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New multiple $_6\psi_6$ summation formulas and related conjectures

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Dedicated to George Andrews on the occasion of his 70th birthday

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Abstract Three new summation formulas for $_6\psi_6$ bilateral basic hypergeometric series attached to root systems are presented. Remarkably, two of these formulae, labelled by the A_{2n-1} and A_{2n} root systems, can be reduced to multiple $_6\phi_5$ sums generalising the well-known van Diejen sum. This latter sum serves as the weight-function normalisation for the BC_n q-Racah polynomials of van Diejen and Stokman. Two $_8\phi_7$-level extensions of the multiple $_6\phi_5$ sums, as well as their elliptic analogues, are conjectured. This opens up the prospect of constructing novel A-type extensions of the Koornwinder–Macdonald theory.

Keywords Basic hypergeometric series · Elliptic hypergeometric series · Root systems · Orthogonal polynomials

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1 Introduction

Bailey's $_6\psi_6$ summation formula [5]

$$\sum_{k=-\infty}^{\infty} \frac{1-aq^{2k}}{1-a} \frac{(b,c,d,e)_k}{(aq/b,aq/c,aq/d,aq/e)_k} \left(\frac{a^2q}{bcde}\right)^k = \frac{(aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de,aq,q/a,q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e,a^2q/bcde)_\infty}, \tag{1.1}$$

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where

$$|q|<1 \quad \text{and} \quad \left|\frac{a^2q}{bcde}\right|<1,$$

is one of the most impressive identities in the theory of bilateral basic hypergeometric series. (Readers unfamiliar with q-series notation are referred to the next section.) Throughout his long and distinguished career George Andrews has been a master in extracting combinatorial information from identities such as (1.1). In a paper on ${}_6\psi_6$ summations dedicated to George it seems appropriate to highlight one of his applications of (1.1) pertaining to two of his favourite subjects, partition theory and the mathematical discoveries of Ramanujan.

Let $p(n)$ be the number of integer partitions of n. Then one of Ramanujan's celebrated congruences states that

$$p(5n+4) \equiv 0 \pmod{5}.$$

For example, $p(4)=5$, $p(9)=30$, $p(14)=135$ and so on. Ramanujan proved his congruence by establishing the beautiful identity

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\,\frac{(q^5;q^5)_\infty^5}{(q)_\infty^6}. \tag{1.2}$$

As noted by Andrews in his SIAM review *Applications of basic hypergeometric functions* [2], identity (1.2) readily follows from (1.1) (for some of the details, see also [6]). Indeed, after replacing

$$(a,b,c,d,e,q) \mapsto \bigl(q^4,q,q,q^3,q^3,q^5\bigr)$$

in Bailey's ${}_6\psi_6$ sum and carrying out some standard manipulations, one obtains

$$\sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{(1-q^n)^2} = q\,\frac{(q^5;q^5)_\infty^5}{(q)_\infty} \tag{1.3a}$$

$$= \bigl(q^5;q^5\bigr)_\infty^5 \sum_{n=0}^{\infty} p(n)q^{n+1}, \tag{1.3b}$$

where $(\frac{n}{p})$, for p an odd prime, is the Legendre symbol [17]. For m a positive integer, let the Hecke operator U_m act on formal power series in q as

$$U_m \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_{nm} q^n.$$

Since $(\frac{n}{p})=0$ when n is a multiple of p,

$$U_p \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \frac{q^n}{(1-q^n)^2} = U_p \sum_{n,m=1}^{\infty} \left(\frac{n}{p}\right) m q^{nm}$$
$$= p \sum_{n,m=1}^{\infty} \left(\frac{n}{p}\right) m q^{nm} = p \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \frac{q^n}{(1-q^n)^2}.$$

Acting with U_5 on (1.3b) thus gives

$$(q)_\infty^5 \sum_{n=0}^{\infty} p(5n+4) q^{n+1} = 5 \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{(1-q^n)^2}.$$

By (1.3a) this proves (1.2).

Another important consequence of the ${}_6\psi_6$ sum pointed out by Andrews in his SIAM review is the Jacobi triple product identity. Specifically, taking the limit $b, c, d, e \to \infty$ in (1.1) and replacing a by z yields[1]

$$\sum_{k=-\infty}^{\infty} (-z)^k q^{\binom{k}{2}} = (z, q/z, q)_\infty, \quad z \neq 0. \tag{1.4}$$

In the landmark paper [20], Macdonald generalised the triple product identity to all (reduced irreducible) affine root systems. For example, for the affine root system of type A_{n-1}, he proved that[2]

$$\sum_{\substack{\lambda\in\mathbb{Z}^n \\ |\lambda|=0}} \prod_{i=1}^{n} z_i^{n\lambda_i} q^{n\binom{\lambda_i}{2}+i\lambda_i} \prod_{1\le i<j\le n} \left(1 - q^{\lambda_i-\lambda_j} z_i/z_j\right)$$
$$= (q)_\infty^{n-1} \prod_{1\le i<j\le n} (z_i/z_j, qz_j/z_i)_\infty \tag{1.5}$$

for $z_1, \dots, z_n \neq 0$.

In view of the above limit reducing (1.1) to (1.4), it is a natural question to ask for a generalisation of (1.5) and other Macdonald identities to multiple ${}_6\psi_6$ Bailey sums. To a large extent this question was settled by Gustafson [10–12], who proved four ${}_6\psi_6$ sums corresponding to the affine root systems of type A_{n-1}, C_n, $\mathrm{B}_n^\vee$ and G_2. By taking various limits, these four identities yield *all* of the infinite families of Macdonald identities, corresponding to A_{n-1}, B_n, $\mathrm{B}_n^\vee$ C_n, $\mathrm{C}_n^\vee$, D_n and BC_n, as well as the Macdonald identity for G_2. For example, if for $z \in (\mathbb{C}^*)^n$ and $\lambda \in \mathbb{Z}^n$,

$$\Delta(z) = \prod_{1\le i<j\le n} (z_i - z_j) \quad \text{and} \quad \Delta\big(zq^\lambda\big) = \prod_{1\le i<j\le n} \big(z_i q^{\lambda_i} - z_j q^{\lambda_j}\big), \tag{1.6}$$

[1] Incidentally, Bailey himself obtained the triple product identity by specialising $b = a^{1/2}$, $c = -a^{1/2}$, then taking the $d, e \to \infty$ limit, and finally replacing $(a, q) \mapsto (zq^{-1/2}, q^{1/2})$.

[2] Equation (1.5) is a particular form of the A_{n-1} Macdonald identity first stated in [22].

then Gustafson's A_{n-1} ${}_6\psi_6$ sum [10] reads

$$\sum_{\substack{\lambda\in\mathbb{Z}^n\\|\lambda|=0}} \frac{\Delta(zq^\lambda)}{\Delta(z)} \prod_{i,j=1}^{n} \frac{(z_j/b_i)_{\lambda_j}}{(qa_iz_j)_{\lambda_j}} = \frac{(qAZ,qB/Z)_\infty}{(q,qAB)_\infty} \prod_{i,j=1}^{n} \frac{(qa_ib_j,qz_i/z_j)_\infty}{(qa_iz_j,qb_i/z_j)_\infty}, \tag{1.7}$$

where $A=a_1\cdots a_n$, $B=b_1\cdots b_n$, $Z=z_1\cdots z_n$,

$$|q|<1 \quad \text{and} \quad |qAB|<1.$$

Letting $a_i,b_i\to 0$ for $1\le i\le n$, one recovers the Macdonald identity (1.5).

Multiple ${}_6\psi_6$ summations are not only important in relation to the Macdonald identities but also have a close connection to q-beta-integrals on root systems, which in turn play a role in the theory of multivariable orthogonal polynomials. The integral counterpart of (1.7), for example, is Gustafson's A_{n-1} integral [14]

$$\int_{\mathbb{T}^{n-1}} \prod_{1\le i<j\le n} (z_i/z_j,z_j/z_i)_\infty \prod_{j=1}^{n} \frac{(ABz_j)_\infty}{\prod_{i=1}^{n+1}(a_iz_j)_\infty \prod_{i=1}^{n}(b_i/z_j)_\infty} \frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_{n-1}}{z_{n-1}}$$
$$= \frac{n!(2\pi\mathrm{i})^{n-1}\prod_{i=1}^{n+1}(AB/a_i)_\infty \prod_{i=1}^{n}(Ab_i)_\infty}{(q)_\infty^{n-1}(B)_\infty \prod_{i=1}^{n+1}(A/a_i)_\infty \prod_{i=1}^{n+1}\prod_{j=1}^{n}(a_ib_j)_\infty}.$$

Here $\mathbb{T}$ is the positively oriented unit circle, $z_1\cdots z_n=1$, $A=a_1\cdots a_{n+1}$, $B=b_1\cdots b_n$, where $a_1,\dots,a_{n+1},b_1,\dots,b_n\in\mathbb{C}$ such that $|a_1|,\dots,|b_n|<1$.

There are a number of known multiple beta integrals for which no corresponding ${}_6\psi_6$ summation has ever been found. Filling this gap in the literature has been the initial motivation for this paper, and in Sects. 4 and 5 three new multiple ${}_6\psi_6$ sums are stated. The consequences of these results go far beyond the completion of the classification of ${}_6\psi_6$ summations. Indeed, as it turns out, two of our new summations corresponding to A_{2n-1} and A_{2n} have rather unexpected consequences for multivariable orthogonal polynomials. In particular, both these ${}_6\psi_6$ summations can be reduced to new multiple ${}_6\phi_5$ sums. Conjecturally, these ${}_6\phi_5$ sums can be interpreted as weight-function normalisations for some yet-to-be found generalisations of the BC_n q-Racah polynomials. Moreover, by conjecturing elliptic extensions of the ${}_6\phi_5$ sums we are led to speculate on the existence of A_{2n-1} and A_{2n} elliptic generalisation of the entire BC_n-Koornwinder–Macdonald theory.

2 Notation

In this section we collect some standard notation from the theory of basic hypergeometric series and partitions.

Throughout this paper we view the base q either as a formal variable or as a fixed complex number such that $|q|<1$. Then the q-shifted factorials $(a)_\infty$ and $(a)_n$ (for $n\in\mathbb{Z}$) are defined as

$$(a)_\infty=(a;q)_\infty:=\prod_{k=0}^{\infty}\bigl(1-aq^k\bigr)$$

and

$$(a)_n = (a;q)_n = \frac{(a)_\infty}{(aq^n)_\infty}.$$

Note that

$$(a)_n = \prod_{k=0}^{n-1} \bigl(1 - aq^k\bigr)$$

for n a nonnegative integer, and

$$(a)_{-n} = \frac{(-q/a)^n}{(q/a)_n} q^{\binom{n}{2}}$$

for all $n \in \mathbb{Z}$. Hence $1/(q^m)_n = 0$ unless $n \geq -m$. We also adopt the usual condensed notations

$$(a_1, \dots, a_k)_m = (a_1, \dots, a_k; q)_m := \prod_{i=1}^{k} (a_i)_m$$

and

$$\bigl(z^{\pm}\bigr)_m = \bigl(z, z^{-1}\bigr)_m, \qquad \bigl(z^{\pm n}\bigr)_m = \bigl(z^n, z^{-n}\bigr)_m,$$
$$\bigl(z^{\pm} w^{\pm}\bigr)_m = \bigl(zw, zw^{-1}, z^{-1}w, z^{-1}w^{-1}\bigr)_m,$$

where $m \in \mathbb{Z} \cup \{\infty\}$.

Because we are dealing with series on root systems, we need some notation pertaining to integer sequences, and for $\lambda = (\lambda_1, \lambda_2, \dots)$ a finite sequence of integers, we set

$$|\lambda| = \sum_{i \geq 1} \lambda_i,$$
$$n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$$

and

$$(a)_\lambda = (a; q, t)_\lambda := \prod_{i \geq 1} \bigl(at^{1-i}\bigr)_{\lambda_i}.$$

In a few instances we also use

$$(a)_{(N^m)} = (a; q, t)_{(N^m)} := \prod_{i=1}^{m} \bigl(at^{1-i}\bigr)_N.$$

We already defined the Vandermonde product $\Delta(z)$ in (1.6). Subsequently we also need the analogous product for the (classical) root system of type C_n, and for $z \in \mathbb{C}^n$, we define

$$\Delta^+(z) = \prod_{i=1}^{n} \bigl(1 - z_i^2\bigr) \prod_{1\le i<j\le n} (z_i - z_j)(1 - z_i z_j).$$

Finally we need some notation concerning partitions. All our partitions will have at most n parts, and we define

$$\Lambda = \bigl\{\lambda \in \mathbb{Z}^n :\ \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0\bigr\}$$

and

$$\Lambda_N = \{\lambda \in \Lambda :\ \lambda_1 \le N\}.$$

As usual we identify a partition with its diagram or Ferrers graph [3, 21]. Given $\lambda, \mu \in \Lambda$, we say that μ is contained in λ, denoted $\mu \subseteq \lambda$, if $\mu_i \le \lambda_i$ for all $1 \le i \le n$. In other words, μ is contained in λ if the graph of μ is a subset of the graph of λ. If $\mu \subseteq \lambda$ we also use the more customary $|\lambda - \mu|$ instead of $|\lambda| - |\mu|$. We write $\mu \preccurlyeq \lambda$ if $\mu \subseteq \lambda$ and the graphs of λ and μ differ by at most one square in each column. (In the terminology of [21], $\mu \preccurlyeq \lambda$ if the skew shape $\lambda - \mu$ is a horizontal strip.) Note that $\mu \preccurlyeq \lambda$ if and only if we have the interlacing property

$$\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \cdots \ge \mu_{n-1} \ge \lambda_n \ge \mu_n \ge 0.$$

3 Some known ${}_6\psi_6$ summations

Our derivations of new multiple ${}_6\psi_6$ summations rely on a technique employed by van Diejen in his proof of Theorem 3.3 below [33]. This method essentially coincides with the one used by Gustafson for proving generalised beta integrals [13], and by Anderson for proving the Selberg integral [1, 4]. At its core is a clever sequential use of existing multiple ${}_6\psi_6$ summations, and for our purposes the following three known summations are crucial.

Recall that throughout this paper it is assumed that $|q| < 1$.

Theorem 3.1 (Gustafson's type I C_n ${}_6\psi_6$ sum [11]) *For* $a_1, \ldots, a_{2n+2}, z_1, \ldots, z_n \in \mathbb{C}^*$,

$$\sum_{\lambda\in\mathbb{Z}^n} (qA)^{|\lambda|} \frac{\Delta^+(zq^\lambda)}{\Delta^+(z)} \prod_{i=1}^{2n+2}\prod_{j=1}^{n} \frac{(z_j/a_i)_{\lambda_j}}{(qa_iz_j)_{\lambda_j}}$$
$$= \frac{(q)_\infty^n}{(qA)_\infty} \prod_{1\le i<j\le 2n+2} (qa_ia_j)_\infty \prod_{j=1}^{n} \frac{(qz_j^{\pm 2})_\infty}{\prod_{i=1}^{2n+2}(qa_iz_j^{\pm})_\infty} \prod_{1\le i<j\le n} \bigl(qz_i^{\pm}z_j^{\pm}\bigr)_\infty, \quad (3.1)$$

where $A = a_1 \cdots a_{2n+2}$ *and* $|qA| < 1$.

We remark that in the limit $a_i \to 0$ for $1 \le i \le 2n+2$ this yields the Macdonald identity of type C_n, in the limit $a_i \to 0$ for $1 \le i \le 2n+1$ and $a_{2n+2} \mapsto -1$ it yields the Macdonald identity of type BC_n, and in the limit $a_i \to 0$ for $1 \le i \le 2n$ and $a_{2n+1} \mapsto -1$, $a_{2n+2} \mapsto -q^{-1/2}$ it yields the Macdonald identity of type $\mathrm{C}_n^\vee$. Similar comments apply to the other ${}_6\psi_6$ summations listed below.

Theorem 3.2 (Gustafson's type I $\mathrm{B}_n^\vee$ ${}_6\psi_6$ sum [11]) *For $a_1, \dots, a_{2n}, z_1, \dots, z_n \in \mathbb{C}^*$ and $\sigma = 0, 1$,*

$$\sum_{\substack{\lambda\in\mathbb{Z}^n \\ |\lambda|\equiv\sigma \pmod 2}} (-A)^{|\lambda|} \frac{\Delta^+(zq^\lambda)}{\Delta^+(z)} \prod_{i=1}^{2n}\prod_{j=1}^{n} \frac{(z_j/a_i)_{\lambda_j}}{(qa_iz_j)_{\lambda_j}}$$

$$= \frac{(q)_\infty^{n-1}(q^2;q^2)_\infty}{(-A)_\infty} \prod_{i=1}^{2n}(qa_i^2;q^2)_\infty \prod_{1\le i<j\le 2n}(qa_ia_j)_\infty$$

$$\times \prod_{j=1}^{n} \frac{(q^2z_j^{\pm 2};q^2)_\infty}{\prod_{i=1}^{2n}(qa_iz_j^{\pm})_\infty} \prod_{1\le i<j\le n}(qz_i^{\pm}z_j^{\pm})_\infty, \tag{3.2}$$

where $A = a_1\cdots a_{2n}$ and $|A| < 1$.

This result in not entirely independent of the C_n ${}_6\psi_6$ sum; setting $a_{2n+1} = q^{-1/2}$ and $a_{2n+2} = -q^{-1/2}$ in the latter, we obtain the former summed over σ.

Theorem 3.3 (van Diejen's type II C_n ${}_6\psi_6$ sum [33]) *For $a_1, \dots, a_4, t, z_1, \dots, z_n \in \mathbb{C}^*$,*

$$\sum_{\lambda\in\mathbb{Z}^n} \left(qt^{2-2n}A\right)^{|\lambda|} \left(\frac{t^2}{q}\right)^{n(\lambda)} \frac{\Delta^+(zq^\lambda)}{\Delta^+(z)} \prod_{i=1}^{4}\prod_{j=1}^{n} \frac{(z_j/a_i)_{\lambda_j}}{(qa_iz_j)_{\lambda_j}}$$

$$\times \prod_{1\le i<j\le n} \frac{(tz_iz_j)_{\lambda_i+\lambda_j}}{(qt^{-1}z_iz_j)_{\lambda_i+\lambda_j}} \frac{(tz_iz_j^{-1})_{\lambda_i-\lambda_j}}{(qt^{-1}z_iz_j^{-1})_{\lambda_i-\lambda_j}}$$

$$= \prod_{j=1}^{n} \frac{(q, qt^{-j}, qz_j^{\pm 2})_\infty \prod_{1\le k<l\le 4}(qt^{1-j}a_ka_l)_\infty}{(qt^{-1}, qt^{2-j-n}A)_\infty \prod_{i=1}^{4}(qa_iz_j^{\pm})_\infty} \prod_{1\le i<j\le n} \frac{(qz_i^{\pm}z_j^{\pm})_\infty}{(qt^{-1}z_i^{\pm}z_j^{\pm})_\infty}, \tag{3.3}$$

where $A = a_1\cdots a_4$ and $\max\{|qt^{2-2n}A|, |q^{2-n}A|\} < 1$.

Other multiple ${}_6\psi_6$ summations that appear not to be amenable to the methods employed in this paper may be found in [18, 27, 28].

In the above we have made reference not only to the root system attached to each ${}_6\psi_6$ sum, but also to its type. In type I hypergeometric sums (or type I multiple beta integrals) the number of free parameters (such as the a_i and b_j) is of the form $2n + m$ where m is a constant and n (or $n - 1$) the rank of the root system. The sums (1.5), (3.1), (3.2) are all of type I. In type II sums or integrals the number of free parameters is assumed to be independent of the rank of the underlying root system. The sum (3.3) is an example of a type II hypergeometric sum. From the point of view of orthogonal polynomial theory, type II sums and integrals are by far the most important. Koornwinder's multivariable Askey–Wilson polynomials, for example, depend on 5 free variables, and the corresponding orthogonality measure is determined by a type II q-beta integral, see e.g., [19, 24, 25, 34] for more details.

4 Type II $\mathrm{B}_n^\vee$ ${}_6\psi_6$ sum

As a warm up to the much more important results of the next section, we prove a type II variant of Gustafson's $\mathrm{B}_n^\vee$ sum.

Theorem 4.1 *For $a_1, a_2, t, z_1, \dots, z_n \in \mathbb{C}^*$ and $\sigma = 0, 1$,*

$$\sum_{\substack{\lambda\in\mathbb{Z}^n \\ |\lambda|\equiv\sigma \pmod 2}} \left(-t^{2-2n}A\right)^{|\lambda|}\left(\frac{t^2}{q}\right)^{n(\lambda)} \frac{\Delta^+(zq^\lambda)}{\Delta^+(z)} \prod_{i=1}^{2}\prod_{j=1}^{n} \frac{(z_j/a_i)_{\lambda_j}}{(qa_iz_j)_{\lambda_j}}$$

$$\times \prod_{1\le i<j\le n} \frac{(tz_iz_j)_{\lambda_i+\lambda_j}}{(qt^{-1}z_iz_j)_{\lambda_i+\lambda_j}} \frac{(tz_iz_j^{-1})_{\lambda_i-\lambda_j}}{(qt^{-1}z_iz_j^{-1})_{\lambda_i-\lambda_j}}$$

$$= \frac{1}{2}\prod_{j=1}^{n}\left(\frac{(q, qt^{-j}, qt^{1-j}A, -t^{1-j})_\infty (q^2z_j^{\pm 2}; q^2)_\infty}{(qt^{-1}, -t^{2-j-n}A)_\infty} \prod_{i=1}^{2} \frac{(qt^{2-2j}a_i^2; q^2)_\infty}{(qa_iz_j^\pm)_\infty}\right)$$

$$\times \prod_{1\le i<j\le n} \frac{(qz_i^\pm z_j^\pm)_\infty}{(qt^{-1}z_i^\pm z_j^\pm)_\infty}, \tag{4.1}$$

where $A = a_1a_2$ such that $\max\{|t^{2-2n}A|, |q^{1-n}A|\} < 1$.

The above identity bears the same relation to (3.3) as (3.2) to (3.1). That is, if we set $a_{2n+1} = q^{-1/2}$ and $a_{2n+2} = -q^{-1/2}$ in (3.3), we obtain (4.1) summed over σ.

Before proving the theorem we list a number of easy consequences. First of all we note that both the $\mathrm{B}_n^\vee$ and D_n ($n \ge 2$) Macdonald identities [20] follow from (4.1); the $\mathrm{B}_n^\vee$ case is obtained if we let $1/a_1, 1/a_2, t \to \infty$ and take $\sigma = 0$, and the D_n case is obtained if we let $t \to \infty$ and take $a_1 = -a_2 = 1$ and $\sigma = 0$.

A collection of rather curious variations of (some of) the Macdonald identities arises if we take Theorems 3.3 and 4.1 and consider the $t \to q$ limit. Depending on the choice of the a_i this yields the following five infinite families, which we, perhaps somewhat misleading, label B_n, $\mathrm{B}_n^\vee$ C_n, $\mathrm{C}_n^\vee$ and BC_n based on the corresponding Macdonald identities (obtained by replacing the $t \to q$ limit by a $t \to \infty$ limit in the proofs).

For $z = (z_1, \dots, z_n)$, let

$$\Xi(zq^\lambda) = \prod_{1\le i<j\le n} \left(1 - z_iz_j^{-1}q^{\lambda_i-\lambda_j}\right)^2\left(1 - z_iz_jq^{\lambda_i+\lambda_j}\right)^2.$$

Corollary 4.2 (B_n identity) *For* $z_1, \dots, z_n \in \mathbb{C}^*$,

$$\sum_{\substack{\lambda\in\mathbb{Z}^n\\ |\lambda|\equiv 0 \pmod 2}} \Xi\big(zq^\lambda\big) \prod_{i=1}^n z_i^{\lambda_i} q^{\binom{\lambda_i}{2}+(2i-2n+1)\lambda_i}\big(1-z_i q^{\lambda_i}\big)$$
$$= 2^{n-1} n! \prod_{i=1}^n z_i^{2(n-i)} \big(q^{-1};q^{-1}\big)_{2i-2}\big(z_i, z_i^{-1}q, q\big)_\infty \big(z_i^{\pm 2}q;q^2\big)_\infty.$$

Proof In (4.1) let $a_1 \to 0$, $t \to q$ and choose $a_2 = -1$, $\sigma = 0$. □

Corollary 4.3 ($B_n^\vee$ identity) *For* $z_1, \dots, z_n \in \mathbb{C}^*$,

$$\sum_{\substack{\lambda\in\mathbb{Z}^n\\ |\lambda|\equiv 0 \pmod 2}} \Xi\big(zq^\lambda\big) \prod_{i=1}^n z_i^{2\lambda_i} q^{2\binom{\lambda_i}{2}+2(i-n)\lambda_i}\big(1-z_i^2 q^{2\lambda_i}\big)$$
$$= 2^{n-1} n! \prod_{i=1}^n z^{2(n-i)} \big(q^{-2};q^{-2}\big)_{i-1}\big(z_i^2, z_i^{-2}q^2, q^2; q^2\big)_\infty.$$

Proof In (4.1) let $a_1, a_2 \to 0$, $t \to q$ and choose $\sigma = 0$. □

Corollary 4.4 (C_n identity) *For* $z_1, \dots, z_n \in \mathbb{C}^*$,

$$\sum_{\lambda\in\mathbb{Z}^n} \Xi\big(zq^\lambda\big) \prod_{i=1}^n z_i^{4\lambda_i} q^{4\binom{\lambda_i}{2}+(2i-2n+1)\lambda_i}\big(1-z_i^2 q^{2\lambda_i}\big)$$
$$= n! \prod_{i=1}^n z_i^{2(n-i)} \big(q^{-1};q^{-1}\big)_{i-1}\big(z_i^2, z_i^{-2}q, q\big)_\infty.$$

Proof In (3.3) let $a_1, \dots, a_4 \to 0$ and $t \to q$. □

Corollary 4.5 ($C_n^\vee$ identity) *For* $z_1, \dots, z_n \in \mathbb{C}^*$,

$$\sum_{\lambda\in\mathbb{Z}^n} \Xi\big(zq^\lambda\big) \prod_{i=1}^n z_i^{2\lambda_i} q^{2\binom{\lambda_i}{2}+(2i-2n+1/2)\lambda_i}\big(1-z_j q^{\lambda_i}\big)$$
$$= n! \prod_{i=1}^n z_i^{2(n-i)} \big(q^{-1/2};q^{-1/2}\big)_{2i-2}\big(z_i, z_i^{-1}q^{1/2}, q^{1/2}; q^{1/2}\big)_\infty.$$

Proof In (3.3) let $a_1, a_2 \to 0$, $t \to q$ and choose $a_3 = -1$, $a_4 = -q^{-1/2}$. □

Corollary 4.6 (BC_n identity) *For* $z_1, \dots, z_n \in \mathbb{C}^*$,

$$\sum_{\lambda\in\mathbb{Z}^n} \Xi\big(zq^\lambda\big) \prod_{i=1}^n z_i^{3\lambda_i} q^{3\binom{\lambda_i}{2}+(2i-2n+1)\lambda_i}\big(1-z_i q^{\lambda_i}\big)$$
$$= n! \prod_{i=1}^n z_i^{2(n-i)} \big(q^{-1};q^{-1}\big)_{i-1}\big(z_i, z_i^{-1}q, q\big)_\infty \big(qz_i^{\pm 2};q^2\big)_\infty.$$

Proof In (3.3) let $a_1,\dots,a_3\to 0$, $t\to q$ and set $a_4=-1$. □

For later comparison, we give one further special case of Theorem 4.1. For $\lambda\in\Lambda$, let $\Delta_\lambda(a)=\Delta_\lambda(a;q,t)$ be defined as

$$\Delta_\lambda(a)=\prod_{i=1}^{n}\frac{1-at^{2-2i}q^{2\lambda_i}}{1-at^{2-2i}}\prod_{1\le i<j\le n}\frac{1-t^{j-i}q^{\lambda_i-\lambda_j}}{1-t^{j-i}}\,\frac{1-at^{2-i-j}q^{\lambda_i+\lambda_j}}{1-at^{2-i-j}}$$
$$\times\prod_{1\le i<j\le n}\frac{(at^{3-i-j})_{\lambda_i+\lambda_j}}{(aqt^{1-i-j})_{\lambda_i+\lambda_j}}\,\frac{(t^{j-i+1})_{\lambda_i-\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}}. \tag{4.2}$$

Then the $a_1=a^{-1/2}t^{n-1}$, $a_2=a^{1/2}b$ and $z_i=a^{1/2}t^{1-i}$ $(1\le i\le n)$ specialisation of (4.1) boils down to

$$\sum_{\substack{\lambda\in\Lambda\\|\lambda|\equiv\sigma \pmod 2}}\Delta_\lambda(a)\,\frac{(at^{1-n},1/b)_\lambda}{(qt^{n-1},abq)_\lambda}\left(-bt^{1-n}\right)^{|\lambda|}t^{2n(\lambda)}$$
$$=\frac{1}{2}\prod_{i=1}^{n}\frac{(aqt^{1-i},-t^{1-i})_\infty}{(abqt^{1-i},-bt^{1-i})_\infty}\,\frac{(ab^2qt^{2-2i};q^2)_\infty}{(aqt^{2-2i};q^2)_\infty}. \tag{4.3}$$

Proof of Theorem 4.1 Making the substitutions

$$a_i\mapsto\begin{cases}uy_iq^{\mu_i}, & \text{for } 1\le i\le n,\\ u(y_iq^{\mu_i})^{-1}, & \text{for } n+1\le i\le 2n,\end{cases}$$

in (3.2) yields

$$\sum_{\substack{\lambda\in\mathbb{Z}^n\\|\lambda|\equiv\sigma \pmod 2}}\left(-u^{2n}\right)^{|\lambda|}\frac{\Delta^+(zq^\lambda)}{\Delta^+(z)}\prod_{i,j=1}^{n}\frac{(z_iy_j^{\pm}/u)_{\lambda_i}}{(quz_iy_j^{\pm})_{\lambda_i}}\,\frac{((q^{\lambda_i}z_i)^{\pm}y_j/u)_{\mu_j}}{(qu(q^{\lambda_i}z_i)^{\pm}y_j)_{\mu_j}}$$
$$=\frac{(q)_\infty^{n-1}(q^2;q^2)_\infty(qu^2)_\infty^n}{(-u^{2n})_\infty}\prod_{i=1}^{n}\left(qu^2y_i^{\pm 2};q^2\right)_\infty\prod_{1\le i<j\le n}\left(qu^2y_i^{\pm}y_j^{\pm}\right)_\infty$$
$$\times\prod_{i=1}^{n}\left(q^2z_i^{\pm 2};q^2\right)_\infty\prod_{1\le i<j\le n}\left(qz_i^{\pm}z_j^{\pm}\right)_\infty\prod_{i,j=1}^{n}\frac{1}{(quy_i^{\pm}z_j^{\pm})_\infty}$$
$$\times\left(-\frac{u^{2(n-1)}}{q}\right)^{|\mu|}\left(\frac{1}{qu^4}\right)^{n(\mu)}\prod_{i=1}^{n}\frac{(qy_i^2/u^2;q^2)_{\mu_i}}{(qu^2y_i^2;q^2)_{\mu_i}}$$
$$\times\prod_{1\le i<j\le n}\frac{(y_iy_j/u^2)_{\mu_i+\mu_j}(y_i/y_ju^2)_{\mu_i-\mu_j}}{(qu^2y_iy_j)_{\mu_i+\mu_j}(qu^2y_i/y_j)_{\mu_i-\mu_j}}.$$

If we multiply this by

$$\big(qb_1b_2u^{2n}\big)^{|\mu|}\,\frac{\Delta^+(yq^\mu)}{\Delta(y)}\prod_{i=1}^{2}\prod_{j=1}^{n}\frac{(y_j/b_i)_{\mu_j}}{(qb_iy_j)_{\mu_j}},$$

where $y=(y_1,\dots,y_n)$, and note that

$$\prod_{i=1}^{n}\frac{(qu^{-2}y_i^2;q^2)_{\mu_i}}{(qu^2y_i^2;q^2)_{\mu_i}}=\prod_{i=1}^{n}\frac{(q^{1/2}u^{-1}y_i,-q^{1/2}u^{-1}y_i)_{\mu_i}}{(q^{1/2}uy_i,-q^{1/2}uy_i)_{\mu_i}},$$

then the left can be summed over μ by (3.1), and the right can be summed over μ by (3.3). The resulting identity corresponds to the claim with $(a_1,a_2,t)\mapsto(ub_1,ub_2,1/u^2)$.

The above application of (3.1), (3.2) and (3.3) is only valid provided that

$$|t|>1,\qquad \big|qt^{1-n}A\big|<1,\qquad \big|q^{1-n}A\big|<1,\quad\text{and}\quad \big|t^{2-2n}A\big|<1,$$

but the first two conditions may be dropped by analytic continuation. □

5 Type II A_{2n-1} and A_{2n} ${}_6\psi_6$ summations

Our next two results, which are the series counterparts of q-beta integrals of Gustafson [15], are new ${}_6\psi_6$ summations for the root systems A_{2n-1} and A_{2n}. Both are much deeper than Theorem 4.1 and a lot more intricate to prove. As alluded to in the introduction, they have some rather surprising consequences, to be discussed in the next section. Because of some intricate convergence issues, which we failed to completely settle, the A_{2n-1} and A_{2n} sums are stated as Claims instead of fully fledged Theorems.

Claim 5.1 (Type II A_{2n-1} ${}_6\psi_6$ sum) *For $a_1,a_2,b_1,b_2,z_1,\dots,z_{2n}\in\mathbb{C}^*$,*

$$\begin{aligned}\sum_{\substack{\lambda\in\mathbb{Z}^{2n}\\|\lambda|=0}}\frac{\Delta(zq^\lambda)}{\Delta(z)}\prod_{1\le i<j\le 2n}\frac{(z_iz_j/t)_{\lambda_i+\lambda_j}}{(qtz_iz_j)_{\lambda_i+\lambda_j}}\prod_{i=1}^{2}\prod_{j=1}^{2n}\frac{(z_j/b_i)_{\lambda_j}}{(qa_iz_j)_{\lambda_j}}\\
&=(q)_\infty^{2n-1}\big(qt^nZ,qt^n/Z,qa_1a_2t^{n-1}Z,qb_1b_2t^{n-1}/Z\big)_\infty\\
&\quad\times\prod_{i=1}^{n-1}\big(qt^{2i},qa_1a_2t^{2i-1},qb_1b_2t^{2i-1}\big)_\infty\prod_{i=1}^{n}\frac{\prod_{j,k=1}^{2}(qa_jb_kt^{2i-2})_\infty}{(qa_1a_2b_1b_2t^{2i+2n-4})_\infty}\\
&\quad\times\prod_{i=1}^{2}\prod_{j=1}^{2n}\frac{1}{(qa_iz_j,qb_i/z_j)_\infty}\prod_{1\le i<j\le 2n}\frac{(qz_i/z_j,qz_j/z_i)_\infty}{(qtz_iz_j,qt/z_iz_j)_\infty},\end{aligned}$$

where $Z=z_1\cdots z_{2n}$ and $|qa_1a_2b_1b_2t^{4n-4}|<1$.

Claim 5.2 (Type II A_{2n} ${}_6\psi_6$ sum) *For* $a_1, a_2, b_1, b_2, z_1, \dots, z_{2n+1} \in \mathbb{C}^*$,

$$\sum_{\substack{\lambda\in\mathbb{Z}^{2n+1}\\|\lambda|=0}} \frac{\Delta(zq^\lambda)}{\Delta(z)} \prod_{1\le i<j\le 2n+1} \frac{(z_iz_j/t)_{\lambda_i+\lambda_j}}{(qtz_iz_j)_{\lambda_i+\lambda_j}} \prod_{i=1}^{2}\prod_{j=1}^{2n+1} \frac{(z_j/b_i)_{\lambda_j}}{(qa_iz_j)_{\lambda_j}}$$

$$= (q)_\infty^{2n} \prod_{i=1}^{2} \big(qa_it^nZ, qb_it^n/Z\big)_\infty$$

$$\times \prod_{i=1}^{n} \left[\frac{(qt^{2i}, qa_1a_2t^{2i-1}, qb_1b_2t^{2i-1})_\infty \prod_{j,k=1}^{2}(qa_jb_kt^{2i-2})_\infty}{(qa_1a_2b_1b_2t^{2i+2n-2})_\infty} \right]$$

$$\times \prod_{i=1}^{2}\prod_{j=1}^{2n+1} \frac{1}{(qa_iz_j, qb_i/z_j)_\infty} \prod_{1\le i<j\le 2n+1} \frac{(qz_i/z_j, qz_j/z_i)_\infty}{(qtz_iz_j, qt/z_iz_j)_\infty},$$

where $Z = z_1\cdots z_{2n+1}$ *and* $|qa_1a_2b_1b_2t^{4n-2}| < 1$.

Proof We first combine the two claims into one statement, for which we give a formal proof.

For $a_1, a_2, b_1, b_2, z_1, \dots, z_n \in \mathbb{C}^*$,

$$\sum_{\substack{\lambda\in\mathbb{Z}^{n}\\|\lambda|=0}} \frac{\Delta(zq^\lambda)}{\Delta(z)} \prod_{1\le i<j\le n} \frac{(z_iz_j/t)_{\lambda_i+\lambda_j}}{(qtz_iz_j)_{\lambda_i+\lambda_j}} \prod_{i=1}^{2}\prod_{j=1}^{n} \frac{(z_j/b_i)_{\lambda_j}}{(qa_iz_j)_{\lambda_j}}$$

$$= (q)_\infty^{n-1} \big(qAt^mZ, qBt^m/Z, q\hat{A}t^{n-m-1}Z, q\hat{B}t^{n-m-1}/Z\big)_\infty$$

$$\times \prod_{i=1}^{n-m-1} \big(qt^{2i}, qa_1a_2t^{2i-1}, qb_1b_2t^{2i-1}\big)_\infty \prod_{i=1}^{m} \frac{\prod_{j,k=1}^{2}(qa_jb_kt^{2i-2})_\infty}{(qa_1a_2b_1b_2t^{2i+2n-2m-4})_\infty}$$

$$\times \prod_{i=1}^{2}\prod_{j=1}^{n} \frac{1}{(qa_iz_j, qb_i/z_j)_\infty} \prod_{1\le i<j\le n} \frac{(qz_i/z_j, qz_j/z_i)_\infty}{(qtz_iz_j, qt/z_iz_j)_\infty}, \tag{5.1}$$

where $m = \lfloor n/2 \rfloor$, $Z = z_1\cdots z_n$, $|qa_1a_2b_1b_2t^{2n-4}| < 1$ and

$$(A, \hat{A}, B, \hat{B}) = \begin{cases} (1, a_1a_2, 1, b_1b_2), & \text{for } n \text{ even},\\ (a_1, a_2, b_1, b_2), & \text{for } n \text{ odd}. \end{cases}$$

To prove this we set $n = 2m + k$ where $k = 0, 1$ in (1.7), and simultaneously replace

$$a_i \mapsto tw_iq^{\nu_i}, \qquad a_{i+m} \mapsto t\big(w_iq^{\nu_i}\big)^{-1} \quad \text{for } 1 \le i \le m,$$

$$b_i \mapsto s^{-1}y_iq^{\mu_i}, \qquad b_{i+m} \mapsto s^{-1}\big(y_iq^{\mu_i}\big)^{-1} \quad \text{for } 1 \le i \le m$$

and

$$a_{2i+m} \mapsto a_i, \qquad b_{2i+m} \mapsto b_i \quad \text{for } 1 \le i \le k.$$

After some elementary manipulations this yields

$$\sum_{\substack{\lambda\in\mathbb{Z}^n\\|\lambda|=0}} \frac{\Delta(zq^\lambda)}{\Delta(z)} \prod_{j=1}^n \left[\prod_{i=1}^m \frac{(sy_i^\pm z_j)_{\lambda_j}}{(qtw_i^\pm z_j)_{\lambda_j}} \prod_{i=1}^k \frac{(z_j/b_i)_{\lambda_j}}{(qa_i z_j)_{\lambda_j}}\right]$$
$$\times \prod_{i=1}^n \prod_{j=1}^m \frac{(sq^{\lambda_i} z_i y_j)_{\mu_j}}{(q^{1-\lambda_i} y_j/sz_i)_{\mu_j}} \frac{(q^{-\lambda_i} w_j/tz_i)_{\nu_j}}{(q^{\lambda_i+1} tz_i w_j)_{\nu_j}}$$
$$= \frac{(qt^{2m}AZ, qB/s^{2m}Z)_\infty}{(q, q(t/s)^{2m}AB)_\infty}$$
$$\times \prod_{j=1}^n \left[\prod_{i=1}^n (qz_i/z_j)_\infty \prod_{i=1}^m \frac{1}{(qy_i^\pm/sz_j, qtw_i^\pm z_j)_\infty} \prod_{i=1}^k \frac{1}{(qb_i/z_j, qa_i z_j)_\infty}\right]$$
$$\times \prod_{i=1}^m \left[\prod_{j=1}^m \big(qty_i^\pm w_j^\pm/s\big)_\infty \prod_{j=1}^k \big(qy_i^\pm a_j/s, qtw_i^\pm b_j\big)_\infty\right] \prod_{i,j=1}^k (qa_i b_j)_\infty$$
$$\times \big(t^{2m}AZ\big)^{|\mu|} \left(\frac{B}{s^{2m}Z}\right)^{|\nu|} \prod_{i,j=1}^m \frac{(q^{\nu_i} sw_i y_j/t, q^{-\nu_i} sy_j/tw_i)_{\mu_j}}{(q^{1-\nu_i} ty_j/sw_i, q^{\nu_i+1} tw_i y_j/s)_{\mu_j}}$$
$$\times \frac{(sy_i^\pm w_j/t)_{\nu_j}}{(qty_i^\pm w_j/s)_{\nu_j}} \prod_{i=1}^k \prod_{j=1}^m \frac{(sy_j/a_i)_{\mu_j}}{(qa_i y_j/s)_{\mu_j}} \frac{(w_j/tb_i)_{\nu_j}}{(qtb_i w_j)_{\nu_j}},$$

where $A = a_1 \cdots a_k$ and $B = b_1 \cdots b_k$. We multiply the above equation by

$$\left(\frac{q\hat{B}}{s^{n+k-2}Z}\right)^{|\mu|} \frac{\Delta^+(yq^\mu)}{\Delta^+(y)} \prod_{i=1}^m \prod_{j=k+1}^2 \frac{(y_i/sb_j)_{\mu_i}}{(qsy_i b_j)_{\mu_i}}$$
$$\times \big(qt^{n+k-2}\hat{A}Z\big)^{|\nu|} \frac{\Delta^+(wq^\nu)}{\Delta^+(w)} \prod_{i=1}^m \prod_{j=k+1}^2 \frac{(tw_i a_j)_{\nu_i}}{(qw_i a_j/t)_{\nu_i}},$$

where $y = (y_1, \ldots, y_m)$, $w = (w_1, \ldots, w_m)$, $\hat{A} = a_{k+1} \cdots a_2$ and $\hat{B} = b_{k+1} \cdots b_2$, and sum both sides over $\mu, \nu \in \mathbb{Z}^m$.

Now we change the order of summations $\sum_{\mu,\nu\in\mathbb{Z}^m}$ and $\sum_{\lambda\in\mathbb{Z}^n,\, |\lambda|=0}$ in the triple sum on the left-hand side. Then the sums over μ and ν on the left can be evaluated with the help of (3.1) with $m \mapsto n$. Evaluating in the same way the resulting sum over

μ on the right-hand side using (3.1), we arrive at the formula

$$
\begin{aligned}
&\sum_{\substack{\lambda\in\mathbb{Z}^n\\|\lambda|=0}} \frac{\Delta(zq^\lambda)}{\Delta(z)} \prod_{1\le i<j\le n} \frac{(s^2z_iz_j)_{\lambda_i+\lambda_j}}{(qt^2z_iz_j)_{\lambda_i+\lambda_j}} \prod_{i=1}^2\prod_{j=1}^n \frac{(z_j/b_i)_{\lambda_j}}{(qa_iz_j)_{\lambda_j}} \\
&\quad= \frac{(qt^{2m}AZ, qB/s^{2m}Z, q\hat{B}/s^{n+k-2}Z, qt^{n+k-2}\hat{A}Z)_\infty (qt^2/s^2)_\infty^m}{(q, q(t/s)^{2m}AB, qt^{2m}A\hat{B}/s^{n+k-2})_\infty (q)_\infty^m} \\
&\qquad\times \prod_{i=1}^m \frac{1}{(qw_i^{\pm 2})_\infty} \prod_{1\le i<j\le m} \frac{(qt^2w_i^{\pm}w_j^{\pm}/s^2)_\infty}{(qw_i^{\pm}w_j^{\pm})_\infty} \\
&\qquad\times \prod_{j=1}^m \Bigg[\prod_{i=1}^k \big(qta_iw_j^{\pm}/s^2\big)_\infty \prod_{i=k+1}^2 \big(qa_iw_j^{\pm}/t\big)_\infty \prod_{i=1}^2 \big(qtw_j^{\pm}b_i\big)_\infty\Bigg] \\
&\qquad\times \prod_{1\le i<j\le n} \frac{1}{(q/s^2z_iz_j, qt^2z_iz_j)_\infty} \prod_{j=1}^n \Bigg[\prod_{i=1}^n (qz_i/z_j)_\infty \prod_{i=1}^2 \frac{1}{(qb_i/z_j, qa_iz_j)_\infty}\Bigg] \\
&\qquad\times \prod_{k+1\le i<j\le 2} \frac{1}{(qa_ia_j/t^2)_\infty} \prod_{1\le i<j\le k} \big(qa_ia_j/s^2\big)_\infty \prod_{i=1}^k\prod_{j=1}^2 (qa_ib_j)_\infty \\
&\qquad\times \sum_{\nu\in\mathbb{Z}^m} \big(q(t/s)^{4m+2k-4}a_1a_2b_1b_2\big)^{|\nu|} \bigg(\frac{s^4}{qt^4}\bigg)^{n(\nu)} \frac{\Delta^+(wq^\nu)}{\Delta^+(w)} \\
&\qquad\times \prod_{1\le i<j\le m} \frac{(s^2w_iw_j/t^2)_{\nu_i+\nu_j}(s^2w_i/t^2w_j)_{\nu_i-\nu_j}}{(qt^2w_iw_j/s^2)_{\nu_i+\nu_j}(qt^2w_i/s^2w_j)_{\nu_i-\nu_j}} \\
&\qquad\times \prod_{j=1}^m \Bigg[\prod_{i=1}^2 \frac{(w_j/tb_i)_{\nu_j}}{(qtb_iw_j)_{\nu_j}} \prod_{i=1}^k \frac{(s^2w_j/a_it)_{\nu_j}}{(qta_iw_j/s^2)_{\nu_j}} \prod_{i=k+1}^2 \frac{(tw_j/a_i)_{\nu_i}}{(qa_iw_j/t)_{\nu_i}}\Bigg].
\end{aligned}
$$

Summing over ν on the right using (3.3) with $n\mapsto m$, formula (5.1) follows upon the substitution $(s^2,t^2)\mapsto(1/t,t)$.

For the convergence of the initial triple sum on the left (and, so, convergence of the series in (5.1)) and of the double sum on the right, the following conditions on the parameters are required:

$$
\begin{aligned}
&\big|q(t/s)^{2m}AB\big|<1, \qquad \big|qt^{n+k-2}\hat{A}Z\big|<1, \qquad \big|qs^{2-n-k}\hat{B}/Z\big|<1, \\
&\big|qt^{2m}s^{2-n-k}A\hat{B}\big|<1, \qquad \big|q^{2-n}(s/t)^{2n}a_1a_2b_1b_2\big|<1, \\
&\big|q(t/s)^{2n-4}a_1a_2b_1b_2\big|<1.
\end{aligned}
$$

By analytic continuation these may be relaxed to yield (after the rescaling $(s^2,t^2)\mapsto(1/t,t)$) the condition $|qt^{2n-4}a_1a_2b_1b_2|\le 1$ imposed on (5.1). □

Unfortunately the above is only a formal proof of (5.1). Although all of the series used converge, we need to take caution since they do not converge absolutely, so that the interchange of the summations $\sum_{\mu,\nu\in\mathbb{Z}^m}$ with $\sum_{\lambda\in\mathbb{Z}^n,\,|\lambda|=0}$ is not justified. It thus remains to be proved rigorously that both series converge to the same function. Indeed, on the left we are looking at evaluating a triple sum of the form

$$\sum_{\mu,\nu\in\mathbb{Z}^m}\sum_{\substack{\lambda\in\mathbb{Z}^n\\|\lambda|=0}} f_{\lambda\mu\nu},$$

but, as pointed out to us by the anonymous referee, neither

$$\sum_{\substack{\lambda\in\mathbb{Z}^n\\|\lambda|=0}}\sum_{\mu\in\mathbb{Z}^m} f_{\lambda\mu\nu} \quad \text{nor} \quad \sum_{\substack{\lambda\in\mathbb{Z}^n\\|\lambda|=0}}\sum_{\nu\in\mathbb{Z}^m} f_{\lambda\mu\nu}$$

converge (for more details, see the appendix in [29]). We do believe it should be possible to give meaning even to these divergent series along the lines described in [16]. That is, one should replace the formal series by an appropriate analytical function which generates formally the corresponding series. It is well known that the summation formulas allow an analytical continuation of functions to the region of parameters where the series representations diverge. If one finds an appropriate analytic continuation of our formal manipulations with the series, then this could lead to a rigorous justification of formula (5.1).

6 New type II ${}_6\phi_5$ summations

We begin this section by reviewing some results from [24, 26, 33, 35–37]. Making the specialisations

$$(a_1,a_2,a_3,a_4)=\big(t^{n-1}/a^{1/2},a^{1/2}/b,a^{1/2}/c,a^{1/2}q^N\big)$$

and

$$z_i=a^{1/2}t^{1-i}\quad \text{for } 1\le i\le n$$

in (3.3) yields van Diejen's type II C_n ${}_6\phi_5$ summation [33]

$$\sum_{\lambda\in\Lambda_N}\Delta_\lambda(a)\frac{(at^{1-n},b,c,q^{-N})_\lambda}{(qt^{n-1},aq/b,aq/c,aq^{N+1})_\lambda}\left(\frac{aq^{N+1}t^{1-n}}{bc}\right)^{|\lambda|}t^{2n(\lambda)}$$
$$=\frac{(aq,aq/bc)_{(N^n)}}{(aq/b,aq/c)_{(N^n)}}, \tag{6.1}$$

where $\Delta_\lambda(a)$ is defined in (4.2). This sum, the $N\to\infty$ limit of which should be compared with (4.3), can be interpreted as the weight-function normalisation for the BC_n q-Racah polynomials of van Diejen and Stokman [36] as follows. Let $\mathcal{H}^{BC_n}$ denote the space of BC_n-symmetric Laurent polynomials (that is, the space of Laurent polynomials symmetric under $W=\mathfrak{S}_n\ltimes(\mathbb{Z}_2)^n$, where $\mathfrak{S}_n$ acts by permuting the variables and $\mathbb{Z}_2$ acts by inversion in the sense of $x\mapsto 1/x$). $\mathcal{H}^{\mathrm{BC}_n}$ is spanned by $\{m_\lambda:\ \lambda\in\Lambda\}$ with m_λ a monomial symmetric function on BC_n:

$$m_\lambda(z)=\sum_\alpha z_1^{\alpha_1}\cdots z_n^{\alpha_n},$$

where the sum is over all distinct signed permutations α of λ. Now define the restricted space $\mathcal{H}_N^{\mathrm{qR}}$ as

$$\mathcal{H}_N^{\mathrm{qR}}=\mathrm{Span}\{m_\lambda:\ \lambda\in\Lambda_N\},$$

and for $t,t_0,\dots,t_3$ such that $t_0t_it^{n-1}=q^{-N}$ for a fixed $i\in\{1,2,3\}$, let the q-Racah weight function be given by

$$\Delta^{\mathrm{qR}}(\lambda)=\Delta_\lambda\bigl(t_0^2t^{2n-2}\bigr)\biggl(\frac{qt^{2-2n}}{t_0t_1t_2t_3}\biggr)^{|\lambda|}t^{2n(\lambda)}\prod_{r=0}^{3}\frac{(t_0t_rt^{n-1})_\lambda}{(qt_0t^{n-1}/t_r)_\lambda}.$$

Note that $\Delta^{\mathrm{qR}}(\lambda)$ is exactly the summand of (6.1) if we identify $a=t_0^2t^{2n-2}$ and

$$\bigl\{b,c,q^{-N}\bigr\}=\bigl\{t_0t_1t^{n-1},t_0t_2t^{n-1},t_0t_3t^{n-1}\bigr\}.$$

With the above notation we can define a bilinear form

$$\langle f,g\rangle_N^{\mathrm{qR}}=\sum_{\lambda\in\Lambda_N}\Delta^{\mathrm{qR}}(\lambda)f\bigl(t_0\langle\lambda\rangle\bigr)g\bigl(t_0\langle\lambda\rangle\bigr)$$

for $f,g\in\mathcal{H}_N^{\mathrm{qR}}$ and

$$\langle\lambda\rangle=\bigl(q^{\lambda_1}t^{n-1},q^{\lambda_2}t^{n-2},\dots,q^{\lambda_n}\bigr)$$

a spectral vector. The BC_n q-Racah (or discrete BC_n Askey–Wilson) polynomials p_λ are the unique monic polynomials in $\mathcal{H}_N^{\mathrm{qR}}$ such that

$$\langle p_\lambda,p_\mu\rangle_N^{\mathrm{qR}}=0,\qquad\lambda\neq\mu,\ \lambda,\ \mu\in\Lambda_N.$$

van Diejen's identity (6.1) may now be put in the equivalent form

$$\langle 1,1\rangle_N^{\mathrm{qR}}=\prod_{i=1}^{n}\frac{(qt_0^2t^{2n-i-1},qt^{1-i}/t_1t_2,qt^{1-i}/t_1t_3,qt^{1-i}/t_2t_3)_\infty}{(qt_0t^{n-i}/t_1,qt_0t^{n-i}/t_2,qt_0t^{n-i}/t_3,qt^{2-i-n}/t_0t_1t_2t_3)_\infty}.$$

In [37] this sum was conjectured to generalise to the elliptic level.[3] To state this now ex-conjecture, we adopt all the notation for q-shifted factorials introduced in Sect. 2 but with $(a)_n$ representing the elliptic shifted factorial [9, 31]:

$$(a)_n=(a;q,p)_n:=\prod_{k=0}^{n-1}\theta\bigl(aq^k\bigr),\tag{6.2}$$

[3]For a recent review of elliptic hypergeometric functions, see [31].

where

$$\theta(x) = \theta(x; p) := (x, p/x; p)_\infty \quad \text{for } |p| < 1.$$

Also defining the elliptic analogue of (4.2) as

$$\begin{aligned}\Delta_\lambda(a) &= \Delta_\lambda(a; q, t; p)\\ &= \prod_{i=1}^{n} \frac{\theta(at^{2-2i}q^{2\lambda_i})}{\theta(at^{2-2i})} \prod_{1\le i<j\le n} \frac{\theta(t^{j-i}q^{\lambda_i-\lambda_j})}{\theta(t^{j-i})} \frac{\theta(at^{2-i-j}q^{\lambda_i+\lambda_j})}{\theta(at^{2-i-j})}\\ &\quad\times \prod_{1\le i<j\le n} \frac{(at^{3-i-j})_{\lambda_i+\lambda_j}}{(aqt^{1-i-j})_{\lambda_i+\lambda_j}} \frac{(t^{j-i+1})_{\lambda_i-\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}},\end{aligned} \tag{6.3}$$

the elliptic generalisation of van Diejen's sum is [37]

$$\begin{aligned}&\sum_{\lambda\in\Lambda_N} \Delta_\lambda(a) \frac{(at^{1-n}, b, c, d, e, q^{-N})_\lambda}{(qt^{n-1}, aq/b, aq/c, aq/d, aq/e, aq^{N+1})_\lambda} q^{|\lambda|} t^{2n(\lambda)}\\ &\quad= \frac{(aq, aq/bc, aq/bd, aq/cd)_{(N^n)}}{(aq/b, aq/c, aq/d, aq/bcd)_{(N^n)}},\end{aligned} \tag{6.4}$$

provided that $bcdet^{n-1} = a^2q^{N+1}$. For $n = 1$ this is Frenkel–Turaev's elliptic extension of the Jackson sum [8], which was shown in [32] to serve as a normalisation condition of the weight function for a family of elliptic biorthogonal rational functions with discrete arguments (for their continuous analogues, see [30]).

For $p = 0$ (and general n), identity (6.4) was first proved in [35] using residue calculus on Gustafson's type II C_n q-Selberg integral. In its full generality (6.4) was proved by Rosengren in [26]. Subsequently Rains [24] and Coskun and Gustafson [7] not only generalised (6.4) to allow for more general partitions than (N^n), but also connected it to the theory of BC_n abelian functions generalising the Koornwinder polynomials (which include the above-discussed q-Racah polynomials) and Macdonald interpolation polynomials to the elliptic level.

After these preliminaries we turn to Claim 5.1. If we make the simultaneous substitutions

$$\begin{aligned} z_{2i} &\mapsto t^{1-2i}(t/a)^{1/2} \quad \text{for } 1\le i\le n-1,\\ z_{2i-1} &\mapsto t^{2i-2}(a/t)^{1/2} \quad \text{for } 1\le i\le n,\\ b_1 &\mapsto q^N(a/t)^{1/2},\\ b_2 &\mapsto (a/t)^{1/2}/b,\\ a_1 &\mapsto t^{2-2n}(t/\hat{a})^{1/2},\\ a_2 &\mapsto a(t/a)^{1/2}/c,\\ z_{2n} &\mapsto (a/t)^{1/2}/\hat{a},\end{aligned}$$

followed by $t^2 \mapsto 1/t$, the summand contains the term

$$\prod_{i=1}^{n-1} \frac{(1)_{\lambda_{2i}+\lambda_{2i+1}}}{(q)_{\lambda_{2i-1}+\lambda_{2i}}} \prod_{i=1}^{n-1} \frac{(q^{-N})_{\lambda_1}}{(q)_{\lambda_{2n-1}}}.$$

Hence it vanishes unless

$$N \geq \lambda_1 \geq -\lambda_2 \geq \lambda_3 \geq \cdots \geq -\lambda_{2n-2} \geq \lambda_{2n-1} \geq 0.$$

If we now relabel $\lambda_{2i} \mapsto -\mu_i$ for $1 \leq i \leq n-1$ followed by $\lambda_{2i-1} \mapsto \lambda_i$ for $1 \leq i \leq n$, we obtain the following new ${}_6\phi_5$ summation. For $m \leq n$ and $\lambda, \mu \in \Lambda$ such that $\mu_{m+1} = \cdots = \mu_n = 0$, let

$$\begin{aligned}
\Delta^{nm}_{\lambda\mu}(a, \hat{a}) &= \Delta^{nm}_{\lambda\mu}(a, \hat{a}; q, t) \\
&:= \prod_{1\leq i<j\leq n} \frac{1 - t^{j-i} q^{\lambda_i - \lambda_j}}{1 - t^{j-i}} \frac{(at^{3-i-j})_{\lambda_i+\lambda_j}}{(aqt^{2-i-j})_{\lambda_i+\lambda_j}} \\
&\quad \times \prod_{1\leq i<j\leq m} \frac{1 - t^{j-i} q^{\mu_i - \mu_j}}{1 - t^{j-i}} \frac{(at^{2-i-j})_{\mu_i+\mu_j}}{(aqt^{1-i-j})_{\mu_i+\mu_j}} \\
&\quad \times \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{1 - at^{2-i-j} q^{\lambda_i + \mu_j}}{1 - at^{2-i-j}} \frac{(t^{j-i+1})_{\lambda_i - \mu_j}}{(qt^{j-i})_{\lambda_i - \mu_j}} \\
&\quad \times \prod_{i=1}^{n} \frac{1 - \hat{a}t^{1-i} q^{\lambda_i + |\lambda-\mu|}}{1 - \hat{a}t^{1-i}} \frac{(at^{2-i}/\hat{a})_{\lambda_i - |\lambda-\mu|}}{(aqt^{1-i}/\hat{a})_{\lambda_i - |\lambda-\mu|}} \\
&\quad \times \prod_{i=1}^{m} \frac{1 - at^{1-i} q^{\mu_i - |\lambda-\mu|}/\hat{a}}{1 - at^{1-i}/\hat{a}} \frac{(\hat{a}t^{1-i})_{\mu_i + |\lambda-\mu|}}{(\hat{a}qt^{-i})_{\mu_i + |\lambda-\mu|}}.
\end{aligned}$$

Corollary 6.1 (Type II A_{2n-1} ${}_6\phi_5$ sum) *For N a nonnegative integer,*

$$\begin{aligned}
&\sum \Delta^{n,n-1}_{\lambda\mu}(a, \hat{a}) \frac{(b, q^{-N})_\lambda}{(aq/b, aq^{N+1})_\mu} \frac{(at^{1-n}, c)_\mu}{(qt^{n-1}, aq/c)_\lambda} \frac{(\hat{a}t^{1-n}, \hat{a}c/a)_{|\lambda-\mu|}}{(\hat{a}q/b, \hat{a}q^{N+1})_{|\lambda-\mu|}} \\
&\quad \times \left(\frac{aq^{N+1}}{bc}\right)^{|\lambda|} q^{n(\lambda)+n(\mu)-(n-1)|\mu|} \\
&= \frac{(aq/bc)_{(N^n)}}{(aq/c)_{(N^n)}} \frac{(aq)_{(N^{n-1})}}{(aq/b)_{(N^{n-1})}} \frac{(\hat{a}q)_N}{(\hat{a}q/b)_N},
\end{aligned}$$

where the sum is over $\lambda, \mu \in \Lambda_N$ such that $\mu_n = 0$ and $\mu \preccurlyeq \lambda$.

Remarkably, the above identity contains van Diejen's ${}_6\phi_5$ sum as a special case, establishing

$$\text{type II } \mathrm{C}_n \ {}_6\phi_5 \text{ sum} \quad \hookrightarrow \quad \text{type II } \mathrm{A}_{2n-1} \ {}_6\phi_5 \text{ sum}.$$

Specifically, if we fix $\hat{a}=at^{1-n}$, then $\Delta_{\lambda\mu}^{n,n-1}(a,\hat{a})$ contains the factor

$$\frac{1}{(q)_{\lambda_n-|\lambda-\mu|}},$$

which implies that it vanishes unless $\mu_i=\lambda_i$ for $1\le i\le n-1$. Assuming that μ is fixed in this manner (so that $|\lambda-\mu|=\lambda_n$), it takes a routine calculation to show that

$$\Delta_{\lambda\mu}^{n,n-1}\big(a,at^{1-n}\big)=\left(\frac{t}{q}\right)^{2n(\lambda)-(n-1)|\lambda|}\Delta_\lambda(a).$$

It is now easily seen that Corollary 6.1 reduces to (6.1).

In view of the above result and our previous discussion of elliptic hypergeometric series, it takes little imagination to make the following conjecture (which has been extensively checked for small values of n and N). Let $(a)_n$ again represent the elliptic shifted factorial (6.2), and for $m\le n$ and $\lambda,\mu\in\Lambda$ such that $\mu_m=\cdots=\mu_n=0$, let

$$\begin{aligned}
\Delta_{\lambda\mu}^{nm}(a,\hat{a}) &= \Delta_{\lambda\mu}^{nm}(a,\hat{a};q,t;p)\\
&:= \prod_{1\le i<j\le n}\frac{\theta(t^{j-i}q^{\lambda_i-\lambda_j})}{\theta(t^{j-i})}\,\frac{(at^{3-i-j})_{\lambda_i+\lambda_j}}{(aqt^{2-i-j})_{\lambda_i+\lambda_j}}\\
&\quad\times\prod_{1\le i<j\le m}\frac{\theta(t^{j-i}q^{\mu_i-\mu_j})}{\theta(t^{j-i})}\,\frac{(at^{2-i-j})_{\mu_i+\mu_j}}{(aqt^{1-i-j})_{\mu_i+\mu_j}}\\
&\quad\times\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{\theta(at^{2-i-j}q^{\lambda_i+\mu_j})}{\theta(at^{2-i-j})}\,\frac{(t^{j-i+1})_{\lambda_i-\mu_j}}{(qt^{j-i})_{\lambda_i-\mu_j}}\\
&\quad\times\prod_{i=1}^{n}\frac{\theta(\hat{a}t^{1-i}q^{\lambda_i+|\lambda-\mu|})}{\theta(\hat{a}t^{1-i})}\,\frac{(at^{2-i}/\hat{a})_{\lambda_i-|\lambda-\mu|}}{(aqt^{1-i}/\hat{a})_{\lambda_i-|\lambda-\mu|}}\\
&\quad\times\prod_{i=1}^{m}\frac{\theta(at^{1-i}q^{\mu_i-|\lambda-\mu|}/\hat{a})}{\theta(at^{1-i}/\hat{a})}\,\frac{(\hat{a}t^{1-i})_{\mu_i+|\lambda-\mu|}}{(\hat{a}qt^{-i})_{\mu_i+|\lambda-\mu|}}.
\end{aligned}$$

Use this to define the new type II elliptic hypergeometric series

$$\begin{aligned}
&V(a,\hat{a};b_1,\dots,b_{r+1};c_1,\dots,c_r)\\
&\quad= V(a,\hat{a};b_1,\dots,b_{r+1};c_1,\dots,c_r;q,t;p)\\
&\quad:= \sum\Delta_{\lambda\mu}^{n,n-1}(a,\hat{a})\frac{(b_1,\dots,b_{r+1})_\lambda}{(aq/b_1,\dots,aq/b_{r+1})_\mu}\,\frac{(at^{1-n},c_1,\dots,c_r)_\mu}{(qt^{n-1},aq/c_1,\dots,aq/c_r)_\lambda}\\
&\qquad\times\frac{(\hat{a}t^{1-n},\hat{a}c_1/a,\dots,\hat{a}c_r/a)_{|\lambda-\mu|}}{(\hat{a}q/b_1,\dots,\hat{a}q/b_{r+1})_{|\lambda-\mu|}}\big(qt^{n-1}\big)^{|\lambda|}q^{n(\lambda)+n(\mu)-(n-1)|\mu|},
\end{aligned}$$

where one of the b_i is of the form q^{-N}, the balancing condition

$$b_1\cdots b_{r+1}c_1\cdots c_rt^{n-1}=a^rq^{r-1}$$

holds, and where the sum is over $\lambda, \mu \in \Lambda_N$ such that $\mu_n = 0$ and $\mu \preccurlyeq \lambda$.

Conjecture 6.2 (Type II A_{2n-1} elliptic ${}_8\phi_7$ sum) *Assuming the balancing condition*

$$bcdet^{n-1} = a^2 q^{N+1},$$

we have

$$\begin{aligned} &V\big(a, \hat{a}; b, c, q^{-N}; d, e\big) \\ &\quad = \frac{(aq/bd, aq/cd)_{(N^n)}}{(aq/d, aq/bcd)_{(N^n)}} \frac{(aq, aq/bc)_{(N^{n-1})}}{(aq/b, aq/c)_{(N^{n-1})}} \frac{(\hat{a}q, \hat{a}q/bc)_N}{(\hat{a}q/b, \hat{a}q/c)_N}. \end{aligned} \tag{6.5}$$

For $n = 1$ this again corresponds to the elliptic Frenkel and Turaev sum [8], and for general n and $(a, \hat{a}) = (a, at^{1-n})$ it reduces to (6.4).

Conjecture 6.3 *Conjecture 6.2 follows from the elliptic type II A_{2n-1} beta integral of* [30] *by an appropriate residue calculus.*

The preceding manipulations can be repeated in the A_{2n} case. That is, if in Claim 5.2 we specialise

$$\begin{aligned} z_{2i} &\mapsto t^{1-2i}(t/a)^{1/2}, \quad 1 \le i \le n, \\ z_{2i-1} &\mapsto t^{2i-2}(a/t)^{1/2}, \quad 1 \le i \le n, \\ c_1 &\mapsto q^N (a/t)^{1/2}, \\ c_2 &\mapsto t^{1-2n}(t/a)^{1/2}, \\ d_1 &\mapsto t(a/t)^{1/2}/b, \\ d_2 &\mapsto t(a/t)^{1/2}/c, \\ z_{2n+1} &\mapsto (a/t)^{1/2}/\hat{a}, \end{aligned}$$

and finally make the substitution $t^2 \mapsto 1/t$, we obtain by a similar reasoning as before the following companion of Corollary 6.1.

Corollary 6.4 (Type II A_{2n} ${}_6\phi_5$ sum) *For N a nonnegative integer,*

$$\begin{aligned} &\sum \Delta^{nn}_{\lambda\mu}(a, \hat{a}) \frac{(at^{1-n}, q^{-N})_\lambda}{(qt^{n-1}, aq^{N+1})_\mu} \frac{(b, c)_\mu}{(aq/b, aq/c)_\lambda} \\ &\qquad \times \frac{(\hat{a}b/a, \hat{a}c/a)_{|\lambda-\mu|}}{(\hat{a}qt^{n-1}/a, \hat{a}q^{N+1})_{|\lambda-\mu|}} \left(\frac{aq^{N+2}}{bct}\right)^{|\lambda|} q^{n(\lambda)+n(\mu)-n|\mu|} \\ &\quad = \frac{(aq/bct, aq)_{(N^n)}}{(aq/b, aq/c)_{(N^n)}} \frac{(\hat{a}q)_N}{(\hat{a}q/t^n)_N}, \end{aligned}$$

where the sum is over $\lambda, \mu \in \Lambda_N$ such that $\mu \preccurlyeq \lambda$.

Once again (6.1) arises through an appropriate specialisation, so that now

$$\text{type II } \mathrm{C}_n \ {}_6\phi_5 \text{ sum} \quad \hookrightarrow \quad \text{type II } \mathrm{A}_{2n} \ {}_6\phi_5 \text{ sum.}$$

To be more precise, $\Delta^{nn}_{\lambda\mu}(a/t,a)$ contains the factor $(1)_{\lambda_1-|\lambda-\mu|}$. Since

$$\lambda_1-|\lambda-\mu|=(\mu_1-\lambda_2)+\cdots+(\mu_{n-1}-\lambda_n)+\mu_n$$

and $\mu_i\geq\lambda_{i+1}$ (recall that $\mu\preccurlyeq\lambda$), $\Delta^{nn}_{\lambda\mu}(a/t,a)$ vanishes unless $\mu_i=\lambda_{i+1}$ for $1\leq i\leq n-1$ and $\mu_n=0$. But for such μ,

$$\Delta^{nn}_{\lambda\mu}(a/t,a)=\left(\frac{t}{q}\right)^{2n(\lambda)-(n-1)|\lambda|}q^{|\lambda|-(n+1)\lambda_1}\frac{(qt^n,at^{1-n})_\lambda}{(qt^{n-1},at^{-n})_\lambda}\,\Delta_\lambda(a).$$

It thus follows that Corollary 6.4 reduces to (6.1) after the substitution $(a,\hat{a},b,c)\mapsto(a/t,a,b/t,c/t)$.

Conjecturally Corollary 6.4 again admits an elliptic generalisation. To state this, we define

$$\begin{aligned}
&V(a,\hat{a};b_1,\dots,b_r;c_1,\dots,c_{r+1})\\
&\quad=V(a,\hat{a};b_1,\dots,b_r;c_1,\dots,c_{r+1};q,t;p)\\
&\quad:=\sum\Delta^{nn}_{\lambda\mu}(a,\hat{a})\frac{(at^{1-n},b_1,\dots,b_r)_\lambda}{(qt^{n-1},aq/b_1,\dots,aq/b_r)_\mu}\frac{(c_1,\dots,c_{r+1})_\mu}{(aq/c_1,\dots,aq/c_{r+1})_\lambda}\\
&\qquad\times\frac{(\hat{a}c_1/a,\dots,\hat{a}c_{r+1}/a)_{|\lambda-\mu|}}{(\hat{a}qt^{n-1}/a,\hat{a}q/b_1,\dots,\hat{a}q/b_r)_{|\lambda-\mu|}}\big(q^2t^{n-1}\big)^{|\lambda|}q^{n(\lambda)+n(\mu)-n|\mu|},
\end{aligned}$$

where one of the b_i is of the form q^{-N}, the balancing condition is

$$b_1\cdots b_rc_1\cdots c_{r+1}t^n=a^rq^{r-1},$$

and where the sum is over $\lambda,\mu\in\Lambda_N$ such that $\mu\preccurlyeq\lambda$.

Conjecture 6.5 (Type II elliptic A_{2n} ${}_8\phi_7$ sum) *Assuming the balancing condition*

$$bcdet^n=a^2q^{N+1},$$

we have

$$V\big(a,\hat{a};b,q^{-N};c,d,e\big)=\frac{(aq,aq/bc,aq/bd,aq/cdt)_{(N^n)}}{(aq/b,aq/c,aq/d,aq/bcdt)_{(N^n)}}\frac{(\hat{a}q,\hat{a}q/bt^n)_N}{(\hat{a}q/t^n,\hat{a}q/b)_N}.$$

If we substitute $(a,\hat{a},c,d,e)\mapsto(a/t,a,c/t,d/t,e/t)$, this again simplifies to the elliptic C_n sum (6.4).

Conjecture 6.6 *Conjecture 6.5 follows from the elliptic type II A_{2n} beta integral of* [30] *by an appropriate residue calculus.*

The sum (6.4) serves as a normalisation of the weight function for Rains' BC_n abelian biorthogonal functions [24, 25] for specifically fixed discrete values of the arguments. It is natural to expect that our two V-function sums conjectured above have similar interpretation for more general biorthogonal functions attached to the root systems A_{2n-1} and A_{2n}. We hope to present a more detailed study of the orthogonal polynomials associated with the sums (6.1) and (6.4), and of the abelian biorthogonal functions based on the type II A_n elliptic beta integrals of [30] in future publications.

7 Further applications of $\mathrm{A}_{n-1}\ {}_6\psi_6$ summations

In this last section we make some final remarks regarding $\mathrm{A}_{n-1}\ {}_6\psi_6$ summations. Such summations contain a sum over $\lambda \in \mathbb{Z}^n$ subject to the restriction $|\lambda| = 0$. It is trivial to lift this restriction to $|\lambda| = M$ for $M \in \mathbb{Z}$ simply be replacing $\lambda_1 \mapsto \lambda_1 - M$ and $z_1 \mapsto z_1 q^M$. For example, in the case of (1.7) one obtains [10]

$$\sum_{\substack{\lambda\in\mathbb{Z}^n\\|\lambda|=M}} \frac{\Delta(zq^\lambda)}{\Delta(z)} \prod_{i,j=1}^n \frac{(z_j/b_i)_{\lambda_j}}{(qa_iz_j)_{\lambda_j}} = \frac{(Z/B)_M}{(qAZ)_M} \frac{(qAZ, qB/Z)_\infty}{(q, qAB)_\infty} \prod_{i,j=1}^n \frac{(qa_ib_j, qz_i/z_j)_\infty}{(qa_iz_j, qb_i/z_j)_\infty},$$

where $A = a_1\cdots a_n$, $B = b_1\cdots b_n$, $Z = z_1\cdots z_n$ and $|qAB| < 1$. This implies the following useful lemma [23].

Lemma 7.1 *Provided that both sides converge*,

$$\begin{aligned}
&\sum_{\lambda\in\mathbb{Z}^n} f_{|\lambda|} \frac{\Delta(zq^\lambda)}{\Delta(z)} \prod_{i,j=1}^n \frac{(z_j/b_i)_{\lambda_j}}{(qa_iz_j)_{\lambda_j}} \\
&\quad = \frac{(qAZ, qB/Z)_\infty}{(q, qAB)_\infty} \prod_{i,j=1}^n \frac{(qa_ib_j, qz_i/z_j)_\infty}{(qa_iz_j, qb_i/z_j)_\infty} \sum_{M=-\infty}^{\infty} f_M \frac{(Z/B)_M}{(qAZ)_M}.
\end{aligned}$$

For example, taking $f_k = t^k$, the sum on the right can be performed using Ramanujan's ${}_1\psi_1$ sum, resulting in a multivariable ${}_1\psi_1$ sum, see [10, 23].

In much the same way, Claims 5.1 and 5.2 imply the following lemma.

Lemma 7.2 *With the same notation as* (5.1),

$$\begin{aligned}
&\sum_{\lambda\in\mathbb{Z}^n} f_{|\lambda|} \frac{\Delta(zq^\lambda)}{\Delta(z)} \prod_{1\le i<j\le n} \frac{(z_iz_j/t)_{\lambda_i+\lambda_j}}{(qtz_iz_j)_{\lambda_i+\lambda_j}} \prod_{i=1}^2\prod_{j=1}^n \frac{(z_j/b_i)_{\lambda_j}}{(qa_iz_j)_{\lambda_j}} \\
&\quad = (q)_\infty^{n-1} \big(qAt^mZ, qBt^m/Z, q\hat{A}t^{n-m-1}Z, q\hat{B}t^{n-m-1}/Z\big)_\infty \\
&\qquad \times \prod_{i=1}^{n-m-1} \big(qt^{2i}, qa_1a_2t^{2i-1}, qb_1b_2t^{2i-1}\big)_\infty \prod_{i=1}^{m} \frac{\prod_{j,k=1}^2 (qa_jb_kt^{2i-2})_\infty}{(qa_1a_2b_1b_2t^{2i+2n-2m-4})_\infty}
\end{aligned}$$

$$\times \prod_{i=1}^{2}\prod_{j=1}^{n} \frac{1}{(qa_i z_j, qb_i/z_j)_\infty} \prod_{1\le i<j\le n} \frac{(qz_i/z_j, qz_j/z_i)_\infty}{(qtz_i z_j, qt/z_i z_j)_\infty}$$

$$\times \sum_{M=-\infty}^{\infty} f_M \frac{(Zt^{-m}/B, t^{m-n+1}Z/\hat{B})_M}{(qAt^m Z, q\hat{A}t^{n-m-1}Z)_M},$$

provided that both sides converge.

For a number of different choices of f_k, the right-hand sides of the above two formulas become explicitly summable. We omit the details and instead refer the interested reader to [23] where applications of Lemma 7.1 are discussed.

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