanishing summand is $k_i = i-1$ for $i = 1,2,\ldots,r$) gives

$$\begin{aligned}(6.3)\qquad &\sum_{0\leqslant k_1<k_2<\cdots<k_r} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\leqslant i<j\leqslant r} (1-q^{k_i-k_j})^2\,(1-aq^{k_i+k_j})^2\\
&\quad\times\prod_{i=1}^r\Big(\frac{a}{q^{2r-2}def}\Big)^{k_i}\frac{(1-aq^{2k_i})\,(a,d,e,f;q)_{k_i}}{(1-a)\,(q,aq/d,aq/e,aq/f;q)_{k_i}}\\
&=q^{-\binom{r}{3}}\Big(\frac{a}{ef}\Big)^{\binom{r}{2}}\prod_{i=1}^r\frac{(q,d,e,f,ef/a,aq^{i-r}/d;q)_{i-1}\,(aq^{2-r}/d;q)_{2i-2}}{(aq/d,aq^{2-r}/de,aq^{2-r}/df,defq^{r-1}/a;q)_{i-1}}\\
&\qquad\times\prod_{i=1}^r\frac{(aq,aq/ef,aq^{2-r}/de,aq^{2-r}/df;q)_\infty}{(aq^{2-r}/d,aq^{2-r}/def,aq/e,aq/f;q)_\infty}.\end{aligned}$$

The case where $b = aq^2/c$ (which has the effect of generating terms $(\lambda d/a;q)_{k_i} = (q^{-r};q)_{k_i}$ in the right-hand side sum of (6.2), in turn implying that the only choices for the summation indices k_i to produce a non-vanishing summand are $k_i = i-1+\chi(i>s)$ for some non-negative integer s, and $i = 1,2,\ldots,r$; here, $\chi(\mathcal{A}) = 1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A}) = 0$ otherwise) gives

$$\begin{aligned}(6.4)\qquad &\sum_{0\leqslant k_1<k_2<\cdots<k_r} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\leqslant i<j\leqslant r} (1-q^{k_i-k_j})^2\,(1-aq^{k_i+k_j})^2\\
&\quad\times\prod_{i=1}^r\Big(\frac{a}{q^{2r-1}def}\Big)^{k_i}\frac{(1-aq^{2k_i})\,(1-cq^{k_i-1})\,(1-aq^{k_i+1}/c)\,(a,d,e,f;q)_{k_i}}{(1-a)\,(1-c/q)\,(1-aq/c)\,(q,aq/d,aq/e,aq/f;q)_{k_i}}\\
&=(-1)^r q^{2\binom{r+1}{3}}\Big(\frac{a}{q^{r-1}ef}\Big)^{\binom{r+1}{2}}\frac{(aq^{2-r}/cd)_r\,(cq^{-r}/d)_r}{(1-aq/c)^r\,(1-c/q)^r\,(q;q)_r\,(aq^{1-r}/d)_r^2}\end{aligned}$$

$$\times \prod_{i=1}^{r} \frac{(q,e,f,aq^{i-r}/d;q)_i\,(d,ef/a;q)_{i-1}\,(aq^{1-r}/d;q)_{2i}}{(aq/d,aq^{1-r}/de,aq^{1-r}/df;q)_i\,(defq^r/a;q)_{i-1}}$$

$$\times \prod_{i=1}^{r} \frac{(aq,aq/ef,aq^{1-r}/de,aq^{1-r}/df;q)_\infty}{(aq^{1-r}/d,aq^{1-r}/def,aq/e,aq/f;q)_\infty}$$

$$\times \sum_{s=0}^{r} \frac{\left(1-\frac{aq^{-r}}{d}q^{2s}\right)}{\left(1-\frac{aq^{-r}}{d}\right)} \frac{(aq^{-r}/d,c/q,aq/c,aq^{1-r}/de,aq^{1-r}/df,q^{-r};q)_s}{(q,aq^{2-r}/cd,cq^{-r}/d,e,f,aq/d;q)_s} \left(\frac{q^r ef}{a}\right)^s.$$

This is a transformation formula between a multiple basic hypergeometric series associated with the root system BC_r and a very-well-poised basic hypergeometric ${}_8\phi_7$-series (see [15] for terminology).

6.1. **The evaluation of $\mathcal{S}_{r,n}(2,1,0)$.**

Proposition 6.2 (D_r summation). *Let q be a real number with $0<q<1$. For all non-negative integers or half-integers m and n and a positive integer r, we have*

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} [k_j-k_i]_q^2\,[k_i+k_j]_q^2 \prod_{i=1}^{r} q^{k_i^2-(2i-\frac{3}{2})k_i} \frac{1+q^{k_i}}{1+q} \begin{bmatrix}2n\\ n+k_i\end{bmatrix}_q \begin{bmatrix}2m\\ m+k_i\end{bmatrix}_q \tag{6.5}$$

$$= r!\left(\frac{2}{[2]_q}\right)^r q^{-2\binom{r+1}{3}+\frac{1}{2}\binom{r}{2}} \prod_{i=1}^{r} \frac{\Gamma_{q^{1/2}}(2i-1)\,\Gamma_q(2n+1)\,\Gamma_q(2m+1)}{\Gamma_q(m+n-i+2)\,\Gamma_q(m+n-i-r+3)} \cdot \frac{\Gamma_{q^{1/2}}(2m+2n-2i-2r+5)}{\Gamma_{q^{1/2}}(2n-2i+3)\,\Gamma_{q^{1/2}}(2m-2i+3)}.$$

Taking the $q\to1$ limit, dividing both sides of the result by $\binom{2m}{m}^r$, and finally taking the limit $m\to\infty$, we arrive at (cf. (2.1))

$$\mathcal{S}_{r,n}(2,1,0) = \sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} (k_i^2-k_j^2)^2 \prod_{i=1}^{r} \binom{2n}{n+k_i} \tag{6.6}$$

$$= 2^{2r(n-r+1)}\Gamma(r+1) \prod_{i=1}^{r-1} \frac{\Gamma(2i+1)\,\Gamma(2n+1)}{\Gamma(2n-2i+1)}.$$

The evaluation of $W_2(n)$ provided after the proof of Theorem 3.2 in [6] is the special case $r=2$ of this identity.

Proof. To begin with, we observe that the summand of the sum on the left-hand side of (6.5) is invariant under permutations of the summation indices. Indeed, writing $S_1(k_1,k_2,\dots,k_r)$ for this summand, for a permutation σ of $\{1,2,\dots,r\}$ we have

$$S_1(k_{\sigma(1)},k_{\sigma(2)},\dots,k_{\sigma(r)}) = q^{E_1(\sigma;k_1,k_2,\dots,k_r)} S_1(k_1,k_2,\dots,k_r), \tag{6.7}$$

where

$$E_1(\sigma;k_1,k_2,\dots,k_r) = 2\sum_{1\leqslant i<j\leqslant r} \chi\big(\sigma(i)>\sigma(j)\big)\big(k_{\sigma(j)}-k_{\sigma(i)}\big) - 2\sum_{i=1}^{r}(ik_{\sigma(i)}-ik_i). \tag{6.8}$$

Here, as before, $\chi(\mathcal{A}) = 1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A}) = 0$ otherwise. Let $I_\sigma(i)$ denote the number of indices j with $1 \leqslant i < j \leqslant r$ and $\sigma(i) > \sigma(j)$. Then, by elementary counting, we have

$$\begin{aligned}\sum_{1\leqslant i<j\leqslant r} \chi\big(\sigma(i) > \sigma(j)\big)(k_{\sigma(j)} - k_{\sigma(i)}) &= \sum_{j=1}^{r} \big(j - \sigma(j) + I_\sigma(j)\big)k_{\sigma(j)} - \sum_{i=1}^{r} I_\sigma(i)k_{\sigma(i)}\\ &= \sum_{i=1}^{r}(i - \sigma(i))k_{\sigma(i)}.\end{aligned}$$

If this is substituted back in (6.8), then one obtains $E_1(\sigma; k_1, k_2, \dots, k_r) = 0$. In combination with (6.7), this implies the claimed invariance of summands under permutations of the summation indices. As a consequence, we may restrict the range of summation on the left-hand side of (6.5) to $k_1 < k_2 < \cdots < k_r$, and in turn multiply this restricted sum by $r!$, thereby not changing the value of the left-hand side of (6.5).

Now, in this (restricted) sum, we replace k_i by $k_i - n$, and we rewrite the arising multiple sum in terms of q-shifted factorials. The result is

$$\begin{aligned}\frac{r!\, q^{rn^2+(r^2-\frac{r}{2})n}(1+q^{-n})^r}{(1-q)^{2r^2-2r}\,(1+q)^r}\begin{bmatrix}2m\\ m-n\end{bmatrix}_q^r \sum_{0\leqslant k_1<\cdots<k_r}\ \prod_{1\leqslant i<j\leqslant r}\big(1-q^{k_i-k_j}\big)^2\big(1-q^{k_i+k_j-2n}\big)^2&\\ \cdot\prod_{i=1}^{r}\frac{(q^{-2n}, -q^{1-n}, q^{-m-n}; q)_{k_i}}{(q, -q^{-n}, q^{m-n+1}; q)_{k_i}}\, q^{(2i-2r+m+n+\frac{1}{2})k_i}&,\end{aligned}$$

where the summation indices k_i now run over integers. Thus, we see that we may apply (6.3) with $a = q^{-2n}$, $d = q^{-m-n}$, $e = q^{-n}$, $f = q^{-n+1/2}$ to evaluate this sum. We have to be a little careful though because of the appearance of the ratio $(aq; q)_\infty/(aq/e; q)_\infty$ on the right-hand side of (6.3), which becomes the indeterminate expression $0/0$ for the above choices of a and e. To be precise, in (6.3) we have to first choose $e = \sqrt{a}$, and subsequently calculate the limit as a tends to q^{-2n}. Doing this, we obtain

$$\begin{aligned}\lim_{a\to q^{-2n}} \frac{(aq; q)_\infty}{(\sqrt{a}q; q)_\infty} &= \lim_{a\to q^{-2n}} \frac{(aq; q)_{2n-1}\,(1 - aq^{2n})\,(aq^{2n+1}; q)_\infty}{(\sqrt{a}q; q)_{n-1}\,(1 - \sqrt{a}q^n)\,(\sqrt{a}q^{n+1}; q)_\infty}\\ &= 2\,\frac{(q^{1-2n}; q)_{2n-1}}{(q^{1-n}; q)_{n-1}} = 2\,(q^{1-2n}; q)_n.\end{aligned}\tag{6.9}$$

After considerable simplification and rewriting of the right-hand side of (6.3) under the above specialisation, we obtain the right-hand side of (6.5). □

6.2. **The evaluation of $\mathcal{S}_{r,n}(2,1,1)$.**

Proposition 6.3. *Let q be a real number with $0<q<1$. For all non-negative integers m and n and a positive integer r, we have*

$$\sum_{k_1,\dots,k_r=-n}^{n}\prod_{1\leqslant i<j\leqslant r}[k_j-k_i]_q^2\,[k_i+k_j]_q^2\prod_{i=1}^{r}q^{k_i^2-(2i-1)k_i}\big|[k_i]_{q^2}\big|\begin{bmatrix}2n\\n+k_i\end{bmatrix}_q\begin{bmatrix}2m\\m+k_i\end{bmatrix}_q$$
$$=r!\Big(\frac{2}{[2]_q}\Big)^r q^{-2\binom{r+1}{3}}\prod_{i=1}^{r}\bigg(\frac{\Gamma_q^2(i)\,\Gamma_q(2n+1)}{\Gamma_q(n-i+2)\,\Gamma_q(n-i+1)}$$
$$\times\frac{\Gamma_q(2m+1)}{\Gamma_q(m-i+2)\,\Gamma_q(m-i+1)}\cdot\frac{\Gamma_q(m+n-i-r+2)}{\Gamma_q(m+n-i+2)}\bigg). \tag{6.10}$$

Taking the $q\to1$ limit, dividing both sides of the result by $\binom{2m}{m}^r$, and finally performing the limit $m\to\infty$, we obtain (cf. (2.1))

$$\mathcal{S}_{r,n}(2,1,1)=\sum_{k_1,\dots,k_r=-n}^{n}\prod_{1\leqslant i<j\leqslant r}(k_i^2-k_j^2)^2\prod_{i=1}^{r}|k_i|\binom{2n}{n+k_i}$$
$$=\prod_{i=1}^{r}\frac{\Gamma(i)\,\Gamma(i+1)\,\Gamma(2n+1)}{\Gamma(n-i+2)\,\Gamma(n-i+1)}. \tag{6.11}$$

Proof. Here, the summand of the multiple sum on the left-hand side of (6.10) is invariant under both permutations of the summation indices and under replacement of k_i by $-k_i$, for some fixed i. To show this, if $S_2(k_1,k_2,\dots,k_r)$ denotes the summand, then we have

$$S_2(k_1,\dots,k_{i-1},-k_i,k_{i+1},\dots,k_r)=q^{E_2(k_1,k_2,\dots,k_r)}S_2(k_1,k_2,\dots,k_r),$$

where

$$E_2(k_1,k_2,\dots,k_r)=2\sum_{1\leqslant i<j\leqslant r}(-k_j-k_i)+2\sum_{1\leqslant i<j\leqslant r}(-k_j+k_i)+\sum_{i=1}^{r}\big(2(2i-1)k_i-2k_i\big)$$
$$=4\sum_{j=1}^{r}\big(-(j-1)k_j\big)+\sum_{i=1}^{r}(4i-4)k_i=0.$$

This proves the invariance of $S_2(k_1,k_2,\dots,k_r)$ under the replacement $k_i\to-k_i$. As a consequence, we may restrict the range of summation on the left-hand side of (6.10) to $1\leqslant k_1<k_2<\dots<k_r$, and in turn multiply this restricted sum by $2^r r!$, thereby not changing the value of the left-hand side of (6.10).

In this (restricted) sum, we replace k_i by k_i+1, and we rewrite the arising multiple sum in terms of q-shifted factorials. The result is

$$\frac{2^r r!}{q^{r^2-r}(1-q)^{2r^2-2r}}\begin{bmatrix}2n\\n+1\end{bmatrix}_q^r\begin{bmatrix}2m\\m+1\end{bmatrix}_q^r\sum_{0\leqslant k_1<\dots<k_r}\prod_{1\leqslant i<j\leqslant r}\big(1-q^{k_i-k_j}\big)^2\big(1-q^{k_i+k_j+2}\big)^2$$
$$\cdot\prod_{i=1}^{r}\frac{(q^2,-q^2,q^{1-n},q^{1-m};q)_{k_i}}{(q,-q,q^{n+2},q^{m+2};q)_{k_i}}\,q^{(2i-2r+m+n)k_i}.$$

Thus, we see that we may apply (6.3) with $a = q^2$, $d = q^{1-n}$, $e = q^{1-m}$, $f = q$ to evaluate this sum. After considerable simplification and rewriting, we obtain the right-hand side of (6.10). □

Proposition 6.4. *Let q be a real number with $0 < q < 1$. For all positive half-integers m and n and a positive integer r, we have*

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} [k_j-k_i]_q^2\,[k_i+k_j]_q^2 \prod_{i=1}^{r} q^{k_i^2-(2i-1)k_i} \big|[k_i]_{q^2}\big| \begin{bmatrix} 2n \\ n+k_i \end{bmatrix}_q \begin{bmatrix} 2m \\ m+k_i \end{bmatrix}_q$$
$$= r!\Big(\frac{2}{[2]_q}\Big)^r q^{-\frac14\binom{2r+1}{3}} \prod_{i=1}^{r}\Bigg(\frac{\Gamma_q^2(i)\,\Gamma_q(2n+1)}{\Gamma_q^2(n-i+\frac32)} \times \frac{\Gamma_q(2m+1)}{\Gamma_q^2(m-i+\frac32)}\cdot\frac{\Gamma_q(m+n-i-r+2)}{\Gamma_q(m+n-i+2)}\Bigg). \tag{6.12}$$

Proof. This can be proved in the same way as Proposition 6.3. The only differences are that, here, the summation index k_i is replaced by $k_i+\frac12$, $i=1,2,\dots,r$, and that the relevant specialisation of (6.3) is $a=q$, $d=q^{1/2-n}$, $e=q^{1/2-m}$, $f=q$. □

From now on, all proofs are similar to one of the proofs of Propositions 6.2–6.4, except for the proof of Proposition 6.7. For the remaining theorems in this section (except for Proposition 6.7), we therefore content ourselves with specifying which choice of parameters in (6.3) has to be used, without providing further details.

6.3. The evaluation of $\mathcal{S}_{r,n}(2,1,2)$.

Proposition 6.5 (B_r summation). *Let q be a real number with $0 < q < 1$. For all non-negative integers or half-integers m and n and a positive integer r, we have*

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} [k_j-k_i]_q^2\,[k_i+k_j]_q^2 \prod_{i=1}^{r} q^{k_i^2-(2i-\frac12)k_i} \big|[k_i]_{q^2}\,[k_i]_q\big| \begin{bmatrix} 2n \\ n+k_i \end{bmatrix}_q \begin{bmatrix} 2m \\ m+k_i \end{bmatrix}_q$$
$$= r!\Big(\frac{2}{[2]_q}\Big)^r [2]_{q^{1/2}}^{-r}\, q^{-2\binom{r+1}{3}-\frac12\binom{r+1}{2}} \prod_{i=1}^{r}\Bigg(\frac{\Gamma_{q^{1/2}}(2i)\,\Gamma_q(2n+1)\,\Gamma_q(2m+1)}{\Gamma_q(m+n-i+2)\,\Gamma_q(m+n-i-r+2)} \times \frac{\Gamma_{q^{1/2}}(2m+2n-2i-2r+3)}{\Gamma_{q^{1/2}}(2n-2i+2)\,\Gamma_{q^{1/2}}(2m-2i+2)}\Bigg). \tag{6.13}$$

Taking the $q\to1$ limit, dividing both sides of the result by $\binom{2m}{m}^r$, and finally performing the limit $m\to\infty$, we obtain (cf. (2.1))

$$\mathcal{S}_{r,n}(2,1,2) = \sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} (k_i^2-k_j^2)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i} = 2^{r(2n-2r-1)} \prod_{i=1}^{r} \frac{\Gamma(2i+1)\,\Gamma(2n+1)}{\Gamma(2n-2i+2)}. \tag{6.14}$$

Proof. The special case of (6.3) which is relevant here is $a=q^{-2n}$, $d=q^{-m-n}$, $e=q^{-n+1}$, and $f=q^{-n+1/2}$. □

6.4. **The evaluation of $\mathcal{S}_{r,n}(2,1,3)$.**

Proposition 6.6. *Let q be a real number with $0<q<1$. For all non-negative integers m and n and a positive integer r, we have*

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} [k_j-k_i]_q^2\,[k_i+k_j]_q^2 \prod_{i=1}^{r} q^{k_i^2-2ik_i} \big|[k_i]_{q^2}\,[k_i]_q^2\big| \begin{bmatrix}2n\\ n+k_i\end{bmatrix}_q \begin{bmatrix}2m\\ m+k_i\end{bmatrix}_q \\ = r!\Big(\frac{2}{[2]_q}\Big)^r q^{-2\binom{r+1}{3}-\binom{r+1}{2}} \prod_{i=1}^{r}\bigg(\frac{\Gamma_q(2n+1)}{\Gamma_q^2(n-i+1)}\cdot\frac{\Gamma_q(2m+1)}{\Gamma_q^2(m-i+1)} \\ \times \frac{\Gamma_q(i)\,\Gamma_q(i+1)\,\Gamma_q(m+n-i-r+1)}{\Gamma_q(m+n-i+2)}\bigg). \tag{6.15}$$

Taking the $q\to 1$ limit, dividing both sides of the result by $\binom{2m}{m}^r$, and finally performing the limit $m\to\infty$, we obtain (cf. (2.1))

$$\mathcal{S}_{r,n}(2,1,3) = \sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r}(k_i^2-k_j^2)^2 \prod_{i=1}^{r}|k_i|^3\binom{2n}{n+k_i} \\ = \prod_{i=1}^{r}\frac{\Gamma^2(i+1)\,\Gamma(2n+1)}{\Gamma^2(n-i+1)}. \tag{6.16}$$

Proof. The special case of (6.3) which is relevant here is $a=q^2$, $d=q^{1-n}$, $e=q^{1-m}$, and $f=q^2$. ☐

Proposition 6.7. *Let q be a real number with $0<q<1$. For all positive half-integers m and n and a positive integer r, we have*

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} [k_j-k_i]_q^2\,[k_i+k_j]_q^2 \prod_{i=1}^{r} q^{k_i^2-2ik_i} \big|[k_i]_{q^2}\,[k_i]_q^2\big| \begin{bmatrix}2n\\ n+k_i\end{bmatrix}_q \begin{bmatrix}2m\\ m+k_i\end{bmatrix}_q \\ = r!\Big(\frac{2}{[2]_q}\Big)^r q^{-2\binom{r+1}{3}-\frac{r^2}{2}-\frac{r}{4}} \prod_{i=1}^{r}\bigg(\frac{\Gamma_q(2n+1)}{\Gamma_q^2(n-i+\frac32)}\cdot\frac{\Gamma_q(2m+1)}{\Gamma_q^2(m-i+\frac32)} \\ \times \frac{\Gamma_q(i)\,\Gamma_q(i+1)\,\Gamma_q(m+n-i-r+1)}{\Gamma_q(m+n-i+2)}\bigg) \\ \times \sum_{s=0}^{r}\frac{(\sqrt{q};q)_s^2}{(1-q)^{2s}}\frac{[n-s-1/2]_q!\,[m-s-1/2]_q!}{[n-r-1/2_q!\,[m-r-1/2]_q!}\begin{bmatrix}r\\ s\end{bmatrix}_q\begin{bmatrix}m+n-r\\ s\end{bmatrix}_q. \tag{6.17}$$

Proof. We start in the same way as in the proof of Proposition 6.3, observing that the summand on the left-hand side of (6.17) is invariant under permutations of the summation indices k_i and under replacement of k_i by $-k_i$, for some fixed i. This allows one to concentrate on the range

$$\frac{1}{2}\leqslant k_1<k_2<\cdots<k_r.$$

The final result is then obtained by multiplying the sum over this range by $2^r r!$.

Next we replace k_i by $k_i + \frac{1}{2}$ for $i = 1, 2, \dots, r$, and rewrite the resulting sum using q-shifted factorials, to obtain

$$\frac{1}{(1-q)^{2r^2-2r}} \begin{bmatrix} 2n \\ n+1/2 \end{bmatrix}_q^r \begin{bmatrix} 2m \\ m+1/2 \end{bmatrix}_q^r \sum_{0 \leqslant k_1 < \dots < k_r} \prod_{1 \leqslant i < j \leqslant r} (1-q^{k_i-k_j})^2 \, (1-q^{k_i+k_j+1})^2$$
$$\times \prod_{i=1}^r \left(q^{(2i-1)k_i - i + 1/4 + k_i(m+n-2r)} \frac{(1-q^{2k_i+1})\,(1-q^{k_i+1/2})^2\,(q, q^{1/2-n}, q^{1/2-m}; q)_{k_i}}{(1-q^2)\,(1-q)^2\,(q, q^{3/2+n}, q^{3/2+m}; q)_{k_i}} \right).$$

We may now transform the multiple sum on the right using (6.4) with $a = q$, $c = q^{3/2}$, $d = q$, $e = q^{1/2-n}$ and $f = q^{1/2-m}$. Using the standard basic hypergeometric notation

$${}_r\phi_s \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, z \right] = \sum_{\ell=0}^{\infty} \frac{(a_1, \dots, a_r; q)_\ell}{(q, b_1, \dots, b_s; q)_\ell} \left((-1)^\ell q^{\binom{\ell}{2}} \right)^{s-r+1} z^\ell,$$

where $(a_1, \dots, a_k; q)_\ell = (a_1; q)_\ell \cdots (a_k; q)_\ell$, we obtain that this sum equals

$$F(m,n,r)\, {}_8\phi_7 \left[\begin{matrix} q^{-r}, q^{1-r/2}, -q^{1-r/2}, q^{1/2-r+n}, q^{1/2-r+m}, q^{1/2}, q^{1/2}, q^{-r} \\ q^{-r/2}, -q^{-r/2}, q^{1/2-n}, q^{1/2-m}, q^{1/2-r}, q^{1/2-r}, q \end{matrix} ; q, q^{r-m-n} \right],$$

where $F(m,n,r)$ is an explicit product, suppressed here in order to focus on the essential part in the expression. To the above ${}_8\phi_7$-series, we may apply Watson's transformation formula between a very-well-poised ${}_8\phi_7$-series and a balanced ${}_4\phi_3$-series (see [15, Appendix (III.17)])

$${}_8\phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix} ; q, \frac{a^2q^2}{bcdef} \right]$$
$$= \frac{(aq, aq/de, aq/df, aq/ef; q)_\infty}{(aq/d, aq/e, aq/f, aq/def; q)_\infty} \, {}_4\phi_3 \left[\begin{matrix} aq/bc, d, e, f \\ aq/b, aq/c, def/a \end{matrix} ; q, q \right],$$

provided the ${}_8\phi_7$-series converges and the ${}_4\phi_3$-series terminates. It is then a routine but tedious task to convert the resulting expression into the right-hand side of (6.17). □

7. Discrete Macdonald–Mehta integrals for $\gamma = 1$ and $\alpha = 1$

The purpose of this section is to present our evaluations of $S_{r,n}(\alpha, \gamma, \delta)$ for $\gamma = \alpha = 1$. In principle, it would seem that such evaluations could also follow from the transformation formula in Theorem 6.1, by considering a limit case where $a \to 0$. Indeed, the case $\delta = 0$, that is, the evaluation of the sum $S_{r,n}(1,1,0)$, is covered by (6.1), and it also produces a q-analogue containing a further parameter. Alas, all our attempts to come up with appropriate further specialisations that would produce the sum $S_{r,n}(1,1,\delta)$ with $\delta = 1$ on the left-hand side of (6.1) failed. Hence, in order to achieve the corresponding summation, we designed an ad hoc approach combining the evaluation of certain Vandermonde- and Cauchy-like determinants with summation formulas from the theory of hypergeometric series. As opposed to the case $\delta = 0$, for $\delta = 1$ we were not able to find a q-analogue.

It is interesting to note that the limit case $a \to 0$ of (6.1) has been worked out earlier in [27, Equation (3.7)], where it was used for the enumeration of standard Young tableaux of certain skew shapes. As is pointed out there, that limit case had explicitly appeared even earlier in [25], where two different proofs had been given (one using a specialisation of an identity for Schur functions, the other using a specialisation of a q-integral evaluation due to Evans), and where it had been applied in an again different context, namely that of the enumeration of domino tilings.

7.1. The evaluation of $\mathcal{S}_{r,n}(1,1,0)$.

Proposition 7.1 (A_{r-1} summation). *Let q be a real number with $0<q<1$. For all non-negative integers or half-integers m and n and a positive integer r, we have*

$$\sum_{k_1,\dots,k_r=-n}^{n}\prod_{1\leqslant i<j\leqslant r}[k_j-k_i]_q^2\prod_{i=1}^{r}q^{k_i^2+(m+n-2i+2)k_i}\begin{bmatrix}2n\\n+k_i\end{bmatrix}_q\begin{bmatrix}2m\\m+k_i\end{bmatrix}_q \\ =r!\,q^{-rmn-\frac16 r(r-1)(2r-3m-3n-1)} \\ \times\prod_{i=1}^{r}\frac{\Gamma_q(i)\,\Gamma_q(2n+1)\,\Gamma_q(2m+1)\,\Gamma_q(2m+2n-r-i+3)}{\Gamma_q(2n-i+2)\,\Gamma_q(2m-i+2)\,\Gamma_q^2(m+n-i+2)}. \tag{7.1}$$

Taking the $q\to1$ limit, dividing both sides of the result by $\binom{2m}{m}^r$, and finally performing the limit $m\to\infty$, we obtain (cf. (2.2))

$$\mathcal{S}_{r,n}(1,1,0)=\sum_{k_1,\dots,k_r=-n}^{n}\prod_{1\leqslant i<j\leqslant r}(k_i-k_j)^2\prod_{i=1}^{r}\binom{2n}{n+k_i} \\ =2^{2rn-r(r-1)}\prod_{i=1}^{r}\frac{\Gamma(i+1)\,\Gamma(2n+1)}{\Gamma(2n-i+2)}. \tag{7.2}$$

Proof. The special case of (6.3) which is relevant here is $d=aq^{n-m}$, $e=q^{-m-n}$, $f=q^{-2n}$, and finally $a\to0$. □

7.2. The evaluation of $\mathcal{S}_{r,n}(1,1,1)$.

Proposition 7.2. *For all non-negative integers m and n and a positive integer r, we have*

$$\sum_{k_1,\dots,k_r=-n}^{n}\prod_{1\leqslant i<j\leqslant r}(k_i-k_j)^2\prod_{i=1}^{r}|k_i|\binom{2n}{n+k_i}\binom{2m}{m+k_i} \\ =r!\prod_{i=1}^{\lceil r/2\rceil}\frac{\Gamma^2(i)\,\Gamma(2n+1)\,\Gamma(2m+1)\,\Gamma(m+n-i-\lceil r/2\rceil+2)}{\Gamma(n-i+2)\,\Gamma(n-i+1)\,\Gamma(m-i+2)\,\Gamma(m-i+1)\,\Gamma(m+n-i+2)} \\ \times\prod_{i=1}^{\lfloor r/2\rfloor}\frac{\Gamma(i)\,\Gamma(i+1)\,\Gamma(2n+1)\,\Gamma(2m+1)\,\Gamma(m+n-i-\lfloor r/2\rfloor+1)}{\Gamma^2(n-i+1)\,\Gamma^2(m-i+1)\,\Gamma(m+n-i+2)}. \tag{7.3}$$

Dividing both sides of (7.3) by $\binom{2m}{m}^r$ and performing the limit $m\to\infty$, we obtain (cf. (2.3))

$$\mathcal{S}_{r,n}(1,1,1)=\sum_{k_1,\dots,k_r=-n}^{n}\prod_{1\leqslant i<j\leqslant r}(k_i-k_j)^2\prod_{i=1}^{r}|k_i|\binom{2n}{n+k_i} \\ =r!\prod_{i=1}^{\lceil r/2\rceil}\frac{\Gamma^2(i)\,\Gamma(2n+1)}{\Gamma(n-i+2)\,\Gamma(n-i+1)}\prod_{i=1}^{\lfloor r/2\rfloor}\frac{\Gamma(i)\,\Gamma(i+1)\,\Gamma(2n+1)}{\Gamma^2(n-i+1)}. \tag{7.4}$$

Proof. We start by writing the Vandermonde products (there are *two* since the Vandermonde product is squared) in the summand as the following determinants:

$$\det_{1\leqslant i,j\leqslant r}\Big(1 \quad k_i \quad (n^2-k_i^2) \quad k_i(n^2-k_i^2) \quad (n^2-k_i^2)\big((n-1)^2-k_i^2\big) \quad \dots\Big),$$

respectively

$$\det_{1\leqslant i,j\leqslant r}\Big(1 \quad k_i \quad (m^2-k_i^2) \quad k_i(m^2-k_i^2) \quad (m^2-k_i^2)\big((m-1)^2-k_i^2\big) \quad \dots\Big),$$

This has to be read in such a way that the individual entries above give the columns of the matrix. More precisely, we have

$$\prod_{1\leqslant i<j\leqslant r}(k_i-k_j)=\pm\det M(N),$$

where $M(N)=\big(M_{i,j}(N)\big)_{1\leqslant i,j\leqslant r}$ is the $r\times r$ matrix defined by

$$M_{i,j}(N)=(-1)^{2\lfloor (j-1)/2\rfloor}k_i^{\chi(j\text{ even})}\,(-N-k_i)_{\lfloor (j-1)/2\rfloor}\,(-N+k_i)_{\lfloor (j-1)/2\rfloor},$$

Here, as before, $\chi(\mathcal{A})=1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A})=0$ otherwise, and the Pochhammer symbol $(\alpha)_m$ is defined by $(\alpha)_m:=\alpha(\alpha+1)\cdots(\alpha+m-1)$ for $m\geqslant 1$, and $(\alpha)_0:=1$. The substitution in (7.3) that we apply is

$$\prod_{1\leqslant i<j\leqslant r}(k_i-k_j)^2=\det M(n)\cdot\det M(m).$$

This turns the left-hand side of (7.3) into

$$\sum_{\sigma,\tau\in S_r}\operatorname{sgn}\sigma\tau\prod_{i=1}^{r}\left(\sum_{k_i=-\infty}^{\infty}\left(|k_i|\,k_i^{\chi(\sigma(i)\text{ even})+\chi(\tau(i)\text{ even})}\right.\right.$$
$$\times\frac{(2n)!}{(n+k_i-\lfloor(\sigma(i)-1)/2\rfloor)!\,(n-k_i-\lfloor(\sigma(i)-1)/2\rfloor)!}$$
$$\left.\left.\times\frac{(2m)!}{(m+k_i-\lfloor(\tau(i)-1)/2\rfloor)!\,(m-k_i-\lfloor(\tau(i)-1)/2\rfloor)!}\right)\right). \tag{7.5}$$

We must now evaluate the sum over k_i. There are three cases to be considered, depending on whether $\sigma(i)$ and $\tau(i)$ are even or odd. For convenience, in the following we shall use the short notation $S=\lfloor(\sigma(i)-1)/2\rfloor$ and $T=\lfloor(\tau(i)-1)/2\rfloor$.

CASE 1: *$\sigma(i)$ and $\tau(i)$ are both odd.* In this case we need to evaluate (writing k instead of k_i)

$$\sum_{k=-\infty}^{\infty}|k|\,\frac{(2n)!}{(n+k-S)!\,(n-k-S)!}\cdot\frac{(2m)!}{(m+k-T)!\,(m-k-T)!} \tag{7.6}$$
$$=2\sum_{k=1}^{\infty}k\,\frac{(2n)!}{(n+k-S)!\,(n-k-S)!}\cdot\frac{(2m)!}{(m+k-T)!\,(m-k-T)!}.$$

We write this sum in terms of the standard hypergeometric notation

$${}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right]=\sum_{\ell=0}^{\infty}\frac{(a_1)_\ell\cdots(a_r)_\ell}{\ell!\,(b_1)_\ell\cdots(b_s)_\ell}\,z^\ell,$$

to obtain the expression

$$2\,\frac{(2n)!}{(n-S+1)!\,(n-S-1)!}\cdot\frac{(2m)!}{(m-T+1)!\,(m-T-1)!}\\ \times {}_3F_2\left[\begin{matrix}2,-n+S+1,-m+T+1\\ n-S+2,m-T+2\end{matrix};1\right].$$

This hypergeometric series can be evaluated by (the terminating version) of Dixon's summation (see [46, Appendix (III.9)])

$${}_3F_2\left[\begin{matrix}a,b,-N\\ 1+a-b,1+a+N\end{matrix};1\right]=\frac{(1+a)_N\,(1+\frac{a}{2}-b)_N}{(1+\frac{a}{2})_N\,(1+a-b)_N},$$

where N is a non-negative integer. Indeed, if we choose $a=2$, $b=-n+S+1$, and $N=m-T-1$ in this summation formula, then our expression becomes

$$\frac{1}{(m+n-S-T)}\cdot\frac{(2n)!}{(n-S)!\,(n-S-1)!}\cdot\frac{(2m)!}{(m-T)!\,(m-T-1)!}\tag{7.7}$$

after some simplification.

Case 2: *$\sigma(i)$ and $\tau(i)$ have different parity.* In this case, we need to evaluate the sum

$$\sum_{k=-\infty}^{\infty}|k|\,k\,\frac{(2n)!}{(n+k-S)!\,(n-k-S)!}\cdot\frac{(2m)!}{(m+k-T)!\,(m-k-T)!}.$$

Since replacement of the summation index k by $-k$ converts this expression into its negative, the sum above vanishes.

Case 3: *$\sigma(i)$ and $\tau(i)$ are both even.* Now we must evaluate the sum

$$\sum_{k=-\infty}^{\infty}|k|\,k^2\,\frac{(2n)!}{(n+k-S)!\,(n-k-S)!}\cdot\frac{(2m)!}{(m+k-T)!\,(m-k-T)!}\\ =2\sum_{k=1}^{\infty}k^3\,\frac{(2n)!}{(n+k-S)!\,(n-k-S)!}\cdot\frac{(2m)!}{(m+k-T)!\,(m-k-T)!}.$$

We write

$$k^2=-\big((n-S)^2-k^2\big)+(n-S)^2=-(n+k-S)(n-k-S)+(n-S)^2\tag{7.8}$$

and substitute this in the summand. Splitting the sum accordingly, we obtain the expression

$$-2\sum_{k=1}^{\infty}k\,\frac{(2n)!}{(n+k-S-1)!\,(n-k-S-1)!}\cdot\frac{(2m)!}{(m+k-T)!\,(m-k-T)!}\\ +2(n-S)^2\sum_{k=1}^{\infty}k\,\frac{(2n)!}{(n+k-S)!\,(n-k-S)!}\cdot\frac{(2m)!}{(m+k-T)!\,(m-k-T)!}.$$

We have evaluated both sums already earlier. To be more specific, the second sum is the sum on the right-hand side of (7.6), and the first sum arises by replacing S by $S+1$

there. The closed-form expression for (7.6) is presented in (7.7). Consequently, our expression above becomes

$$
\begin{aligned}
(7.9)\qquad & -\frac{1}{(m+n-S-T-1)}\cdot\frac{(2n)!}{(n-S-1)!\,(n-S-2)!}\cdot\frac{(2m)!}{(m-T)!\,(m-T-1)!}\\
&\quad+\frac{(n-S)^2}{(m+n-S-T)}\cdot\frac{(2n)!}{(n-S)!\,(n-S-1)!}\cdot\frac{(2m)!}{(m-T)!\,(m-T-1)!}\\
&\quad=\frac{1}{(m+n-S-T-1)(m+n-S-T)}\cdot\frac{(2n)!}{(n-S-1)!^2}\cdot\frac{(2m)!}{(m-T-1)!^2}.
\end{aligned}
$$

If we summarise our findings so far (combine (7.5), (7.7) and (7.9)), then we have seen that the left-hand side of (7.3) equals

$$
\sum_{\sigma,\tau\in S_r}\operatorname{sgn}\sigma\tau\prod_{i=1}^{r}A_{\sigma(i),\tau(i)}, \tag{7.10}
$$

where, using the shorthand notation $K=\lfloor (k-1)/2\rfloor$ and $L=\lfloor (l-1)/2\rfloor$,

$$
A_{k,l}=\begin{cases}\frac{1}{(m+n-K-L)}\cdot\frac{(2n)!}{(n-K)!\,(n-K-1)!}\cdot\frac{(2m)!}{(m-L)!\,(m-L-1)!}, & \text{if } k \text{ and } l \text{ are odd},\\ \frac{1}{(m+n-K-L-1)(m+n-K-L)}\cdot\frac{(2n)!}{(n-K-1)!^2}\cdot\frac{(2m)!}{(m-L-1)!^2}, & \text{if } k \text{ and } l \text{ are even},\\ 0, & \text{otherwise.}\end{cases}
$$

We may reorder the product in (7.10),

$$
\sum_{\sigma,\tau\in S_r}\operatorname{sgn}\tau\sigma^{-1}\prod_{i=1}^{r}A_{i,\tau\sigma^{-1}(i)}.
$$

Writing $\rho=\tau\sigma^{-1}$, we may as well sum over all σ and ρ. Thereby we obtain

$$
\sum_{\sigma,\rho\in S_r}\operatorname{sgn}\rho\prod_{i=1}^{r}A_{i,\rho^{-1}(i)}=r!\det_{1\leqslant i,j\leqslant r}(A_{i,j}).
$$

Thus, the remaining task is to evaluate the determinant of the $A_{i,j}$'s.

If we recall the definition of $A_{i,j}$, then we see that the matrix $(A_{i,j})_{1\leqslant i,j\leqslant r}$ has a checkerboard structure. By reordering rows and columns, the matrix can be brought into a block form, from which it follows that

$$
\det_{1\leqslant i,j\leqslant r}(A_{i,j})=\det_{1\leqslant i,j\leqslant\lceil r/2\rceil}(A_{2i-1,2j-1})\cdot\det_{1\leqslant i,j\leqslant\lfloor r/2\rfloor}(A_{2i,2j}). \tag{7.11}
$$

Explicitly, the first determinant on the right-hand side of (7.11) is

$$
\begin{aligned}
&\det_{1\leqslant i,j\leqslant\lceil r/2\rceil}\left(\frac{1}{(m+n-i-j+2)}\cdot\frac{(2n)!}{(n-i+1)!\,(n-i)!}\cdot\frac{(2m)!}{(m-j+1)!\,(m-j)!}\right)\\
&=\prod_{i=1}^{\lceil r/2\rceil}\left(\frac{(2n)!}{(n-i+1)!\,(n-i)!}\cdot\frac{(2m)!}{(m-i+1)!\,(m-i)!}\right)\det_{1\leqslant i,j\leqslant\lceil r/2\rceil}\left(\frac{1}{m+n-i-j+2}\right).
\end{aligned}
$$

Clearly, the last determinant is a special case of Cauchy's double alternant (take $X_i = n-i+1$ and $Y_j = m-j+2$ in Eq. (2.7) of [24]). Substitution of the result leads to

$$\det_{1\leqslant i,j\leqslant\lceil r/2\rceil}(A_{2i-1,2j-1}) \tag{7.12}$$
$$=\prod_{i=1}^{\lceil r/2\rceil}\frac{(2n)!}{(n-i+1)!\,(n-i)!}\cdot\frac{(2m)!}{(m-i+1)!\,(m-i)!}\cdot\frac{(i-1)!^2\,(m+n-i-\lceil r/2\rceil+1)!}{(m+n-i+1)!}$$

after some manipulation.

On the other hand, the second determinant on the right-hand side of (7.11) is

$$\begin{aligned}
&\det_{1\leqslant i,j\leqslant\lfloor r/2\rfloor}\left(\frac{1}{(m+n-i-j+1)(m+n-i-j+2)}\cdot\frac{(2n)!}{(n-i)!^2}\cdot\frac{(2m)!}{(m-j)!^2}\right)\\
&\quad=\prod_{i=1}^{\lfloor r/2\rfloor}\left(\frac{(2n)!}{(n-i)!^2}\cdot\frac{(2m)!}{(m-i)!^2}\right)\det_{1\leqslant i,j\leqslant\lfloor r/2\rfloor}\left(\frac{1}{(m+n-i-j+1)(m+n-i-j+2)}\right)\\
&\quad=\prod_{i=1}^{\lfloor r/2\rfloor}\left(\frac{(2n)!}{(n-i)!^2}\cdot\frac{(2m)!}{(m-i)!^2}\cdot\frac{(m+n-i-\lfloor r/2\rfloor)!}{(m+n-i+1)!}\right)\\
&\qquad\qquad\times\det_{1\leqslant i,j\leqslant\lfloor r/2\rfloor}\Big((m+n-i-j+3)_{j-1}\,(m+n-i-\lfloor r/2\rfloor+1)_{\lfloor r/2\rfloor-j}\Big).
\end{aligned}$$

In order to evaluate this determinant, we have to put $n=\lfloor r/2\rfloor$, $X_i=m+n-i$, $A_j=-j+1$, and $B_j=-j+3$ in [24, Lemma 3]. Substitution of the result gives

$$\det_{1\leqslant i,j\leqslant\lfloor r/2\rfloor}(A_{2i,2j})=\prod_{i=1}^{\lfloor r/2\rfloor}\frac{(2n)!}{(n-i)!^2}\cdot\frac{(2m)!}{(m-i)!^2}\cdot\frac{(m+n-i-\lfloor r/2\rfloor)!\,(i-1)!\,i!}{(m+n-i+1)!} \tag{7.13}$$

after some manipulation.

If we finally combine (7.11), (7.12), and (7.13), then we obtain the right-hand side of (7.3). ☐

Proposition 7.3. *For all positive half-integers m and n and a positive integer r, we have*

(7.14)
$$\begin{aligned}
&\sum_{k_1,\dots,k_r=-n}^{n}\prod_{1\leqslant i<j\leqslant r}(k_i-k_j)^2\prod_{i=1}^{r}|k_i|\binom{2n}{n+k_i}\binom{2m}{m+k_i}\\
&\quad=r!\prod_{i=1}^{\lceil r/2\rceil}\frac{\Gamma(2n+1)}{\Gamma^2(n-i+\frac32)}\cdot\frac{\Gamma(2m+1)}{\Gamma^2(m-i+\frac32)}\cdot\frac{\Gamma^2(i)\,\Gamma(m+n-i-\lceil\frac r2\rceil+2)}{\Gamma(m+n-i+2)}\\
&\qquad\times\prod_{i=1}^{\lfloor r/2\rfloor}\frac{\Gamma(2n+1)}{\Gamma^2(n-i+\frac32)}\cdot\frac{\Gamma(2m+1)}{\Gamma^2(m-i+\frac32)}\cdot\frac{\Gamma(i)\,\Gamma(i+1)\,\Gamma(m+n-i-\lfloor\frac r2\rfloor+1)}{\Gamma(m+n-i+2)}\\
&\qquad\times\sum_{s=0}^{\lfloor r/2\rfloor}(-1)^{\lfloor r/2\rfloor-s}2^{-4(\lfloor r/2\rfloor-s)}\frac{m!\,n!\,(\lfloor\frac r2\rfloor)!\,(m+n-s)!}{s!\,(m-s)!\,(n-s)!\,(m+n-\lfloor\frac r2\rfloor)!}\binom{2\lfloor\frac r2\rfloor-2s}{\lfloor\frac r2\rfloor-s}^2.
\end{aligned}$$

Proof. The proof follows along the lines of the previous proof. In fact, not much needs to be changed. Until we reach Case 1, everything is identical. The sum to be evaluated in Case 1 is now

$$2\sum_{k=1/2}^{\infty} k\,\frac{(2n)!}{(n+k-S)!\,(n-k-S)!}\cdot\frac{(2m)!}{(m+k-T)!\,(m-k-T)!},$$

with the understanding that k ranges over half-integers. In hypergeometric terms, this sum equals

$$\frac{(2n)!}{(n-S+\frac{1}{2})!\,(n-S-\frac{1}{2})!}\cdot\frac{(2m)!}{(m-T+\frac{1}{2})!\,(m-T-\frac{1}{2})!}\\ \times {}_4F_3\left[\begin{matrix}1,\frac{3}{2},-n+S+\frac{1}{2},-m+T+\frac{1}{2}\\ \frac{1}{2},n-S+\frac{3}{2},m-T+\frac{3}{2}\end{matrix};1\right].$$

This hypergeometric series can be summed by means of the summation formula (see [46, Appendix (III.22)])

$${}_4F_3\left[\begin{matrix}a,\frac{a}{2}+1,b,c\\ \frac{a}{2},1+a-b,1+a-c\end{matrix};1\right]=\frac{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(\frac{1}{2}+\frac{a}{2})\,\Gamma(\frac{1}{2}+\frac{a}{2}-b-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)\,\Gamma(\frac{1}{2}+\frac{a}{2}-b)\,\Gamma(\frac{1}{2}+\frac{a}{2}-c)}$$

with $a=1$, $b=-n+S+\frac{1}{2}$, and $c=-m+T+\frac{1}{2}$. The result is that our sum simplifies to

$$\frac{1}{(m+n-S-T)}\cdot\frac{(2n)!}{(n-S-\frac{1}{2})!^2}\cdot\frac{(2m)!}{(m-T-\frac{1}{2})!^2}.$$

In Case 2 we obtain zero, as before. Finally, the result in Case 3 is

$$-\frac{1}{(m+n-S-T-1)}\cdot\frac{(2n)!}{(n-S-\frac{3}{2})!^2}\cdot\frac{(2m)!}{(m-T-\frac{1}{2})!^2}\\ +\frac{(n-S)^2}{(m+n-S-T)}\cdot\frac{(2n)!}{(n-S-\frac{1}{2})!^2}\cdot\frac{(2m)!}{(m-T-\frac{1}{2})!^2}\\ =\frac{(2n)!}{(n-S-\frac{1}{2})!^2}\cdot\frac{(2m)!}{(m-T-\frac{1}{2})!^2}\left(\frac{(n-S)^2}{m+n-S-T}-\frac{(n-S-\frac{1}{2})^2}{m+n-S-T-1}\right).$$

Consequently, as in the previous proof, the left-hand side of (7.14) can be written as a product of two determinants multiplied by $r!$. More precisely, it is equal to

$$r!\cdot\det_{1\leqslant i,j\leqslant\lceil r/2\rceil}\left(B^{(1)}_{i,j}\right)\cdot\det_{1\leqslant i,j\leqslant\lfloor r/2\rfloor}\left(B^{(2)}_{i,j}\right),$$

where

$$B^{(1)}_{i,j}=\frac{1}{(m+n-i-j+2)}\cdot\frac{(2n)!}{(n-i+\frac{1}{2})!^2}\cdot\frac{(2m)!}{(m-j+\frac{1}{2})!^2},$$

and

$$B^{(2)}_{i,j}=\frac{(2n)!}{(n-i+\frac{1}{2})!^2}\cdot\frac{(2m)!}{(m-j+\frac{1}{2})!^2}\left(\frac{(n-i+1)^2}{m+n-i-j+2}-\frac{(n-i+\frac{1}{2})^2}{m+n-i-j+1}\right).$$

The first determinant is again evaluated by applying Cauchy's double alternant,

$$\det_{1\leqslant i,j\leqslant \lceil r/2\rceil}\left(B^{(1)}_{i,j}\right)=\prod_{i=1}^{\lceil r/2\rceil}\frac{(2n)!}{(n-i+\frac{1}{2})!^2}\cdot\frac{(2m)!}{(m-i+\frac{1}{2})!^2}\cdot\frac{(i-1)!^2\,(m+n-i-\lceil r/2\rceil+1)!}{(m+n-i+1)!}.$$

In order to evaluate the second determinant, we observe that, after having factored out the terms which depend only on the row index i or only on the column index j, each entry is a sum of two terms. We use linearity of the determinant in the rows to decompose it into a sum of simpler determinants. In principle, this leads to $2^{\lfloor r/2\rfloor}$ terms. However, one readily sees that in most of these two successive rows are linearly dependent, and hence these terms vanish. More precisely, we have

$$\det_{1\leqslant i,j\leqslant \lceil r/2\rceil}\left(B^{(2)}_{i,j}\right)=\prod_{i=1}^{\lfloor r/2\rfloor}\frac{(2n)!}{(n-i+\frac{1}{2})!^2}\cdot\frac{(2m)!}{(m-i+\frac{1}{2})!^2}\\
\times\sum_{s=0}^{\lfloor r/2\rfloor}(-1)^{\lfloor r/2\rfloor-s}\det_{1\leqslant i,j\leqslant\lfloor r/2\rfloor}\left(\frac{(n-i+1-\chi(i>s)\frac{1}{2})^2}{m+n-i-\chi(i>s)-j+2}\right).$$

The last determinant can be evaluated by appealing to Cauchy's double alternant another time, and the result is

$$\det_{1\leqslant i,j\leqslant \lfloor r/2\rfloor}\left(B^{(2)}_{i,j}\right)=\prod_{i=1}^{\lfloor r/2\rfloor}\frac{(2n)!}{(n-i+\frac{1}{2})!^2}\cdot\frac{(2m)!}{(m-i+\frac{1}{2})!^2}\cdot\frac{(i-1)!\,i!\,(m+n-i-\lfloor\frac{r}{2}\rfloor)!}{(m+n-i+1)!}\\
\times\sum_{s=0}^{\lfloor r/2\rfloor}(-1)^{\lfloor r/2\rfloor-s}\frac{(n-s+1)_s^2\,(n-\lfloor\frac{r}{2}\rfloor+\frac{1}{2})^2_{\lfloor r/2\rfloor-s}\,(m+n-s)!}{s!\,(\lfloor\frac{r}{2}\rfloor-s)!\,(m+n-s-\lfloor\frac{r}{2}\rfloor)!}.$$

In order to make the symmetry in m and n of the final result immediately obvious, we convert the last sum into a different form. This is done by first writing it in hypergeometric notation,

$$\sum_{s=0}^{R}(-1)^{R-s}\frac{(n-s+1)_s^2\,(n-R+\frac{1}{2})^2_{R-s}\,(m+n-s)!}{s!\,(R-s)!\,(m+n-s-R)!}\\
=(-1)^R\frac{(n-R+\frac{1}{2})_R^2\,(m+n)!}{R!\,(m+n-R)!}\,{}_4F_3\left[\begin{matrix}-n,-n,-m-n+R,-R\\ -n+\frac{1}{2},-n+\frac{1}{2},-m-n\end{matrix};1\right]$$

(here, R is short for $\lfloor r/2\rfloor$), apply one of Whipple's balanced ${}_4F_3$-transformation formulas (see [46, Equation (4.3.5.1)]),

$${}_4F_3\left[\begin{matrix}a,b,c,-N\\ e,f,1+a+b+c-e-f-N\end{matrix};1\right]=\frac{(-a+e)_N\,(-a+f)_N}{(e)_N\,(f)_N}\\
\times{}_4F_3\left[\begin{matrix}-N,a,1+a+c-e-f-N,1+a+b-e-f-N\\ 1+a+b+c-e-f-N,1+a-e-N,1+a-f-N\end{matrix};1\right]$$

where N is a non-negative integer, to obtain

$$\sum_{s=0}^{R}(-1)^{R-s}\frac{(n-s+1)_s^2\,(n-R+\frac{1}{2})_{R-s}^2\,(m+n-s)!}{s!\,(R-s)!\,(m+n-s-R)!}$$
$$=\sum_{s=0}^{R}(-1)^{R-s}2^{-4(R-s)}\frac{\Gamma(m+1)\,\Gamma(n+1)\,R!\,(m+n-s)!}{s!\,\Gamma(m-s+1)!\,\Gamma(n-s+1)!\,(m+n-R)!}\binom{2R-2s}{R-s}^2.$$

Combining everything, we arrive at the right-hand side of (7.14). □

8. Discussion

We conclude our paper with a discussion of some open problems, additional results and future work.

8.1. **Arbitrary values of γ.** We have only proved discrete analogues of the Macdonald–Mehta integral (1.6) for $\gamma = 1/2$ and 1, values which in type A correspond to the *Gaußian orthogonal* and *Gaußian unitary* random matrix ensembles GOE and GUE, see e.g., [11]. For more general integer or half-integer values of γ, the sum (1.7) is not expressible in terms of a simple ratio of gamma functions. One of the reasons for this is that we have insisted on the simplest-possible discrete analogue of the G-Vandermonde product $|\Delta(x^\alpha)|^{2\gamma}$ as $|\Delta(k^\alpha)|^{2\gamma}$. To obtain formulas for more general choices of γ, more complicated analogues are required. For example, the A_{r-1} identity (7.2), pertaining to $\gamma = 1$, may be generalised to all non-negative integer values of γ as[3]

$$\sum_{k_1,\dots,k_r=-n}^{n}\ \prod_{1\leqslant i<j\leqslant r}\big|(k_i-k_j)_\gamma\,(k_j-k_i)_\gamma\big|\prod_{i=1}^{r}\binom{2n}{n+k_i}$$
$$=2^{2rn-\gamma r(r-1)}\prod_{i=1}^{r}\frac{\Gamma(1+i\gamma)}{\Gamma(1+\gamma)}\cdot\frac{\Gamma(2n+1)}{\Gamma(2n-(i-1)\gamma+1)}.$$

For $\gamma = 2$ this choice of Vandermonde-type product is in agreement with the *discrete symplectic ensemble* considered by Borodin and Strahov [2]. We did not, however, succeed in finding analogous generalisations for the other summations presented in this paper.

8.2. **More general reflection groups.** Another notable omission has been the treatment of reflection groups other than A_{r-1}, B_r and D_r. So far we have not found nice closed-form discrete analogues of Macdonald's integral (1.4) for any of the exceptional reflection groups or for the remaining infinite series, made up of the dihedral groups $I_2(m)$, $m \geqslant 3$ (the automorphism groups of the regular m-gons). It is difficult to conclude with certainty that no nice discrete analogues actually exist for any of these missing cases. In writing down the polynomials $P_G(x)$ for A_{r-1}, B_r and D_r in (1.5), we implicitly used the fact that Macdonald's integral *does not* depend on the choice of

[3]To prove this, we can take [49, Theorem 4.1] with $a = q^{-n-m}$, $b \to \infty$, $c = q^{m-n+1}$ and $k = \gamma$, where it is assumed without loss of generality that $m \geqslant n$. Symmetrising the summand using Lemma 3.1 of that same paper we obtain a generalisation of (7.1) in which $[k_j - k_i]_q^2$ is replaced by $(q^{k_j-k_i}, q^{1-\gamma+k_j-k_i}; q)_\gamma$ and $q^{k_i^2+(m+n-2i+2)k_i}$ by $q^{k_i^2+(m+n-2(i-1)\gamma)k_i}$. The rest follows as in the proof of (7.2).

$P_G(x)$. The actual form of $P_G(x)$ *does* depend on the choice of normals a_i in (1.3), and hence on the choice of reflecting hyperplanes $H_1, \dots, H_m$ generating G. For a given G, the set of hyperplanes, and hence the set of normals, is fixed up to a global rotation R of $\mathbb{R}^r$. If $a_i' = R(a_i)$ for $i = 1, \dots, m$, then

$$\int_{\mathbb{R}^r} \Big| \prod_{i=1}^m (a_i' \cdot y_i) \Big|^{2\gamma} \mathrm{d}\varphi(y) \overset{y=R(x)}{=} \int_{\mathbb{R}^r} \Big| \prod_{i=1}^m (a_i \cdot x_i) \Big|^{2\gamma} \mathrm{d}\varphi(x),$$

since the measure $\varphi(x)$ is rotationally invariant. At the discrete level, however, rotational invariance is lost, and hence the choice of $P_G(x)$ crucially affects the definition of a discrete analogue. Since there are infinitely many inequivalent choices of $P_G(x)$, there are infinitely many discrete analogues one may wish to try.

8.3. **Expressing the discrete Macdonald–Mehta integrals uniformly.** Another loose end concerns the question as to whether the six integral evaluations of Table 1 corresponding to A_{r-1}, B_r and D_r can be expressed in a single expression using only data coming from the underlying reflection group. Obviously, each case contains the factor

$$\prod_{i=1}^r \frac{\Gamma(1+d_i\gamma)}{\Gamma(1+\gamma)}$$

(with $\{d_i\} = \{1, \dots, r\}$ for A_{r-1}, $\{2, 4, \dots, 2r\}$ for B_r and $\{2, 4, \dots, 2r-2, r\}$ for $G = \mathrm{D}_r$), since the discrete evaluations reproduce the Macdonald–Mehta integral in the limit. We have however not been able to write the n-dependent factors in a uniform manner.

8.4. **Missing q-analogues of $\mathcal{S}_{r,n}(\alpha, \gamma, \delta)$.** We have obtained q-analogues for all evaluations listed in Table 1 except for (α, γ, δ) given by $(2, \frac{1}{2}, 0)$ and $(1, 1, 1)$. We can easily write down a q-analogue for the first of these two cases (given in (5.6)). Instead of $\sum_{\lambda \subseteq (r^n)} \mathrm{so}_{2n,\lambda}(1^n)$ we have to consider

$$\sum_{\lambda \subseteq (r^n)} \mathrm{so}_{2n,\lambda}\big(q^{1/2}, q^{3/2}, \dots, q^{n-1/2}\big).$$

Closed-form expressions for the summand as well as the actual sum are available in (3.36) and (4.7a). However, neither of these completely factor. A more natural q-analogue might result from summing

$$\sum_{\lambda \subseteq (r^n)} \mathrm{o}_{2n,\lambda}\big(q^{1/2}, q^{3/2}, \dots, q^{n-1/2}\big)$$

(cf. (3.34) for a fully factored expression for the summand). Unfortunately, we do not know a simple formula for the above character sum.

The problem of finding a q-analogue of our evaluation of $\mathcal{S}_{r,n}(1,1,1)$ (given in (2.3) as well as (7.4)) lies with identities such as (7.8) used in the proof of Proposition 7.2. It seems highly non-trivial to come up with an appropriate q-analogue of (7.8) such that in the next step of our calculations a q-analogue of Dixon's summation may be applied. In any case, the form of the evaluation of $\mathcal{S}_{r,n}(1,1,1)$, with its inherent distinction between even and odd r values, is an indication that this particular case is an outlier.

8.5. **Alternating sums.** As a variation on the main theme of the paper, we also considered the *alternating* sums

$$\widehat{\mathcal{S}}_{r,n}(\alpha,\gamma,\delta) := \sum_{k_1,\dots,k_r=-n}^{n} |\Delta(k^\alpha)|^{2\gamma} \prod_{i=1}^{r} (-1)^{k_i} |k_i|^\delta \binom{2n}{n+k_i}. \tag{8.1}$$

This differs from $\mathcal{S}_{r,n}(\alpha,\gamma,\delta)$ only in the sign $\prod_{i=1}^{r}(-1)^{k_i}$, but importantly, does not have a continuous analogue. The sum (8.1) admits a closed-form evaluation for all ten choices of α,β and γ considered in Table 1. (In some cases these evaluations are simply 0.) Since in each case a suitable adaptation of the arguments leading to the evaluation of $\mathcal{S}_{r,n}(\alpha,\gamma,\delta)$ suffices,[4] we refrain from presenting the corresponding identities and proofs here. We remark that it is often possible to prove alternating versions of most of our parametric extensions and q-analogues as well. As a typical example, we here just state one such result.

Proposition 8.1. *Let q be a real number with $0<q<1$. For all non-negative integers n, m, and p, and a positive integer r, we have*

$$\begin{aligned}
&\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} [k_j-k_i]_q^2\,[k_i+k_j]_q^2 \\
&\qquad\times \prod_{i=1}^{r} (-1)^{k_i} q^{\frac{3}{2}k_i^2-(2i-\frac{1}{2})k_i} \big|[k_i]_{q^2}\,[k_i]_q\big| \begin{bmatrix} 2n \\ n+k_i \end{bmatrix}_q \begin{bmatrix} 2m \\ m+k_i \end{bmatrix}_q \begin{bmatrix} 2p \\ p+k_i \end{bmatrix}_q \\
&= (-1)^{\binom{r+1}{2}} r! \Big(\frac{2}{[2]_q}\Big)^r q^{-\binom{r+1}{3}} \prod_{i=1}^{r} \bigg(\frac{\Gamma_q(2n+1)\,\Gamma_q(2m+1)\,\Gamma_q(2p+1)}{\Gamma_q(n-i+1)\,\Gamma_q(m-i+1)\,\Gamma_q(p-i+1)} \\
&\qquad\qquad \times \frac{\Gamma_q(i)\,\Gamma_q(n+m+p-i-r+2)}{\Gamma_q(n+m-i+2)\,\Gamma_q(m+p-i+2)\,\Gamma_q(p+n-i+2)} \bigg).
\end{aligned}$$

Proof. This follows by specialising $a=q^{-2n}$, $d=q^{-m-n}$, $e=q^{-p-n}$ and $f=q^{-n+1}$ in (6.3). ☐

Sending p to ∞ in Proposition 8.1, we obtain

$$\begin{aligned}
&\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\leqslant i<j\leqslant r} [k_j-k_i]_q^2\,[k_i+k_j]_q^2 \\
&\qquad\times \prod_{i=1}^{r} (-1)^{k_i} q^{\frac{3}{2}k_i^2-(2i-\frac{1}{2})k_i} \big|[k_i]_{q^2}\,[k_i]_q\big| \begin{bmatrix} 2n \\ n+k_i \end{bmatrix}_q \begin{bmatrix} 2m \\ m+k_i \end{bmatrix}_q \\
&= (-1)^{\binom{r+1}{2}} r! \Big(\frac{2}{[2]_q}\Big)^r q^{-\binom{r+1}{3}} \prod_{i=1}^{r} \frac{\Gamma_q(2n+1)\,\Gamma_q(2m+1)\,\Gamma_q(i)}{\Gamma_q(n-i+1)\,\Gamma_q(m-i+1)\,\Gamma_q(n+m-i+2)}.
\end{aligned}$$

[4]For example, in the proofs in Section 5 we have to insert $(-1)^{|\lambda|}$ in the summands of the appropriate character sums, while in the derivations in Sections 6 and 7 one typically has to specialise one of the indeterminates d,e,f in (6.3) to $-\sqrt{aq}$.

Upon letting $q \to 1$, dividing both sides by $\binom{2m}{m}^r$, and finally also letting m tend to ∞, we arrive at

$$\widehat{\mathcal{S}}_{r,n}(2,1,2) = \begin{cases} (-1)^{\binom{r+1}{2}} r!\big((2r)!\big)^r & \text{if } n = r, \\ 0 & \text{otherwise.} \end{cases}$$

8.6. **Additional character identities.** In Section 5 we evaluated the sum $\mathcal{S}_{r,n}(\alpha, \frac{1}{2}, \delta)$ using identities for classical group characters. Our evaluations of $\mathcal{S}_{r,n}(\alpha, 1, \delta)$ in Sections 6 and 7 were entirely different, relying on a transformation formula for elliptic hypergeometric series. It is nevertheless natural to wonder whether there are also character identities hidden behind the $\gamma = 1$ formulas. The answer to this question is, at least partially, affirmative. If one specialises all variables x_i to 1 in the identities given in [36, Theorem 2.2], then one obtains (6.5), (6.13) and (7.1) in the integer-n case, *all for* $q = 1$. We discovered this fact in a rather roundabout way as follows. Helmut Prodinger suggested to the first author that non-intersecting lattice paths may have a role to play in proving some of the discrete Macdonald–Mehta integrals, an idea we initially discarded. Subsequently we realised that the combinatorics of non-intersecting lattice paths can indeed be used to prove the evaluations of $\mathcal{S}_{r,n}(\alpha, 1, \delta)$ for $(\alpha, \delta) \in \{(1,0), (2,0), (2,2)\}$. However, we did not see how to use this approach to also prove corresponding q-analogues. Clearly, to obtain these one would have to introduce appropriate q-weights for the non-intersecting lattice paths. By introducing weights, we however discovered numerous identities for classical group characters, which Soichi Okada quickly identified as [36, Theorem 2.2]. While we still do not see how to specialise these identities appropriately to produce q-analogues, using our combinatorial machinery we did find one identity for Proctor's odd symplectic characters [40] missed by Okada. The full details of this part of the story of discrete analogues of Macdonald–Mehta integrals will be presented in [5].

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