HALL–LITTLEWOOD FUNCTIONS AND THE A_2 ROGERS–RAMANUJAN IDENTITIES

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ABSTRACT. We prove an identity for Hall–Littlewood symmetric functions labelled by the Lie algebra A_2. Through specialization this yields a simple proof of the A_2 Rogers–Ramanujan identities of Andrews, Schilling and the author.

1. Introduction

The Rogers–Ramanujan identities, given by [14]

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \tag{1.1a}$$

and

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-2})(1-q^{5n-3})}, \tag{1.1b}$$

are two of the most famous q-series identities, with deep connections with number theory, representation theory, statistical mechanics and various other branches of mathematics.

Many different proofs of the Rogers–Ramanujan identities have been given in the literature, some bijective, some representation theoretic, but the vast majority basic hypergeometric. In 1990, J. Stembridge, building on work of I. Macdonald, found a proof of the Rogers–Ramanujan identities quite unlike any of the previously known proofs. In particular he discovered that Rogers–Ramanujan-type identities may be obtained by appropriately specializing identities for Hall–Littlewood polynomials. The Hall–Littlewood polynomials and, more generally, Hall–Littlewood functions are an important class of symmetric functions, generalizing the well-known Schur functions. Stembridge's Hall–Littlewood approach to Rogers–Ramanujan identities has been further generalized in recent work by Fulman [2], Ishikawa *et al.* [8] and Jouhet and Zeng [10].

Several years ago Andrews, Schilling and the present author generalized the two Rogers–Ramanujan identities to three identities labelled by the Lie algebra A_2 [1]. The simplest of these, which takes the place of (1.1a) when A_1 is replaced by A_2

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reads

$$(1.2)\qquad (q;q)_\infty \sum_{n_1,n_2=0}^{\infty} \frac{q^{n_1^2-n_1n_2+n_2^2}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_1+n_2}}$$

$$= \sum_{n_1,n_2=0}^{\infty} \frac{q^{n_1^2-n_1n_2+n_2^2}}{(q;q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-1})^2(1-q^{7n-3})(1-q^{7n-4})(1-q^{7n-6})^2},$$

where $(q;q)_0 = 1$ and $(q;q)_n = \prod_{i=1}^n (1-q^i)$ is a q-shifted factorial, and

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \dfrac{(q^{n-m+1};q)_m}{(q;q)_m} & \text{for } m \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

is a q-binomial coefficient. The equivalence of the two expressions on the left of (1.2) follows from a straightforward application of Jackson's terminating ${}_2\phi_1$ transformation [4, Equation (III.7)], see [1].

The A_2 characteristics of (1.2) are (i) the exponent of q of the two summands, which may alternatively be put as $\frac{1}{2}\sum_{i,j=1}^2 C_{ij}n_in_j$ with $C = ((2,-1),(-1,2))$ the A_2 Cartan matrix, and (ii) the infinite product on the right, which can be identified with a branching function of the coset pair $(\mathrm{A}_2^{(1)} \oplus \mathrm{A}_2^{(1)}, \mathrm{A}_2^{(1)})$ at levels $-9/4$, 1 and $-5/4$, see [1].

An important question is whether (1.2) and its companions can again be understood in terms of Hall–Littlewood functions. This question is especially relevant since the A_n analogues of the Rogers–Ramanujan identities have so far remained elusive, and an understanding of (1.2) in the context of symmetric functions might provide further insight into the structure of the full A_n generalization of (1.1).

In this paper we will show that the theory of Hall–Littlewood functions may indeed be applied to yield a proof of (1.2). In particular we will prove the following A_2-type identity for Hall–Littlewood functions.

Theorem 1.1. *Let $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$ and let $P_\lambda(x;q)$ and $P_\mu(y;q)$ be Hall–Littlewood functions indexed by the partitions λ and μ. Then*

$$(1.3)\qquad \sum_{\lambda,\mu} q^{n(\lambda)+n(\mu)-(\lambda'|\mu')} P_\lambda(x;q)P_\mu(y;q) = \prod_{i\geq 1} \frac{1}{(1-x_i)(1-y_i)} \prod_{i,j\geq 1} \frac{1-x_iy_j}{1-q^{-1}x_iy_j}.$$

In the above λ' and μ' are the conjugates of λ and μ, $(\lambda|\mu) = \sum_{i\geq 1}\lambda_i\mu_i$, and $n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$.

For $q = 1$ the Hall–Littlewood function $P_\lambda(x;q)$ reduces to the monomial symmetric function $m_\lambda(x)$, and the identity (1.3) factorizes into a product of the well-known

$$\sum_\lambda m_\lambda(x) = \prod_{i\geq 1} \frac{1}{1-x_i}.$$

An appropriate specialization of Theorem 1.1 leads to a q-series identity of [1] which is the key-ingredient in proving (1.2). In fact, the steps leading from (1.3) to (1.2) suggests that what is needed for the A_n version of the Rogers–Ramanujan identities is an identity for the more general sum

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} \prod_{i=1}^{n} q^{n(\lambda^{(i)})-(\lambda^{(i)\prime}|\lambda^{(i+1)\prime})} P_{\lambda^{(i)}}(x^{(i)};q), \tag{1.4}$$

where $\lambda^{(1)},\dots,\lambda^{(n+1)}$ are partitions with $\lambda^{(n+1)}=0$ the empty or zero partition, and $x^{(i)}=(x_1^{(i)},x_2^{(i)},\dots)$. What makes this sum difficult to handle is that no factorized right-hand side exists for $n>2$.

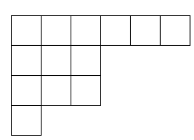
In the next section we give the necessary background material on Hall–Littlewood functions. In Section 3 some immediate consequences of Theorem 1.1 are derived, the most interesting one being the new q-series identity claimed in Corollary 3.4. Section 4 contains a proof of Theorem 1.1 and Section 5 contains a proof of the A_2 Rogers–Ramanujan identities (1.2) based on Corollary 3.4. Finally, in Section 6 we present some open problems related to the results of this paper.

2. Hall-Littlewood functions

We review some basic facts from the theory of Hall-Littlewood functions. For more details the reader may wish to consult Chapter III of Macdonald's book on symmetric functions [13].

Let $\lambda=(\lambda_1,\lambda_2,\dots)$ be a partition, i.e., $\lambda_1\geq\lambda_2\geq\dots$ with finitely many λ_i unequal to zero. The length and weight of λ, denoted by $\ell(\lambda)$ and $|\lambda|$, are the number and sum of the non-zero λ_i (called parts), respectively. The unique partition of weight zero is denoted by 0, and the multiplicity of the part i in the partition λ is denoted by $m_i(\lambda)$.

We identify a partition with its diagram or Ferrers graph in the usual way, and, for example, the diagram of $\lambda=(6,3,3,1)$ is given by



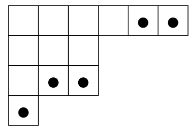
The conjugate λ' of λ is the partition obtained by reflecting the diagram of λ in the main diagonal. Hence $m_i(\lambda)=\lambda'_i-\lambda'_{i+1}$.

A standard statistic on partitions needed repeatedly is

$$n(\lambda)=\sum_{i\geq 1}(i-1)\lambda_i=\sum_{i\geq 1}\binom{\lambda'_i}{2}.$$

We also need the usual scalar product $(\lambda|\mu)=\sum_{i\geq 1}\lambda_i\mu_i$ (which in the notation of [13] would be $|\lambda\mu|$). We will occasionally use this for more general sequences of integers, not necessarily partitions.

If λ and μ are two partions then $\mu\subset\lambda$ iff $\lambda_i\geq\mu_i$ for all $i\geq 1$, i.e., the diagram of λ contains the diagram of μ. If $\mu\subset\lambda$ then the skew-diagram $\lambda-\mu$ denotes the set-theoretic difference between λ and μ, and $|\lambda-\mu|=|\lambda|-|\mu|$. For example, if $\lambda=(6,3,3,1)$ and $\mu=(4,3,1)$ then the skew diagram $\lambda-\mu$ is given by the marked squares in



and $|\lambda-\mu|=5$.

For $\theta=\lambda-\mu$ a skew diagram, its conjugate $\theta'=\lambda'-\mu'$ is the (skew) diagram obtained by reflecting θ in the main diagonal. Following [13] we define the components of θ and θ' by $\theta_i=\lambda_i-\mu_i$ and $\theta'_i=\lambda'_i-\mu'_i$. Quite often we only require knowledge of the sequence of components of a skew diagram θ, and by abuse of notation we will occasionally write $\theta=(\theta_1,\theta_2,\dots)$, even though the components θ_i alone do not fix θ.

A skew diagram θ is a horizontal strip if $\theta'_i\in\{0,1\}$, i.e., if at most one square occurs in each column of θ. The skew diagram in the above example is a horizontal strip since $\theta'=(1,1,1,0,1,1,0,0,\dots)$.

Let S_n be the symmetric group, $\Lambda_n=\mathbb{Z}[x_1,\dots,x_n]^{S_n}$ be the ring of symmetric polynomials in n independent variables and Λ the ring of symmetric functions in countably many independent variables.

For $x=(x_1,\dots,x_n)$ and λ a partition such that $\ell(\lambda)\le n$ the Hall–Littlewood polynomials $P_\lambda(x;q)$ are defined by

$$P_\lambda(x;q)=\sum_{w\in S_n/S_n^\lambda} w\Big(x^\lambda\prod_{\lambda_i>\lambda_j}\frac{x_i-qx_j}{x_i-x_j}\Big). \tag{2.1}$$

Here S_n^λ is the subgroup of S_n consisting of the permutations that leave λ invariant, and $w(f(x))=f(w(x))$. When $\ell(\lambda)>n$,

$$P_\lambda(x;q)=0. \tag{2.2}$$

The Hall–Littlewood polynomials are symmetric polynomials in x, homogeneous of degree $|\lambda|$, with coefficients in $\mathbb{Z}[q]$, and form a $\mathbb{Z}[q]$ basis of $\Lambda_n[q]$. Thanks to the stability property $P_\lambda(x_1,\dots,x_n,0;q)=P_\lambda(x_1,\dots,x_n;q)$ the Hall–Littlewood polynomials may be extended to the Hall–Littlewood functions in an infinite number of variables $x_1,x_2,\dots$ in the usual way, to form a $\mathbb{Z}[q]$ basis of $\Lambda[q]$. The indeterminate q in the Hall–Littlewood symmetric functions serves as a parameter interpolating between the Schur functions and monomial symmetric functions; $P_\lambda(x;0)=s_\lambda(x)$ and $P_\lambda(x;1)=m_\lambda(x)$.

We will also need the symmetric functions $Q_\lambda(x;q)$ (also referred to as Hall-Littlewood functions) defined by

$$Q_\lambda(x;q)=b_\lambda(q)P_\lambda(x;q), \tag{2.3}$$

where

$$b_\lambda(q)=\prod_{i=1}^{\lambda_1}(q;q)_{m_i(\lambda)}.$$

We already mentioned the homogeneity of the Hall–Littlewood functions;

$$P_\lambda(ax;q)=a^{|\lambda|}P_\lambda(x;q), \tag{2.4}$$

where $ax=(ax_1,ax_2,\dots)$. Another useful result is the specialization

$$P_\lambda(1,q,\dots,q^{n-1};q)=\frac{q^{n(\lambda)}(q;q)_n}{(q;q)_{n-\ell(\lambda)}b_\lambda(q)}, \tag{2.5}$$

where $1/(q;q)_{-m}=0$ for m a positive integer, so that $P_\lambda(1,q,\dots,q^{n-1};q)=0$ if $\ell(\lambda)>n$ in accordance with (2.2). By (2.3) this also implies the particularly simple

$$Q_\lambda(1,q,q^2,\dots;q)=q^{n(\lambda)}. \tag{2.6}$$

The Cauchy identity for Hall–Littlewood functions states that

$$\sum_\lambda P_\lambda(x;q)Q_\lambda(y;q)=\prod_{i,j\geq 1}\frac{1-qx_iy_j}{1-x_iy_j}. \tag{2.7}$$

Taking $y_j=q^{j-1}$ for all $j\geq 1$ and using the specialization (2.6) yields

$$\sum_\lambda q^{n(\lambda)}P_\lambda(x;q)=\prod_{i\geq 1}\frac{1}{1-x_i}. \tag{2.8}$$

We remark that this is the A_1 analogue of Theorem 1.1, providing an evaluation for the sum (1.4) when $n=1$.

The skew Hall–Littlewood functions $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$ are defined by

$$P_\lambda(x,y;q)=\sum_\mu P_{\lambda/\mu}(x;q)P_\mu(y;q) \tag{2.9}$$

and

$$Q_\lambda(x,y;q)=\sum_\mu Q_{\lambda/\mu}(x;q)Q_\mu(y;q),$$

so that

$$Q_{\lambda/\mu}(x;q)=\frac{b_\lambda(q)}{b_\mu(q)}P_{\lambda/\mu}(x;q). \tag{2.10}$$

An important property is that $P_{\lambda/\mu}$ is zero if $\mu\not\subset\lambda$. Some trivial instances of the skew functions are given by $P_{\lambda/0}=P_\lambda$ and $P_{\lambda/\lambda}=1$. By (2.10) similar statements apply to $Q_{\lambda/\mu}$.

The Cauchy identity (2.7) can be generalized to the skew case as [16, Lemma 3.1]

$$\sum_\lambda P_{\lambda/\mu}(x;q)Q_{\lambda/\nu}(y;q)=\sum_\lambda P_{\nu/\lambda}(x;q)Q_{\mu/\lambda}(y;q)\prod_{i,j\geq 1}\frac{1-qx_iy_j}{1-x_iy_j}. \tag{2.11}$$

Taking $\nu=0$ and specializing $y_j=q^{j-1}$ for all $j\geq 1$ extends (2.8) to

$$\sum_\lambda q^{n(\lambda)}P_{\lambda/\mu}(x;q)=q^{n(\mu)}\prod_{i\geq 1}\frac{1}{1-x_i}. \tag{2.12}$$

We conclude our introduction of the Hall–Littlewood functions with the following two important definitions. Let $\lambda\supset\mu$ be partitions such that $\theta=\lambda-\mu$ is a horizontal strip, i.e., $\theta_i'\in\{0,1\}$. Let I be the set of integers $i\geq 1$ such that $\theta_i'=1$ and $\theta_{i+1}'=0$. Then

$$\phi_{\lambda/\mu}(q)=\prod_{i\in I}(1-q^{m_i(\lambda)}).$$

Similarly, let J be the set of integers $j\geq 1$ such that $\theta_j'=0$ and $\theta_{j+1}'=1$. Then

$$\psi_{\lambda/\mu}(q)=\prod_{j\in J}(1-q^{m_j(\mu)}).$$

For example, if $\lambda=(5,3,2,2)$ and $\mu=(3,3,2)$ then θ is a horizontal strip and $\theta'=(1,1,0,1,1,0,0,\dots)$. Hence $I=\{2,5\}$ and $J=\{3\}$, leading to

$$\phi_{\lambda/\mu}(q)=(1-q^{m_2(\lambda)})(1-q^{m_5(\lambda)})=(1-q^2)(1-q)$$

and

$$\psi_{\lambda/\mu}(q)=(1-q^{m_3(\mu)})=(1-q^2).$$

The skew Hall–Littlewood functions $Q_{\lambda/\mu}(x;q)$ and $P_{\lambda/\mu}(x;q)$ can be expressed in terms of $\phi_{\lambda/\mu}(q)$ and $\psi_{\lambda/\mu}(q)$ [13, p. 229]. For our purposes we only require a special instance of this result corresponding to the case that x represents a single variable. Then

$$Q_{\lambda/\mu}(x;q)=\begin{cases}\phi_{\lambda/\mu}(q)x^{|\lambda-\mu|} & \text{if } \lambda-\mu \text{ is a horizontal strip,}\\ 0 & \text{otherwise}\end{cases} \tag{2.13a}$$

and

$$P_{\lambda/\mu}(x;q)=\begin{cases}\psi_{\lambda/\mu}(q)x^{|\lambda-\mu|} & \text{if } \lambda-\mu \text{ is a horizontal strip,}\\ 0 & \text{otherwise.}\end{cases} \tag{2.13b}$$

3. Consequences of Theorem 1.1

Before we present a proof of Theorem 1.1 we will establish some simple corollaries of the A_2 sum for Hall–Littlewood functions.

We begin by noting that (1.3) simplifies to (2.8) when all components of y are set to zero. Our first corollary of Theorem 1.1 corresponds to a slight generalization that also includes (2.12).

Corollary 3.1. *For ν a partition,*

$$\begin{aligned}\sum_{\lambda,\mu} & q^{n(\lambda)+n(\mu)-(\lambda'|\mu')}P_{\lambda/\nu}(x;q)P_\mu(y;q)\\ &=\sum_{\lambda} q^{n(\lambda)+n(\nu)-(\lambda'|\nu')}P_\lambda(y;q)\prod_{i\geq 1}\frac{1}{1-x_i}\prod_{i,j\geq 1}\frac{1-x_iy_j}{1-q^{-1}x_iy_j}.\end{aligned} \tag{3.1}$$

When $\nu=0$ the sum over λ on the right may be performed by (2.8) and one recovers (1.3).

Proof of Corollary 3.1. Multiplying both sides of (3.1) by $P_\nu(z;q)$ and summing over ν gives

$$\begin{aligned}\sum_{\lambda,\mu} & q^{n(\lambda)+n(\mu)-(\lambda'|\mu')}P_{\lambda}(x,z;q)P_\mu(y;q)\\ &=\sum_{\lambda,\nu} q^{n(\lambda)+n(\nu)-(\lambda'|\nu')}P_\lambda(y;q)P_\nu(z;q)\prod_{i\geq 1}\frac{1}{1-x_i}\prod_{i,j\geq 1}\frac{1-x_iy_j}{1-q^{-1}x_iy_j},\end{aligned}$$

where on the left we have used (2.9). The truth of this identity is readily verified upon noting that both sides can be summed by (1.3). Since the $P_\nu(z;q)$ form a basis of $\Lambda[q]$ the identity (3.1) itself must be true. ☐

It is suggestive that a yet more general symmetric expansion should hold for

$$\sum_{\lambda,\mu} q^{n(\lambda)+n(\mu)-(\lambda'|\mu')}P_{\lambda/\nu}(x;q)P_{\mu/\eta}(y;q),$$

but we were only able to obtain the following asymmetric sum.

Corollary 3.2. *For ν and η partitions,*

$$\sum_{\lambda,\mu} q^{n(\lambda)+n(\mu)-(\lambda'|\mu')} P_{\lambda/\nu}(x;q) P_{\mu/\eta}(y;q)$$
$$= \sum_{\lambda,\mu} q^{n(\lambda)+n(\nu)-(\lambda'|\nu')} Q_{\eta/\mu}(x/q;q) P_{\lambda/\mu}(y;q) \prod_{i\geq 1} \frac{1}{1-x_i} \prod_{i,j\geq 1} \frac{1-x_iy_j}{1-q^{-1}x_iy_j}.$$

When all the y_i are set to zero this yields (after a change of variables)

$$\sum_{\nu} q^{n(\lambda)+n(\nu)-(\lambda'|\nu')} P_{\nu/\mu}(x;q) = \sum_{\nu} q^{n(\mu)+n(\nu)-(\mu'|\nu')} Q_{\lambda/\nu}(x/q;q) \prod_{i\geq 1} \frac{1}{1-x_i}. \tag{3.2}$$

The case when x represents a single variable will play an important role in the proof of Theorem 1.1.

Proof of Corollary 3.2. After multiplying both sides by $P_\nu(z;q)P_\eta(w;q)$ and summing over ν and η we get

$$\sum_{\lambda,\mu} q^{n(\lambda)+n(\mu)-(\lambda'|\mu')} P_\lambda(x,z;q) P_\mu(y,w;q)$$
$$= \sum_{\lambda,\mu,\nu,\eta} q^{n(\lambda)+n(\nu)-(\lambda'|\nu')} Q_{\eta/\mu}(x/q;q) P_{\lambda/\mu}(y;q) P_\nu(z;q) P_\eta(w;q)$$
$$\times \prod_{i\geq 1} \frac{1}{1-x_i} \prod_{i,j\geq 1} \frac{1-x_iy_j}{1-q^{-1}x_iy_j}.$$

By the Cauchy identity for skew Hall–Littlewood functions (2.11) the sum over η on the right simplifies to

$$P_\mu(w;q) \prod_{i,j\geq 1} \frac{1-x_iw_j}{1-q^{-1}x_iw_j}.$$

This then allows for the sum over μ to be carried out using (2.9), yielding

$$\sum_{\lambda,\mu} q^{n(\lambda)+n(\mu)-(\lambda'|\mu')} P_\lambda(x,z;q) P_\mu(y,w;q)$$
$$= \sum_{\lambda,\nu} q^{n(\lambda)+n(\nu)-(\lambda'|\nu')} P_\lambda(y,w;q) P_\nu(z;q)$$
$$\times \prod_{i\geq 1} \frac{1}{1-x_i} \prod_{i,j\geq 1} \frac{(1-x_iy_j)(1-x_iw_j)}{(1-q^{-1}x_iy_j)(1-q^{-1}x_iw_j)}.$$

The rest again follows from (1.3). $\square$

As a third corollary we can include a linear term in the exponent of q in (2.8).

Corollary 3.3. *For j a non-negative integer,*

$$\sum_{\lambda} q^{n(\lambda)-\sum_{l=1}^{j} \lambda'_l} P_\lambda(x;q) = \Big(1 + (1-q)\sum_{k=1}^{j} q^{-k} P_{(k)}(x;q)\Big) \prod_{i\geq 1} \frac{1}{1-x_i}.$$

This result implies some nice q-series identities. By specializing $x_i = zq^i$ for $1 \le i \le n$ and $x_i = 0$ for $i > n$, and using (2.4) and (2.5) we find

$$\begin{aligned}\sum_{\lambda} \frac{z^{|\lambda|} q^{(\lambda'|\lambda') - \sum_{l=1}^{j} \lambda'_l} (q;q)_n}{(q;q)_{n-\ell(\lambda)} b_\lambda(q)} &= \frac{1}{(zq;q)_n} \Bigl(1 + (1-q^n) \sum_{k=1}^{j} z^k\Bigr) \\ &= \frac{1}{(z;q)_n} - (1-q^n) \frac{z^{j+1}}{(z;q)_{n+1}},\end{aligned}$$

where we have used that $2n(\lambda) + |\lambda| = (\lambda'|\lambda')$. By the q-binomial theorem

$$\sum_{j=0}^{\infty} z^j \begin{bmatrix} n+j-1 \\ j \end{bmatrix} = \frac{1}{(z;q)_n}$$

the coefficient of z^k can easily be found as

$$\sum_{\lambda \vdash k} \frac{q^{(\lambda|\lambda) - \sum_{l=1}^{j} \lambda_l} (q;q)_n}{(q;q)_{n-\lambda_1(\lambda)} b_{\lambda'}(q)} = \begin{bmatrix} n+k-1 \\ k \end{bmatrix} - (1-q^n) \begin{bmatrix} n+k-j-1 \\ k-j-1 \end{bmatrix}$$

for $0 \le j \le k$. Here we have changed the summation index λ by its conjugate. For $j = k$ or (after simplifying the right) $j = 0$ this is a well-known q-series identity of Hall [6], see also [12, 15, 16]. Letting n tend to infinity finally gives

$$\sum_{\lambda \vdash k} \frac{q^{(\lambda|\lambda) - \sum_{l=1}^{j} \lambda_j}}{b_{\lambda'}(q)} = \frac{1}{(q;q)_k} - \frac{1}{(q;q)_{k-j-1}}.$$

Proof of Corollary 3.3. Equation (1.3) with $y_1 = z$ and $y_i = 0$ for $i \ge 2$ yields

$$\sum_{j=0}^{\infty} z^j \sum_{\lambda} q^{n(\lambda) - \sum_{l=1}^{j} \lambda'_l} P_\lambda(x;q) = \frac{1}{1-z} \prod_{i \ge 1} \frac{1}{1-x_i} \prod_{i \ge 1} \frac{1 - x_i z}{1 - q^{-1} x_i z}.$$

From the Cauchy identity (2.7) it follows that the last product on the right can be expanded as

$$\sum_{k=0}^{\infty} Q_{(k)}(z/q;q) P_{(k)}(x;q) = 1 + (1-q) \sum_{k=0}^{\infty} P_{(k)}(x;q)(z/q)^k.$$

Then equating coefficients of z^j leads to the desired result. □

Finally we come to what is by far the most important corollary of Theorem 1.1. Let $(a;q)_0 = 1$, $(a;q)_n = \prod_{i=1}^{n} (1 - aq^{i-1})$ and $(a_1, \ldots, a_k; q)_n = (a_1;q)_n \cdots (a_k;q)_n$.

Corollary 3.4. *There holds*

$$\sum_{\lambda,\mu} \frac{a^{|\lambda|} b^{|\mu|} q^{(\lambda'|\lambda') + (\mu'|\mu') - (\lambda'|\mu')}}{(q;q)_{n-\ell(\lambda)} (q;q)_{m-\ell(\mu)} b_\lambda(q) b_\mu(q)} = \frac{(abq;q)_{n+m}}{(q,aq,abq;q)_n (q,bq,abq;q)_m} \tag{3.3}$$

Proof. Taking $x_i = aq^i$ for $1 \le i \le n$, $x_i = 0$ for $i > n$, $y_j = bq^j$ for $1 \le j \le m$ and $y_j = 0$ for $j > m$, using the homogeneity (2.4) and specialization (2.5), and noting that $2n(\lambda) + |\lambda| = (\lambda'|\lambda')$, we obtain (3.3). □

In Section 5 we will show how Corollary 3.4 relates to the A_2 Rogers–Ramanujan identity (1.2). For now let us remark that (3.3) is a bounded version of the A_2 case of the following identity for the A_n root system due to Hua [7] (and corrected in [3]):

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} \frac{q^{\frac{1}{2}\sum_{i,j=1}^n C_{ij}(\lambda^{(i)\prime}|\lambda^{(j)\prime})} \prod_{i=1}^n a_i^{|\lambda^{(i)}|}}{\prod_{i=1}^n b_{\lambda^{(i)}}(q)} = \prod_{\alpha\in\Delta_+} \frac{1}{(a^\alpha q;q)_\infty}. \tag{3.4}$$

Here $C_{ij}=2\delta_{i,i}-\delta_{i,j-1}-\delta_{i,j+1}$ is the (i,j) entry of the A_n Cartan matrix and Δ_+ is the set of positive roots of A_n, i.e., the set (of cardinality $\binom{n+1}{2}$) of roots of the form $\alpha_i+\alpha_{i+1}+\dots+\alpha_j$ with $1\le i\le j\le n$, where $\alpha_1,\dots,\alpha_n$ are the simple roots of A_n. Furthermore, if $\alpha=\alpha_i+\alpha_{i+1}+\dots+\alpha_j$ then $a^\alpha=a_ia_{i+1}\cdots a_j$.

4. Proof of Theorem 1.1

Throughout this section z represents a single variable.

To establish (1.3) it is enough to show its truth for $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_m)$, and by induction on m it then easily follows that we only need to prove

$$\begin{aligned}\sum_{\lambda,\mu} q^{n(\lambda)+n(\mu)-(\lambda'|\mu')} P_\lambda(x;q)P_\mu(y,z;q) \\ = \frac{1}{1-z}\prod_{i=1}^n \frac{1-zx_i}{1-q^{-1}zx_i} \sum_{\lambda,\mu} q^{n(\lambda)+n(\mu)-(\lambda'|\mu')} P_\lambda(x;q)P_\mu(y;q),\end{aligned} \tag{4.1}$$

where we have replaced y_{m+1} by z.

If on the left we replace μ by ν and use (2.9) (with $\lambda\to\nu$ and $x\to z$) we get

$$\mathrm{LHS}(4.1)=\sum_{\lambda,\mu,\nu} q^{n(\lambda)+n(\nu)-(\lambda'|\nu')} P_\lambda(x;q)P_\mu(y;q)P_{\nu/\mu}(z;q).$$

From (2.11) with $\mu=0$, $x=(x_1,\dots,x_n)$ and $y\to z/q$ it follows that

$$P_\nu(x;q)\prod_{i=1}^n \frac{1-zx_i}{1-q^{-1}zx_i} = \sum_\lambda Q_{\lambda/\nu}(z/q;q)P_\lambda(x;q).$$

Using this on the right of (4.1) with λ replaced by ν yields

$$\mathrm{RHS}(4.1)=\frac{1}{1-z}\sum_{\lambda,\mu,\nu} q^{n(\mu)+n(\nu)-(\mu'|\nu')} P_\lambda(x;q)P_\mu(y;q)Q_{\lambda/\nu}(z/q;q).$$

Therefore, by equating coefficients of $P_\lambda(x;q)P_\mu(y;q)$ we find that the problem of proving (1.3) boils down to showing that

$$\sum_\nu q^{n(\lambda)+n(\nu)-(\lambda'|\nu')} P_{\nu/\mu}(z;q) = \frac{1}{1-z}\sum_\nu q^{n(\mu)+n(\nu)-(\mu'|\nu')} Q_{\lambda/\nu}(z/q;q),$$

which is (3.2) with $x\to z$.

Next we use (2.13) to arrive at the equivalent but more combinatorial statement that

$$\sum_{\substack{\nu\supset\mu\\ \nu-\mu \text{ hor. strip}}} q^{n(\lambda)+n(\nu)-(\lambda'|\nu')} z^{|\nu-\mu|}\psi_{\nu/\mu}(q) = \frac{1}{1-z}\sum_{\substack{\nu\subset\lambda\\ \lambda-\nu \text{ hor. strip}}} q^{n(\mu)+n(\nu)-(\mu'|\nu')}(z/q)^{|\lambda-\nu|}\phi_{\lambda/\nu}(q). \tag{4.2}$$

This identity is reminiscient of the well-known but much simpler [13, Eq. (1); p. 230]

$$\sum_{\substack{\mu\supset\lambda\\ \mu-\lambda \text{ hor. strip}}} z^{|\mu-\lambda|}\psi_{\mu/\lambda}(q) = \frac{1}{1-z}\sum_{\substack{\mu\subset\lambda\\ \lambda-\mu \text{ hor. strip}}} z^{|\lambda-\mu|}\phi_{\lambda/\mu}(q).$$

To make further progress we need a lemma.

Lemma 4.1. *For k a positive integer let $\omega=(\omega_1,\dots,\omega_k)\in\{0,1\}^k$, and let $J=J(\omega)$ be the set of integers j such that $\omega_j=0$ and $\omega_{j+1}=1$. For $\lambda\supset\mu$ partitions let $\theta'=\lambda'-\mu'$ be a skew diagram. Then*

$$\sum_{\substack{\lambda\supset\mu\\ \lambda-\mu \text{ hor. strip}\\ \theta'_i=\omega_i,\ i\in\{1,\dots,k\}}} q^{n(\lambda)}z^{|\lambda-\mu|}\psi_{\lambda/\mu}(q) = \frac{q^{n(\mu)+(\mu'|\omega)}z^{|\omega|}}{1-z}(1-z(1-\omega_k)q^{\mu'_k})\prod_{j\in J}(1-q^{m_j(\mu)}).$$

The restriction $\theta'_i=\omega_i$ for $i\in\{1,\dots,k\}$ in the sum over λ on the left means that the first k parts of λ' are fixed. The remaining parts are free subject only to the condition that $\lambda-\mu$ is a horizontal strip, i.e., that $\lambda'_i-\mu'_i\in\{0,1\}$.

Proof. From (2.12) with $x\to z$ combined with (2.13b) we have

$$\sum_{\substack{\lambda\supset\mu\\ \lambda-\mu \text{ hor. strip}}} q^{n(\lambda)}z^{|\lambda-\mu|}\psi_{\lambda/\mu}(q) = \frac{q^{n(\mu)}}{1-z}. \tag{4.3}$$

We will use this to first prove the lemma for $k=1$. When $k=1$ and $\omega_1=1$ we need to show that

$$\sum_{\substack{\lambda\supset\mu\\ \lambda-\mu \text{ hor. strip}\\ \theta'_1=1}} q^{n(\lambda)}z^{|\lambda-\mu|}\psi_{\lambda/\mu}(q) = \frac{q^{n(\mu)+\mu'_1}z}{1-z}. \tag{4.4}$$

Now let $\bar\lambda$ and $\bar\mu$ be the partitions obtained from λ and μ by removal of the first column of their respective diagrams; $\bar\lambda=(\lambda'_2,\lambda'_3,\dots)'$ and $\bar\mu=(\mu'_2,\mu'_3,\dots)'$. Since $\theta'_1=1$ we have $\psi_{\lambda/\mu}(q)=\psi_{\bar\lambda/\bar\mu}(q)$, $|\lambda-\mu|=|\bar\lambda-\bar\mu|+1$ and $\lambda'_1=\mu'_1+1$ so that

$n(\lambda)=n(\bar{\lambda})+\binom{\mu_1'+1}{2}$. Hence

$$\begin{aligned}\sum_{\substack{\lambda\supset\mu\\ \lambda-\mu\text{ hor. strip}\\ \theta_1'=1}} q^{n(\lambda)}z^{|\lambda-\mu|}\psi_{\lambda/\mu}(q)&=zq^{\binom{\mu_1'+1}{2}}\sum_{\substack{\bar{\lambda}\supset\bar{\mu}\\ \bar{\lambda}-\bar{\mu}\text{ hor. strip}}} q^{n(\bar{\lambda})}z^{|\bar{\lambda}-\bar{\mu}|}\psi_{\bar{\lambda}/\bar{\mu}}(q)\\ &=zq^{\binom{\mu_1'+1}{2}}\times\frac{q^{n(\bar{\mu})}}{1-z}\\ &=\frac{q^{n(\mu)+\mu_1'}z}{1-z},\end{aligned}$$

where the second equality follows from (4.3).

When $k=1$ and $\omega_1=0$ we need to show that

$$\sum_{\substack{\lambda\supset\mu\\ \lambda-\mu\text{ hor. strip}\\ \theta_1'=0}} q^{n(\lambda)}z^{|\lambda-\mu|}\psi_{\lambda/\mu}(q)=\frac{q^{n(\mu)}}{1-z}(1-zq^{\mu_1'}).$$

This time we cannot simply relate $\psi_{\lambda/\mu}(q)$ to $\psi_{\bar{\lambda}/\bar{\mu}}(q)$, but by inclusion-exclusion we have

$$\begin{aligned}&\sum_{\substack{\lambda\supset\mu\\ \lambda-\mu\text{ hor. strip}\\ \theta_1'=0}} q^{n(\lambda)}z^{|\lambda-\mu|}\psi_{\lambda/\mu}(q)\\ &\qquad=\Bigg(\sum_{\substack{\lambda\supset\mu\\ \lambda-\mu\text{ hor. strip}}}-\sum_{\substack{\lambda\supset\mu\\ \lambda-\mu\text{ hor. strip}\\ \theta_1'=1}}\Bigg)q^{n(\lambda)}z^{|\lambda-\mu|}\psi_{\lambda/\mu}(q)\\ &\qquad=\frac{q^{n(\mu)}}{1-z}-\frac{q^{n(\mu)+\mu_1'}z}{1-z}\\ &\qquad=\frac{q^{n(\mu)}}{1-z}(1-zq^{\mu_1'}),\end{aligned}$$

where the second equality follows from (4.3) and (4.4).

The remainder of the proof proceeds by induction on k. Let us assume the lemma to be true for all $1\le k\le K-1$ with $K\ge 2$, and use this to show its truth for $k=K$. To do so we need to again distinguish two cases: $\omega=(\omega_1,\dots,\omega_K)$ with (i) $\omega_1=1$ or $\omega_1=\omega_2=0$ and (ii) $\omega_1=0$ and $\omega_2=1$.

First consider (i) and attach the same meaning to $\bar{\lambda}$ and $\bar{\mu}$ as before. We also set $\bar{\omega}=(\bar{\omega}_1,\dots,\bar{\omega}_{K-1})=(\omega_2,\dots,\omega_K)$ and $\bar{\theta}_i=\theta_{i+1}$. Then, since $\omega_1=1$ or $\omega_1=\omega_2=0$,

$$\psi_{\lambda/\mu}(q)=\psi_{\bar{\lambda}/\bar{\mu}}(q)\quad\text{and}\quad\prod_{j\in J(\omega)}(1-q^{m_j(\mu)})=\prod_{j\in J(\bar{\omega})}(1-q^{m_j(\bar{\mu})}).$$

Moreover, $|\lambda-\mu|=|\bar\lambda-\bar\mu|+\omega_1$ and $\lambda'_1=\mu'_1+\omega_1$ so that $n(\lambda)=n(\bar\lambda)+\binom{\mu'_1+\omega_1}{2}$. Therefore

$$\begin{aligned}
&\sum_{\substack{\lambda\supset\mu\\ \lambda-\mu\text{ hor. strip}\\ \theta'_i=\omega_i,\ i\in\{1,\dots,K\}}} q^{n(\lambda)}z^{|\lambda-\mu|}\psi_{\lambda/\mu}(q)\\
&\quad= z^{\omega_1}q^{\binom{\mu'_1+\omega_1}{2}} \sum_{\substack{\bar\lambda\supset\bar\mu\\ \bar\lambda-\bar\mu\text{ hor. strip}\\ \bar\theta'_i=\bar\omega_i,\ i\in\{1,\dots,K-1\}}} q^{n(\bar\lambda)}z^{|\bar\lambda-\bar\mu|}\psi_{\bar\lambda/\bar\mu}(q)\\
&\quad= z^{\omega_1}q^{\binom{\mu'_1+\omega_1}{2}}\times\frac{q^{n(\bar\mu)+(\bar\mu'|\bar\omega)}z^{|\bar\omega|}}{1-z}\big(1-z(1-\bar\omega_{K-1})q^{\bar\mu'_{K-1}}\big)\prod_{j\in J(\bar\omega)}(1-q^{m_j(\bar\mu)})\\
&\quad=\frac{q^{n(\mu)+(\mu'|\omega)}z^{|\omega|}}{1-z}\big(1-z(1-\omega_K)q^{\mu'_K}\big)\prod_{j\in J(\omega)}(1-q^{m_j(\mu)}).
\end{aligned}$$

In the case of (ii) the proof requires only minor changes, and this time we need

$$\psi_{\lambda/\mu}(q)=\psi_{\bar\lambda/\bar\mu}(q)(1-q^{m_1(\mu)}),$$

and

$$\prod_{j\in J(\omega)}(1-q^{m_j(\mu)})=(1-q^{m_1(\mu)})\prod_{j\in J(\bar\omega)}(1-q^{m_j(\bar\mu)}).$$

(Note that both sides of the first of these equations vanish if $m_1(\mu)=0$ as it should. Indeed, if $\mu'_1=\mu'_2$ there is no partition $\lambda\supset\mu$ such that $\theta'=\lambda'-\mu'=(0,1,\omega_3,\dots,\omega_K)$ since it would require that $\lambda'_1<\lambda_2$.) $\square$

In view of Lemma 4.1 it is natural to rewrite the left side of (4.2) as

$$\mathrm{LHS}(4.2)=\sum_{\omega\in\{0,1\}^{\lambda_1}}\sum_{\substack{\nu\supset\mu\\ \nu-\mu\text{ hor. strip}\\ \theta'_i=\omega_i,\ i\in\{1,\dots,\lambda_1\}}} q^{n(\lambda)+n(\nu)-(\lambda'|\mu')-(\lambda'|\omega)}z^{|\nu-\mu|}\psi_{\nu/\mu}(q),$$

where $\theta=\nu-\mu$, and where we have used that $\theta'_i\in\{0,1\}$ as follows from the fact that $\nu-\mu$ is a horizontal strip.

Now the sum over ν can be performed by application of Lemma 4.1 with $\lambda\to\nu$ and $k\to\lambda_1$, resulting in

$$\begin{aligned}
\mathrm{LHS}(4.2)=\frac{q^{n(\lambda)+n(\mu)-(\lambda'|\mu')}}{1-z}&\sum_{\omega\in\{0,1\}^{\lambda_1}} q^{(\mu'|\omega)-(\lambda'|\omega)}z^{|\omega|}\\
&\times\big(1-z(1-\omega_{\lambda_1})q^{\mu'_{\lambda_1}}\big)\prod_{j\in J}(1-q^{m_j(\mu)})
\end{aligned}$$

with $J=J(\omega)\subset\{1,\dots,\lambda_1-1\}$ the set of integers j such that $\omega_j<\omega_{j+1}$.

For the right-hand side of (4.2) we introduce the notation $\tau_i=\lambda'_i-\nu'_i$, so that the sum over ν can be rewritten as a sum over $\tau\in\{0,1\}^{\lambda_1}$. Using that

$$n(\nu)=\sum_{i=1}^{\lambda_1}\binom{\nu'_i}{2}=\sum_{i=1}^{\lambda_1}\binom{\lambda'_i-\tau_i}{2}=n(\lambda)-(\lambda'|\tau)+|\tau|$$

this yields

$$\mathrm{RHS}(4.2)=\frac{q^{n(\lambda)+n(\mu)-(\lambda'|\mu')}}{1-z}\sum_{\tau\in\{0,1\}^{\lambda_1}} q^{(\mu'|\tau)-(\lambda'|\tau)}z^{|\tau|}\prod_{i\in I}(1-q^{m_i(\lambda)}),$$

with $I=I(\tau)\subset\{1,\dots,\lambda_1\}$ the set of integers i such that $\tau_i>\tau_{i+1}$ (with the convention that $\lambda_1\in I$ if $\tau_{\lambda_1}=1$).

Equating the above two results for the respective sides of (4.2) gives

$$\sum_{\omega\in\{0,1\}^{\lambda_1}} q^{(\mu'|\omega)-(\lambda'|\omega)}z^{|\omega|}(1-z(1-\omega_{\lambda_1})q^{\mu'_{\lambda_1}})\prod_{j\in J}(1-q^{m_j(\mu)}) \\ =\sum_{\tau\in\{0,1\}^{\lambda_1}} q^{(\mu'|\tau)-(\lambda'|\tau)}z^{|\tau|}\prod_{i\in I}(1-q^{m_i(\lambda)}).$$

Using that $m_i(\lambda)=\lambda'_i-\lambda'_{i+1}$ it is not hard to see that this is the

$$k\to\lambda_1,\quad b_{k+1}\to 1,\quad a_i\to zq^{\mu'_i},\quad b_i\to q^{\lambda'_i},\quad i\in\{1,\dots,\lambda_1\}$$

specialization of the more general

$$\sum_{\omega\in\{0,1\}^k}(a/b)^{\omega}(1-(1-\omega_k)a_k/b_{k+1})\prod_{j\in J}(1-a_j/a_{j+1}) \\ =\sum_{\tau\in\{0,1\}^k}(a/b)^{\tau}\prod_{i\in I}(1-b_i/b_{i+1}),$$

where $(a/b)^{\omega}=\prod_{i=1}^k(a_i/b_i)^{\omega_i}$ and $(a/b)^{\tau}=\prod_{i=1}^k(a_i/b_i)^{\tau_i}$. Obviously, the set $J\subset\{1,\dots,k-1\}$ should now be defined as the set of integers j such that $\omega_j<\omega_{j+1}$ and the the set $I\subset\{1,\dots,k\}$ as the set of integers i such that $\tau_i>\tau_{i+1}$ (with the convention that $k\in I$ if $\tau_k=1$).

Next we split both sides into the sum of two terms as follows:

$$\Biggl(\sum_{\omega\in\{0,1\}^k}-(a_k/b_{k+1})\sum_{\substack{\omega\in\{0,1\}^k\\ \omega_k=0}}\Biggr)(a/b)^{\omega}\prod_{j\in J}(1-a_j/a_{j+1}) \\ =\Biggl(\sum_{\tau\in\{0,1\}^k}-(b_k/b_{k+1})\sum_{\substack{\tau\in\{0,1\}^k\\ \tau_k=1}}\Biggr)(a/b)^{\tau}\prod_{\substack{i\in I\\ i\neq k}}(1-b_i/b_{i+1}).$$

Equating the first sum on the left with the first sum on the right yields

$$\sum_{\omega\in\{0,1\}^k}(a/b)^{\omega}\prod_{j\in J}(1-a_j/a_{j+1})=\sum_{\tau\in\{0,1\}^k}(a/b)^{\tau}\prod_{\substack{i\in I\\ i\neq k}}(1-b_i/b_{i+1}).\tag{4.5}$$

If we equate the second sum on the left with the second sum on the right and use that $k-1\not\in J(\omega)$ if $\omega_k=0$ and $k-1\not\in I(\tau)$ if $\tau_k=1$, we obtain $(a_k/b_{k+1})((4.5)_{k\to k-1})$.

Slightly changing our earlier convention we thus need to prove that

$$\sum_{\omega\in\{0,1\}^k}(a/b)^{\omega}\prod_{j\in J}(1-a_j/a_{j+1})=\sum_{\tau\in\{0,1\}^k}(a/b)^{\tau}\prod_{i\in I}(1-b_i/b_{i+1}),\tag{4.6}$$

where from now on $I\subset\{1,\dots,k-1\}$ denotes the set of integers i such that $\tau_i>\tau_{i+1}$ (so that no longer $k\in I$ if $\tau_k=1$). It is not hard to see by multiplying out the respective products that boths sides yield $((1+\sqrt{2})^{k+1}-(1-\sqrt{2})^{k+1})/(2\sqrt{2})$ terms.

To see that the terms on the left and right are in one-to-one correspondence we again resort to induction. First, for $k=1$ it is readily checked that both sides yield $1+a_1/b_1$. For $k=2$ we on the left get

$$\underbrace{1}_{\omega=(0,0)} + \underbrace{(a_1/b_1)}_{\omega=(1,0)} + \underbrace{(a_2/b_2)(1-a_1/a_2)}_{\omega=(0,1)} + \underbrace{(a_1a_2/b_1b_2)}_{\omega=(1,1)}$$

and on the right

$$\underbrace{1}_{\tau=(0,0)} + \underbrace{(a_1/b_1)(1-b_1/b_2)}_{\tau=(1,0)} + \underbrace{(a_2/b_2)}_{\tau=(0,1)} + \underbrace{(a_1a_2/b_1b_2)}_{\tau=(1,1)}$$

which both give

$$1+a_1/b_1+a_2/b_2-a_1/b_2+a_1a_2/b_1b_2.$$

Let us now assume that (4.6) has been shown to be true for $1\le k\le K-1$ with $K\ge 3$ and prove the case $k=K$.

On the left of (4.6) we split the sum over ω according to

$$\sum_{\omega\in\{0,1\}^k} = \sum_{\substack{\omega\in\{0,1\}^k\\ \omega_1=1}} + \sum_{\substack{\omega\in\{0,1\}^k\\ \omega_1=\omega_2=0}} + \sum_{\substack{\omega\in\{0,1\}^k\\ \omega_1=0,\ \omega_2=1}}.$$

Defining $\bar\omega\in\{0,1\}^{k-1}$ and $\bar{\bar\omega}\in\{0,1\}^{k-2}$ by $\bar\omega=(\omega_2,\dots,\omega_k)$ and $\bar{\bar\omega}=(\omega_3,\dots,\omega_k)$, and also setting and $\bar a_j=a_{j+1}$, $\bar b_j=b_{j+1}$, and $\bar{\bar a}_j=a_{j+2}$, $\bar{\bar b}_j=b_{j+2}$, this leads to

$$\begin{aligned}
\mathrm{LHS}(4.6)&=(a_1/b_1)\sum_{\bar\omega\in\{0,1\}^{k-1}}(\bar a/\bar b)^{\bar\omega}\prod_{j\in J(\bar\omega)}(1-\bar a_j/\bar a_{j+1})\\
&\quad+\sum_{\substack{\bar\omega\in\{0,1\}^{k-1}\\ \bar\omega_1=0}}(\bar a/\bar b)^{\bar\omega}\prod_{j\in J(\bar\omega)}(1-\bar a_j/\bar a_{j+1})\\
&\quad+(1-a_1/a_2)\sum_{\substack{\bar\omega\in\{0,1\}^{k-1}\\ \bar\omega_1=1}}(\bar a/\bar b)^{\bar\omega}\prod_{j\in J(\bar\omega)}(1-\bar a_j/\bar a_{j+1})\\
&=(1+a_1/b_1)\sum_{\bar\omega\in\{0,1\}^{k-1}}(\bar a/\bar b)^{\bar\omega}\prod_{j\in J(\bar\omega)}(1-\bar a_j/\bar a_{j+1})\\
&\quad-(a_1/a_2)\sum_{\substack{\bar\omega\in\{0,1\}^{k-1}\\ \bar\omega_1=1}}(\bar a/\bar b)^{\bar\omega}\prod_{j\in J(\bar\omega)}(1-\bar a_j/\bar a_{j+1})\\
&=(1+a_1/b_1)\sum_{\bar\omega\in\{0,1\}^{k-1}}(\bar a/\bar b)^{\bar\omega}\prod_{j\in J(\bar\omega)}(1-\bar a_j/\bar a_{j+1})\\
&\quad-(a_1/b_2)\sum_{\bar{\bar\omega}\in\{0,1\}^{k-2}}(\bar{\bar a}/\bar{\bar b})^{\bar{\bar\omega}}\prod_{j\in J(\bar{\bar\omega})}(1-\bar{\bar a}_j/\bar{\bar a}_{j+1}).
\end{aligned}$$

On the right of (4.6) we split the sum over τ according to

$$\sum_{\tau\in\{0,1\}^k} = \sum_{\substack{\tau\in\{0,1\}^k\\ \tau_1=0}} + \sum_{\substack{\tau\in\{0,1\}^k\\ \tau_1=\tau_2=1}} + \sum_{\substack{\tau\in\{0,1\}^k\\ \tau_1=1,\ \tau_2=0}}.$$

Defining $\bar{\tau} \in \{0,1\}^{k-1}$ and $\bar{\bar{\tau}} \in \{0,1\}^{k-2}$ by $\bar{\tau} = (\tau_2, \dots, \tau_k)$ and $\bar{\bar{\tau}} = (\tau_3, \dots, \tau_k)$, this yields

$$\begin{aligned}
\mathrm{RHS}(4.6) = &\sum_{\bar{\tau}\in\{0,1\}^{k-1}} (\bar{a}/\bar{b})^{\bar{\tau}} \prod_{j\in J(\bar{\tau})} (1 - \bar{b}_j/\bar{b}_{j+1}) \\
&+ (a_1/b_1) \sum_{\substack{\bar{\tau}\in\{0,1\}^{k-1}\\ \bar{\tau}_1=1}} (\bar{a}/\bar{b})^{\bar{\tau}} \prod_{j\in J(\bar{\tau})} (1 - \bar{b}_j/\bar{b}_{j+1}) \\
&+ (a_1/b_1)(1 - b_1/b_2) \sum_{\substack{\bar{\tau}\in\{0,1\}^{k-1}\\ \bar{\tau}_1=0}} (\bar{a}/\bar{b})^{\bar{\tau}} \prod_{j\in J(\bar{\tau})} (1 - \bar{b}_j/\bar{b}_{j+1}) \\
= &(1 + a_1/b_1) \sum_{\bar{\tau}\in\{0,1\}^{k-1}} (\bar{a}/\bar{b})^{\bar{\tau}} \prod_{j\in J(\bar{\tau})} (1 - \bar{b}_j/\bar{b}_{j+1}) \\
&- (a_1/b_2) \sum_{\substack{\bar{\tau}\in\{0,1\}^{k-1}\\ \bar{\tau}_1=0}} (\bar{a}/\bar{b})^{\bar{\tau}} \prod_{j\in J(\bar{\tau})} (1 - \bar{b}_j/\bar{b}_{j+1}) \\
= &(1 + a_1/b_1) \sum_{\bar{\tau}\in\{0,1\}^{k-1}} (\bar{a}/\bar{b})^{\bar{\tau}} \prod_{j\in J(\bar{\tau})} (1 - \bar{b}_j/\bar{b}_{j+1}) \\
&- (a_1/b_2) \sum_{\bar{\bar{\tau}}\in\{0,1\}^{k-2}} (\bar{\bar{a}}/\bar{\bar{b}})^{\bar{\bar{\tau}}} \prod_{j\in J(\bar{\bar{\tau}})} (1 - \bar{\bar{b}}_j/\bar{\bar{b}}_{j+1}).
\end{aligned}$$

By our induction hypothesis this equates with the previous expression for the left-hand side of (4.6), completing the proof.

5. Corollary 3.4 and the A_2 Rogers–Ramanujan identities

For $M = (M_1, \dots, M_n)$ with M_i a non-negative integer, and C the A_n Cartan matrix we define the following bounded analogue of the sum in (3.4):

$$R_M(a_1, \dots, a_n; q) = \sum_{\lambda^{(1)},\dots,\lambda^{(n)}} \frac{q^{\frac{1}{2}\sum_{i,j=1}^n C_{ij}(\lambda^{(i)\prime}|\lambda^{(j)\prime})} \prod_{i=1}^n a_i^{|\lambda^{(i)}|}}{\prod_{i=1}^n (q;q)_{M_i - \ell(\lambda^{(i)})} b_{\lambda^{(i)}}(q)}.$$

By construction $R_M(a_1, \dots, a_n; q)$ satisfies the following invariance property.

Lemma 5.1. *We have*

$$\sum_{r_1=0}^{M_1} \cdots \sum_{r_n=0}^{M_n} \frac{q^{\frac{1}{2}\sum_{i,j=1}^n C_{ij} r_i r_j} \prod_{i=1}^n a_i^{r_i}}{\prod_{i=1}^n (q;q)_{M_i - r_i}} R_r(a_1, \dots, a_n; q) = R_M(a_1, \dots, a_n; q).$$

Proof. Take the definition of R_M given above and replace each of $\lambda^{(1)}, \dots, \lambda^{(n)}$ by its conjugate. Then introduce the non-negative integer r_i and the partition $\mu^{(i)}$ with largest part not exceeding r_i through $\lambda^{(i)} = (r_i, \mu_1^{(i)}, \mu_2^{(i)}, \dots)$. Since $b_{\lambda'}(q) = (q;q)_{r-\mu_1} b_{\mu'}(q)$ for $\lambda = (r, \mu_1, \mu_2, \dots)$ this implies the identity of the lemma after again replacing each of $\mu^{(1)}, \dots, \mu^{(n)}$ by its conjugate. $\square$

Next is the observation that the left-hand side of (3.3) corresponds to $R_{(n,m)}(a, b; q)$. Hence we may reformulate the A_2 instance of Lemma 5.1.

Theorem 5.1. *For M_1 and M_2 non-negative integers*

$$\sum_{r_1=0}^{M_1}\sum_{r_2=0}^{M_2}\frac{a^{r_1}b^{r_2}q^{r_1^2-r_1r_2+r_2^2}}{(q;q)_{M_1-r_1}(q;q)_{M_2-r_2}}\frac{(abq;q)_{r_1+r_2}}{(q,aq,abq;q)_{r_1}(q,bq,abq;q)_{r_2}} = \frac{(abq;q)_{M_1+M_2}}{(q,aq,abq;q)_{M_1}(q,bq,abq;q)_{M_2}}. \tag{5.1}$$

To see how this leads to the A_2 Rogers–Ramanujan identity (1.2) and its higher moduli generalizations, let k_1,k_2,k_3 be integers such that $k_1+k_2+k_3=0$. Making the substitutions

$$\begin{aligned}
&r_1\to r_1-k_1-k_2, && a\to aq^{k_2-k_3}, && M_1\to M_1-k_1-k_2,\\
&r_2\to r_2-k_1, && b\to bq^{k_1-k_2}, && M_2\to M_2-k_1,
\end{aligned}$$

in (5.1), we obtain

$$\begin{aligned}
&\sum_{r_1=0}^{M_1}\sum_{r_2=0}^{M_2}\frac{a^{r_1}b^{r_2}q^{r_1^2-r_1r_2+r_2^2}}{(q;q)_{M_1-r_1}(q;q)_{M_2-r_2}}\\
&\qquad\times\frac{(abq)_{r_1+r_2}}{(q;q)_{r_1+k_3}(aq;q)_{r_1+k_2}(abq;q)_{r_1+k_1}(q;q)_{r_2-k_1}(bq;q)_{r_2-k_2}(abq;q)_{r_2-k_3}}\\
&=\frac{a^{k_1+k_2}b^{k_1}q^{\frac{1}{2}(k_1^2+k_2^2+k_3^2)}(abq)_{M_1+M_2}}{(q;q)_{M_1+k_3}(aq;q)_{M_1+k_2}(abq;q)_{M_1+k_1}(q;q)_{M_2-k_1}(bq;q)_{M_2-k_2}(abq;q)_{M_2-k_3}},
\end{aligned}$$

which is equivalent to the type-II A_2 Bailey lemma of [1, Theorem 4.3]. Taking $a=b=1$ this simplifies to

$$\sum_{r_1=0}^{M_1}\sum_{r_2=0}^{M_2}\frac{q^{r_1^2-r_1r_2+r_2^2}}{(q;q)_{M_1-r_1}(q;q)_{M_2-r_2}(q;q)_{r_1+r_2}^2}\begin{bmatrix}r_1+r_2\\r_1+k_1\end{bmatrix}\begin{bmatrix}r_1+r_2\\r_1+k_2\end{bmatrix}\begin{bmatrix}r_1+r_2\\r_1+k_3\end{bmatrix} = \frac{q^{\frac{1}{2}(k_1^2+k_2^2+k_3^2)}}{(q)_{M_1+M_2}^2}\begin{bmatrix}M_1+M_2\\M_1+k_1\end{bmatrix}\begin{bmatrix}M_1+M_2\\M_1+k_2\end{bmatrix}\begin{bmatrix}M_1+M_2\\M_1+k_3\end{bmatrix}. \tag{5.2}$$

The idea is now to apply this transformation to the A_2 Euler identity [1, Equation (5.15)]

$$\sum_{k_1+k_2+k_3=0}q^{\frac{3}{2}(k_1^2+k_2^2+k_3^2)}\times\sum_{w\in S_3}\epsilon(w)\prod_{i=1}^{3}q^{\frac{1}{2}(3k_i-w_i+i)^2-w_ik_i}\begin{bmatrix}M_1+M_2\\M_1+3k_i-w_i+i\end{bmatrix}=\begin{bmatrix}M_1+M_2\\M_1\end{bmatrix}, \tag{5.3}$$

where $w\in S_3$ is a permutation of $(1,2,3)$ and $\epsilon(w)$ denotes the signature of w.

Replacing M_1,M_2 by r_1,r_2 in (5.3), then multiplying both sides by

$$\frac{q^{r_1^2-r_1r_2+r_2^2}}{(q;q)_{M_1-r_1}(q;q)_{M_2-r_2}(q;q)_{r_1+r_2}^2},$$

and finally summing over r_1 and r_2 using (5.2) (with $k_i \to 3k_i - w_i + i$), yields

$$\sum_{k_1+k_2+k_3=0} q^{\frac{3}{2}(k_1^2+k_2^2+k_3^2)} \sum_{w\in S_3} \epsilon(w) \prod_{i=1}^{3} q^{(3k_i-w_i+i)^2-w_ik_i} \begin{bmatrix} M_1+M_2 \\ M_1+3k_i-w_i+i \end{bmatrix} = \sum_{r_1=0}^{M_1}\sum_{r_2=0}^{M_2} \frac{q^{r_1^2-r_1r_2+r_2^2}(q;q)_{M_1+M_2}^2}{(q;q)_{M_1-r_1}(q;q)_{M_2-r_2}(q;q)_{r_1}(q;q)_{r_2}(q;q)_{r_1+r_2}}. \tag{5.4}$$

Letting M_1 and M_2 tend to infinity, and using the Vandermonde determinant

$$\sum_{w\in S_3} \epsilon(w) \prod_{i=1}^{3} x_i^{i-w_i} = \prod_{1\le i<j\le 3} (1-x_jx_i^{-1})$$

with $x_i \to q^{7k_i+2i}$, gives

$$\frac{1}{(q;q)_\infty^3} \sum_{k_1+k_2+k_3=0} q^{\frac{21}{2}(k_1^2+k_2^2+k_3^2)-k_1-2k_2-3k_3} \times (1-q^{7(k_2-k_1)+2})(1-q^{7(k_3-k_2)+2})(1-q^{7(k_3-k_1)+4}) = \sum_{r_1,r_2=0}^{\infty} \frac{q^{r_1^2-r_1r_2+r_2^2}}{(q;q)_{r_1}(q;q)_{r_2}(q;q)_{r_1+r_2}}.$$

Finally, by the A_2 Macdonald identity [11]

$$\sum_{k_1+k_2+k_3=0} \prod_{i=1}^{3} x_i^{3k_i} q^{\frac{3}{2}k_i^2-ik_i} \prod_{1\le i<j\le 3} (1-x_jx_i^{-1}q^{k_j-k_i}) = (q;q)_\infty^2 \prod_{1\le i<j\le 3} (x_i^{-1}x_j, qx_ix_j^{-1};q)_\infty$$

with $q \to q^7$ and $x_i \to q^{2i}$ this becomes

$$\sum_{r_1,r_2=0}^{\infty} \frac{q^{r_1^2-r_1r_2+r_2^2}}{(q;q)_{r_1}(q;q)_{r_2}(q;q)_{r_1+r_2}} = \frac{(q^2,q^2,q^3,q^4,q^5,q^5,q^7,q^7;q^7)_\infty}{(q;q)_\infty^3}.$$

This result is easily recognized as the A_2 Rogers–Ramanujan identity (1.2).

The identity (5.4) can be further iterated using (5.2). Doing so and repeating the above calculations (requiring the Vandermonde determinant with $x_i \to q^{(3n+1)k_i+ni}$ and the Macdonald identity with $q \to q^{3n+1}$ and $x_i \to q^{ni}$) yields the following A_2 Rogers–Ramanujan-type identity for modulus $3n+1$ [1, Theorem 5.1; $i=k$]:

$$\sum_{\substack{\lambda,\mu \\ \ell(\lambda),\ell(\mu)\le n-1}} \frac{q^{(\lambda|\lambda)+(\mu|\mu)-(\lambda|\mu)}}{b_{\lambda'}(q)b_{\mu'}(q)(q;q)_{\lambda_{n-1}+\mu_{n-1}}} = \frac{(q^n,q^n,q^{n+1},q^{2n},q^{2n+1},q^{2n+1},q^{3n+1},q^{3n+1};q^{3n+1})_\infty}{(q;q)_\infty^3}.$$

In the large n limit ones recovers the A_2 case of Hua's identity (3.4) with $a_1=a_2=1$.

To obtain identities corresponding to the modulus $3n-1$ we replace $q \to 1/q$ in (5.3) to get

$$\sum_{k_1+k_2+k_3=0} q^{-\frac{3}{2}(k_1^2+k_2^2+k_3^2)} \sum_{w\in S_3} \epsilon(w) \prod_{i=1}^3 q^{\frac{1}{2}(3k_i-w_i+i)^2+w_ik_i} \begin{bmatrix} M_1+M_2 \\ M_1+3k_i-w_i+i \end{bmatrix} \\ = q^{2M_1M_2} \begin{bmatrix} M_1+M_2 \\ M_1 \end{bmatrix}.$$

Iterating this using (5.2) and then taking the limit of large M_1 and M_2 yields [1, Theorem 5.3; $i=k$]

$$\sum_{\substack{\lambda,\mu \\ \ell(\lambda),\ell(\mu)\le n-1}} \frac{q^{(\lambda|\lambda)+(\mu|\mu)-(\lambda|\mu)+2\lambda_{n-1}\mu_{n-1}}}{b_{\lambda'}(q)b_{\mu'}(q)(q;q)_{\lambda_{n-1}+\mu_{n-1}}} \\ = \frac{(q^{n-1},q^n,q^n,q^{2n-1},q^{2n-1},q^{2n},q^{3n-1},q^{3n-1};q^{3n-1})_\infty}{(q;q)_\infty^3}.$$

Finally, the modulus $3n$ arises by iterating [5, Equation (6.18)]

$$\sum_{k_1+k_2+k_3=0} \sum_{w\in S_3} \epsilon(w) \prod_{i=1}^3 q^{\frac{1}{2}(3k_i-w_i+i)^2} \begin{bmatrix} M_1+M_2 \\ M_1+3k_i-w_i+i \end{bmatrix} = \begin{bmatrix} M_1+M_2 \\ M_1 \end{bmatrix}_{q^3}.$$

A repeat of the earlier calculation then gives [1, Theorem 5.4; $i=k$]

$$\sum_{\substack{\lambda,\mu \\ \ell(\lambda),\ell(\mu)\le n-1}} \frac{q^{(\lambda|\lambda)+(\mu|\mu)-(\lambda|\mu)}(q;q)_{\lambda_{n-1}}(q;q)_{\mu_{n-1}}}{b_{\lambda'}(q)b_{\mu'}(q)(q;q)^2_{\lambda_{n-1}+\mu_{n-1}}} \begin{bmatrix} \lambda_{n-1}+\mu_{n-1} \\ \lambda_{n-1} \end{bmatrix}_{q^3} \\ = \frac{(q^n,q^n,q^n,q^{2n},q^{2n},q^{2n},q^{3n},q^{3n};q^{3n})_\infty}{(q;q)_\infty^3}.$$

6. Some open problems

In this final section we pose several open problems related to the results of this paper.

6.1. **Macdonald's symmetric function.** The Hall–Littlewood functions $P_\lambda(x;t)$ are special cases of Macdonald's celebrated symmetric functions $P_\lambda(x;q,t)$, obtained from the latter by taking $q=0$. An obvious question is whether Theorem 1.1 can be generalized to the Macdonald case.

From the Cauchy identity [13, Sec. VI, Eqn. (4.13)]

$$\sum_\lambda P_\lambda(x;q,t)Q_\lambda(y;q,t) = \prod_{i,j\ge 1} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}$$

(see [13] for definitions related to Macdonald's symmetric function) and the specialization

$$Q_\lambda(1,t,t^2,\dots;q,t) = \frac{t^{n(\lambda)}}{c'_\lambda(q,t)}$$

we have

$$\sum_\lambda \frac{t^{n(\lambda)}P_\lambda(x;q,t)}{c'_\lambda(q,t)} = \prod_{i\ge 1} \frac{1}{(x_i;q)_\infty}. \tag{6.1}$$

Here c'_λ is the generalized hook-polynomial

$$c'_\lambda(q,t)=\prod_{s\in\lambda}(1-q^{a(s)+1}t^{\ell(s)})$$

with $a(s)=\lambda_i-j$ and $\ell(s)=\lambda'_j-i$ the arm-length and leg-length of the square $s=(i,j)$ of λ. Note that $c'_\lambda(0,t)=1$.

In view of the above we pose the problem of finding a (q,t)-analogue of Theorem 1.1 which simplifies to (6.1) when $y_i=0$ for all $i\geq 1$ and to (1.3) (with $q\to t$) when $q=0$.

Alternatively we may ask for a (q,t)-analogue of (4.2). From (6.1) and standard properties of Macdonald polynomials it follows that

$$\sum_{\substack{\nu\supset\mu\\ \nu-\mu \text{ hor. strip}}}\frac{t^{n(\nu)}z^{|\nu-\mu|}\psi_{\nu/\mu}(q,t)}{c'_\nu(q,t)}=\frac{1}{(z;q)_\infty}\frac{t^{n(\mu)}}{c'_\mu(q,t)}. \tag{6.2}$$

Here $\psi_{\lambda/\mu}(q,t)$ is generalization of $\psi_{\lambda/\mu}(t)$ (such that $\psi_{\lambda/\mu}(0,t)=\psi_{\lambda/\mu}(t)$) given by

$$\psi_{\lambda/\mu}(q,t)=\prod_{s\in\mu}{}'\,\frac{b_\mu(s)}{b_\lambda(s)},$$

where the product is over all squares $s=(i,j)$ of μ such that $\theta_i>0$ and $\theta'_j=0$ for $\theta=\lambda-\mu$. Moreover

$$b_\lambda(s)=\frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}$$

for $s\in\lambda$.

Hence a (q,t)-version of (4.2) should reduce to (6.2) when $\lambda=0$ and to (4.2) (with $q\to t$) when $q=0$. Moreover, its right-hand side should involve the rational function

$$\phi_{\lambda/\mu}(q,t)=\prod_{s\in\lambda}{}'\,\frac{b_\mu(s)}{b_\lambda(s)},$$

where the product is over all squares $s=(i,j)$ of λ such that $\theta'_i>0$ with $\theta=\lambda-\mu$ (and $b_\mu(s)=1$ if $s\notin\mu$).

6.2. **The A_n version of Theorem 1.1.** In the introduction we already mentioned the problem of evaluating the A_n sum (1.4). For $n>2$ this sum does not equate to an infinite product and a possible scenario is that for general n the right-hand takes the form of an n by n determinant with infinite-product entries.

A specialized case of the sum (1.4) does however exhibit a simple closed form evaluation, and the following extension of Theorem 1.1 holds. Let $x^{(1)}=x=(x_1,x_2,\dots)$, $x^{(n)}=y=(y_1,y_2,\dots)$ and $x_j^{(i)}=a_iq^j$ for $2\leq i\leq n-1$ and $j\geq 1$. Also, let Δ'_+ be the set (of cardinality $\binom{n-1}{2}$) of positive roots of A_n not containing the simple roots α_1 and α_n, i.e., the set of roots of the form $\alpha_i+\alpha_{i+1}+\cdots+\alpha_j$

with $2 \le i \le j \le n-1$. Then

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} \prod_{i=1}^{n} q^{n(\lambda^{(i)})-(\lambda^{(i)\prime}|\lambda^{(i+1)\prime})} P_{\lambda^{(i)}}(x^{(i)};q) \tag{6.3}$$

$$= \prod_{\alpha\in\Delta'_+} \frac{1}{(a^\alpha q;q)_\infty} \prod_{i\ge 1}\prod_{j=1}^{n-1} \frac{1}{(1-a_2\cdots a_j x_i)(1-a_{n-j+1}\cdots a_{n-1}y_i)} \times \prod_{i,j\ge 1} \frac{1-a_2\cdots a_{n-1}x_iy_j}{1-q^{-1}a_2\cdots a_{n-1}x_iy_j}.$$

When $x_i=a_1q^i$ and $y_i=a_nq^i$ for $i\ge 1$ this yields (3.4).

Similarly, we have an isolated result for A_3 of the form

$$\sum_{\lambda,\mu,\nu} q^{n(\lambda)+n(\mu)+n(\nu)-(\lambda'|\mu')-(\mu'|\nu')} \times P_\lambda(aq,aq^2,\dots;q)P_\mu(x;q)P_\nu(bq,bq^2,\dots;q) \tag{6.4}$$

$$= \frac{1}{(aq,bq;q)_\infty} \prod_{i\ge 1} \frac{(1-abx_i^2)}{(1-x_i)(1-ax_i)(1-bx_i)(1-abx_i)} \prod_{i<j} \frac{1-abx_ix_j}{1-q^{-1}abx_ix_j}.$$

6.3. **Bounds on Theorem 1.1.** The way we have applied (1.3) to obtain the A_2 Rogers–Ramanujan identity (1.2) is rather different from Stembridge's Hall–Littlewood approach to the classical Rogers–Ramanujan identities [16]. Specifically, Stembridge took [13, p. 231]

$$\sum_{\lambda} P_{2\lambda}(x;q) = \prod_{i=1}^{n} \frac{1}{1-x_i^2} \prod_{1\le i<j\le n} \frac{1-qx_ix_j}{1-x_ix_j} =: \Psi(x;q)$$

for $x=(x_1,\dots,x_n)$, and generalized this to

$$\sum_{k=0}^{\infty} u^k \sum_{\substack{\lambda \\ \lambda_1\le k}} P_{2\lambda}(x;q) = \sum_{\epsilon\in\{-1,1\}^n} \frac{\Psi(x^\epsilon;q)}{1-ux^{1-\epsilon}}, \tag{6.5}$$

where $f(x^\epsilon)=f(x_1^{\epsilon_1},\dots,x_n^{\epsilon_n})$ and $x^{1-\epsilon}=x_1^{1-\epsilon_1}\cdots x_n^{1-\epsilon_n}$. By specializing $x_i = z^{1/2}q^{i-1}$ for all $1\le i\le n$ this yields

$$\sum_{\substack{\lambda \\ \lambda_1\le k}} \frac{z^{|\lambda|}q^{2n(\lambda)}(q;q)_n}{(q;q)_{n-\ell(\lambda)}b_\lambda(q)} = \sum_{r=0}^{n} (-1)^r(1-zq^{2r-1})z^{(k+1)r}q^{(2k+3)\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix} \frac{(z/q;q)_r}{(z/q;q)_{n+r+1}}. \tag{6.6}$$

Letting n tend to infinity and taking $k=1$ and $z=q$ or $z=q^2$ gives the Rogers–Ramanujan identities (1.1) by an appeal to the Jacobi triple-product identity to transform the sum on the right into a product.

An obvious question is whether the identity (1.3) also admits a version in which the partitions λ and μ are summed restricted to $\lambda_1\le k_1$ and $\mu\le k_2$, and if so, whether such an identity would yield further A_2 q-series identities upon specialization. At present we have been unable to answer these questions. It is to be noted, however, that since (2.8) is the special case of (1.3) — obtained by setting all y_i equal to zero — a bounded form of (2.8) would be a precursor to a bounded form of (1.3).

Defining

$$\Phi(x;q)=\prod_{i=1}^{n}\frac{1}{1-x_i}\prod_{1\le i<j\le n}\frac{1-qx_ix_j}{1-x_ix_j}$$

Macdonald has shown that [13, p. 231–234]

$$\sum_{\lambda} P_\lambda(x;q)=\Phi(x;q) \tag{6.7}$$

and

$$\sum_{k=0}^{\infty}u^k\sum_{\substack{\lambda\\ \lambda_1\le k}}P_\lambda(x;q)=\sum_{\epsilon\in\{-1,1\}^n}\frac{\Phi(x^\epsilon;q)}{1-ux^{(1-\epsilon)/2}}. \tag{6.8}$$

With the above notation, (2.8) takes a form rather similar to (6.7);

$$\sum_{\lambda} q^{n(\lambda)}P_\lambda(x;q)=\Phi(x;1).$$

But more can be done as the following bounded analogue of (2.8) holds.

Theorem 6.1. *Let $[n]=\{1,\dots,n\}$. For I a subset of $[n]$ let $|I|$ be its cardinality and $J=[n]-I$ its complement. Then*

$$\begin{aligned}\sum_{k=0}^{\infty}u^k&\sum_{\substack{\lambda\\ \lambda_1\le k}}q^{n(\lambda)}P_\lambda(x;q)\\ &=\sum_{I\subset[n]}\frac{1}{1-uq^{\binom{|I|}{2}}\prod_{i\in I}x_i}\prod_{i\in I}\frac{1}{1-x_i^{-1}q^{1-|I|}}\prod_{j\in J}\frac{1}{1-x_jq^{|I|}}\prod_{\substack{i\in I\\ j\in J}}\frac{x_i-qx_j}{x_i-x_j}.\end{aligned} \tag{6.9}$$

If we specialize $x_i=zq^{i-1}$ — but do not yet use (2.5) — and equate coefficients of u^k, this leads to

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\le k}}&q^{n(\lambda)}z^{|\lambda|}P_\lambda(1,q,\dots,q^{n-1};q)\\ &=\sum_{r=0}^{n}(-1)^r(1-zq^{2r-1})z^{(k+1)r}q^{(2k+3)\binom{r}{2}}\begin{bmatrix}n\\ r\end{bmatrix}\frac{(z/q;q)_r}{(z/q;q)_{n+r+1}}.\end{aligned} \tag{6.10}$$

This is a finite-n analogue of [2, Theorem 2] of Fulman. (To get Fulman's theorem take $z=q$ or q^2, replace $k\to k-1$, $q\to q^{-1}$ and let n tend to infinity. The Jacobi triple-product identity does the rest). However, the reader should also note that the above right-hand side coincides with the right-hand side of Stembridge's (6.6). Indeed, using the specialization formula (2.5), (6.10) is readily seen to be equivalent to (6.6) — the reason for this coincidence being that $P_{2\lambda}(z^{1/2},z^{1/2}q,\dots,z^{1/2}q^{n-1};q)=q^{n(\lambda)}P_\lambda(z,zq,\dots,zq^{n-1};q)$.

Proof of (6.10). The left-hand side simply follows by extracting the coefficient of u^k in (6.9) and by making the required specialization.

To get to the claimed right-hand side we note that after specialization the term

$$\prod_{\substack{i\in I\\ j\in J}} \frac{x_i - qx_j}{x_i - x_j}$$

will vanish if there is an $i \in I$ and a $j \in J$ such that $i - j = 1$. Hence the only I that will contribute to the sum are the sets $\{1, \dots, r\}$ with $0 \le r \le n$, resulting in

$$\begin{aligned}\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} &\frac{1}{(z^{-1}q^{2-2r};q)_r (zq^{2r};q)_{n-r}} \frac{1}{1 - uz^r q^{2\binom{r}{2}}} \\ &= \sum_{r=0}^{n} (-1)^r (1 - zq^{2r-1}) z^r q^{3\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix} \frac{(z/q;q)_r}{(z/q;q)_{n+r+1}} \frac{1}{1 - uz^r q^{2\binom{r}{2}}}.\end{aligned}$$

The observation that the coefficient of u^k of this series is given by the right-hand side of (6.10) completes the proof. □

Proof of Theorem 6.1. The proof proceeds along the lines of Macdonald's partial fraction proof of (6.8) [13] and Stembridge's proof of (6.5) (see also [8–10]).

For any subset E of $X = \{x_1, \dots, x_n\}$, let $p(E)$ denote the product of the elements of E. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be of the form $(\mu_1^{r_1}, \dots, \mu_k^{r_k})$, with $\mu_1 > \mu_2 > \dots > \mu_k \ge 0$ and $r_1, \dots, r_k > 0$ such that $\sum_i \mu_i = n$. Then the defining expression (2.1) of the Hall–Littlewood polynomials can be rewritten as

$$P_\lambda(x;q) = \sum_f \prod_{i=1}^{k} p(f^{-1}(i))^{\mu_i} \prod_{f(x_i)<f(x_j)} \frac{x_i - qx_j}{x_i - x_j}, \tag{6.11}$$

where the sum is over all surjections $f : X \to \{1, \dots, k\}$ such that $|f^{-1}(i)| = r_i$. Each such surjection f corresponds to a filtration

$$\mathscr{F} : \emptyset = F_0 \subset F_1 \subset \dots \subset F_k = X,$$

according to the rule that $x \in F_i$ iff $f(x) \le i$, and each filtration of length k such that $|F_i - F_{i-1}| = r_i$ corresponds to a surjection $f : X \to \{1, \dots, k\}$ such that $|f^{-1}(i)| = r_i$. Hence (6.11) can be put as

$$P_\lambda(x;q) = \sum_{\mathscr{F}} \pi_{\mathscr{F}}(X) \prod_{i=1}^{k} p(F_i - F_{i-1})^{\mu_i} = \sum_{\mathscr{F}} \pi_{\mathscr{F}}(X) \prod_{i=1}^{k} p(F_i)^{\mu_i - \mu_{i+1}}$$

Here $\mu_{k+1} := 0$, the sum over $\mathscr{F}$ is a sum over all filtrations of length k such that $|F_i - F_{i-1}| = r_i$, and

$$\pi_{\mathscr{F}}(X) = \prod_{f(x_i)<f(x_j)} \frac{x_i - qx_j}{x_i - x_j},$$

with f the surjection corresponding to $\mathscr{F}$.

Now given λ, the statistic $n(\lambda)$ may be expressed in terms of the above defined quantities as

$$n(\lambda) = \sum_{i=1}^{k} (\mu_i - \mu_{i+1}) \binom{|F_i|}{2}.$$

Hence, denoting the sum on the left of (6.9) by $S(u)$,

$$S(u) = \sum_{\mathscr{F}} \pi_{\mathscr{F}}(X) \sum u^k \prod_{i=1}^{k} \Bigl(q^{\binom{|F_i|}{2}} p(F_i) \Bigr)^{\mu_i - \mu_{i+1}},$$

where the sum over $\mathscr{F}$ is a sum over filtrations of arbitrary length k and where the inner sum is a sum over integers $k', \mu_1, \ldots, \mu_k$ such that $k' \geq \mu_1 > \cdots > \mu_k \geq 0$. Introducing the new variables $\nu_0 = k' - \mu_1$ and $\nu_i = \mu_i - \mu_{i+1}$ for $i \in \{1, \ldots, k\}$, so that $\nu_0, \nu_k \geq 0$ and all other $\nu_i > 0$, the inner sum can readily be carried out yielding

$$S(u) = \frac{1}{1-u} \sum_{\mathscr{F}} \pi_{\mathscr{F}}(X) A_{\mathscr{F}}(X, u), \tag{6.12}$$

with

$$A_{\mathscr{F}}(X, u) = \frac{1}{1 - p(X) q^{\binom{n}{2}} u} \prod_{i=1}^{k-1} \frac{p(F_i) q^{\binom{|F_i|}{2}} u}{1 - p(F_i) q^{\binom{|F_i|}{2}} u}. \tag{6.13}$$

In the remainder it will be convenient not to work with the filtrations $\mathscr{F}$ but with the filtrations $\mathscr{G}$

$$\mathscr{G} : \emptyset = G_0 \subset G_1 \subset \cdots \subset G_k = [n],$$

where $\mathscr{G}$ is determined from $\mathscr{F}$ by $G_i = \{j | x_j \in F_i\}$. Instead of $\pi_{\mathscr{F}}$ and $A_{\mathscr{F}}$ we will write $\pi_{\mathscr{G}}$ and $A_{\mathscr{G}}$ and so on.

From (6.12) and (6.13) it follows that the following partial fraction expansion for $S(u)$ must hold:

$$S(u) = \sum_{I \subset [n]} \frac{a_I}{1 - x_I q^{\binom{|I|}{2}} u},$$

where x_I stands for $\prod_{i \in I} x_i$. After comparing this with (6.9), the remaining task is to show that

$$\begin{aligned} a_I &= \lim_{u \to x_I^{-1} q^{-\binom{|I|}{2}}} (1 - x_I q^{\binom{|I|}{2}} u) S(u) \\ &= \lim_{u \to x_I^{-1} q^{-\binom{|I|}{2}}} \frac{1 - x_I q^{\binom{|I|}{2}} u}{1-u} \sum_{\mathscr{G}} \pi_{\mathscr{G}}(X) A_{\mathscr{G}}(X, u) \end{aligned} \tag{6.14}$$

is given by

$$a_I = \prod_{i \in I} \frac{1}{1 - x_i^{-1} q^{1-|I|}} \prod_{j \in J} \frac{1}{1 - x_j q^{|I|}} \prod_{\substack{i \in I \\ j \in J}} \frac{x_i - q x_j}{x_i - x_j}. \tag{6.15}$$

Since

$$S(u) = \sum_{\lambda} q^{n(\lambda)} P_\lambda(x; q) \sum_{k=\lambda_1}^{\infty} u^k = \frac{1}{1-u} \sum_{\lambda} u^{\lambda_1} q^{n(\lambda)} P_\lambda(x; q),$$

we have

$$a_\emptyset = \lim_{u \to 1} (1-u) S(u) = \sum_{\mathscr{G}} \pi_{\mathscr{G}}(X) A_{\mathscr{G}}(X, 1) = \Phi(X) = \Phi(x; 1), \tag{6.16}$$

where, for later reference, we have introduced

$$\Phi(Y) = \prod_{y \in Y} \frac{1}{1-y}$$

for arbitrary sets Y. Now let us use (6.16) to compute a_I for general sets I.

The only filtrations that contribute to the sum in (6.14) are those $\mathscr{G}$ that contain a G_r (with $0 \le r \le k$) such that $G_r = I$. Any such $\mathscr{G}$ may be decomposed into two filtrations $\mathscr{G}_1$ and $\mathscr{G}_2$ of length r and $k-r$ by

$$\mathscr{G}_1 : \emptyset = I - G_r \subset I - G_{r-1} \subset \cdots \subset I - G_1 \subset I - G_0 = I$$

and

$$\mathscr{G}_2 : \emptyset = G_r - I \subset G_{r+1} - I \subset \cdots \subset G_{k-1} - I \subset G_k - I = [n] - I = J,$$

and given $\mathscr{G}_1$ and $\mathscr{G}_2$ we can clearly reconstruct $\mathscr{G}$.

For fixed I and $J = [n] - I$ let $\bar{X}_I, X_J \subset X$ be the sets $\{x_i^{-1} q^{1-|I|} | i \in I\}$ and $\{x_j q^{|I|} | j \in J\}$, respectively. Then it is not hard to verify that

$$\pi_{\mathscr{G}}(X) = \pi_{\mathscr{G}_1}(\bar{X}_I)\pi_{\mathscr{G}_2}(X_J) \prod_{\substack{i\in I\\ j \in J}} \frac{x_i - q x_j}{x_i - x_j}.$$

Here we should perhaps remark that due to the homogeneity of the terms making up $\pi_{\mathscr{G}}(X)$, the factors $q^{1-|I|}$ and $q^{|I|}$ occurring in the definitions of $\bar{X}_I$ and X_J simply cancel out. Similarly, it follows that

$$\frac{1 - x_I q^{\binom{|I|}{2}} u}{1-u} A_{\mathscr{G}}(X,u) = x_I q^{\binom{|I|}{2}} u \times A_{\mathscr{G}_1}(\bar{X}_I, x_I q^{\binom{|I|}{2}} u) \times A_{\mathscr{G}_2}(X_J, x_I q^{\binom{|I|}{2}} u).$$

Substituting the above two decompositions in (6.14) and taking the limit yields

$$\begin{aligned} a_I &= \sum_{\mathscr{G}_1} \pi_{\mathscr{G}_1}(\bar{X}_I) A_{\mathscr{G}_1}(\bar{X}_I, 1) \sum_{\mathscr{G}_2} \pi_{\mathscr{G}_2}(X_J) A_{\mathscr{G}_2}(X_J, 1) \prod_{\substack{i\in I\\ j \in J}} \frac{x_i - q x_j}{x_i - x_j} \\ &= \Phi(\bar{X}_I)\Phi(X_J) \prod_{\substack{i\in I\\ j \in J}} \frac{x_i - q x_j}{x_i - x_j} \\ &= \prod_{i \in I} \frac{1}{1 - x_i^{-1} q^{1-|I|}} \prod_{j \in I} \frac{1}{1 - x_j^{-1} q^{|J|}} \prod_{\substack{i\in I\\ j \in J}} \frac{x_i - q x_j}{x_i - x_j} \end{aligned}$$

in accordance with (6.15). □

Unfortunately, Macdonald's the partial fraction method fails to provide an expression for

$$\sum_{k_1,k_2=0}^{\infty} u^{k_1} v^{k_2} \sum_{\substack{\lambda,\mu \\ \lambda_1 \le k_1,\ \mu_1 \le k_2}} q^{n(\lambda)+n(\mu)-(\lambda'|\mu')} P_\lambda(x;q) P_\mu(y;q)$$

when not all y_i (or x_i) are equal to zero.

In fact, even for the special 1-dimensional subcase of Corollary 3.3 no simple closed form expression is apparent for

$$\sum_{k=0}^{\infty} u^k \sum_{\substack{\lambda \\ \lambda_1 \le k}} q^{n(\lambda) - \sum_{l=1}^{j} \lambda'_l} P_\lambda(x;q).$$

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