

Integral: An easy approach after Kurzweil and Henstock. (English) [B] Australian Mathematical Society Lecture Series. 14. Cambridge: Cambridge University Press. xii, 311 p. \$ 39.95 (2000). [ISBN 0-521-77968-5/pbk]

As the authors note whereas the integral most commonly employed by mathematicians is the Lebesgue integral, due to its intuitive appeal and ease of presentation, the integral most familiar to non-specialists is the Riemann integral. However, there is an integral discovered by J. Kurzweil and R. Henstock and called the Kurzweil-Henstock (KH) integral in this text that is a slight variant of the Riemann integral but which possesses the power of the Lebesgue integral. The authors argue – quite convincingly – that the KH integral is a good candidate for the non-mathematician or undergraduate/beginning graduate student to study instead of the classical Riemann integral.

A function $f : [a, b] \rightarrow R$ is Riemann integrable if there exists A such that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\sum_{i=1}^n f(t_i)(x_{i+1} - x_i) - A| < \varepsilon$ whenever $a = x_1 < x_2 < \dots < x_n = b$, $x_i \leq t_i \leq x_{i+1}$ and $x_{i+1} - x_i < \delta$. The KH integral essentially results from replacing the constant δ in the definition of the Riemann integral by a positive function $\delta : [a, b] \rightarrow (0, \infty)$. Although this may appear to be a trivial alteration, the effect on the resulting integration theory is profound, leading to an integral more general than the Lebesgue integral and equivalent to the Perron and Denjoy integrals.

In Chapters 2 and 3 the authors define and develop the basis properties of the KH integral [Chapter 1 contains a discussion of the Riemann integral which affords a comparison with the later development of the KH integral, but this chapter is independent of the following chapters]. In particular, it is shown that the Fundamental Theorem of Calculus holds for the KH integral in full generality; that is, in contrast to the Riemann or Lebesgue integrals, the KH integral integrates all derivatives. Moreover, there are no “improper integrals” for the KH integral (this, of course, implies that the KH integral is a “non-absolute integral”) and the Monotone and Dominated Convergence Theorems hold for the KH integral. All of these results are established using elementary methods, involving no measure theory except the notion of a null set. The last half of Chapter 3 uses the KH integral to introduce Lebesgue measure and contains more technical results including a descriptive characterization of the KH integral and comparisons with the McShane and Lebesgue integrals.

Chapter 6 contains a discussion of the KH integral the on n -dimensional Euclidean space including versions of the Fubini and Tonelli Theorems. Chapter 7 contains applications of the KH integral to line integrals, Green’s Theorem, differentiation of series, Dirichlet’s Problem, Fourier series and L^2 .

Chapters 4 and 5 contain more technical results for the KH integral including another descriptive characterization and a convergence theorem, called the Controlled Convergence Theorem, which is more general than the Dominated Convergence Theorem.

This text is very thorough and well-written. Chapter 2, the first half of Chapter 3, and Chapter 6, along with their abundant supply of exercises, would make an excellent text for an introduction to the KH integral. The remaining topics discussed in the text would make interesting reading for one interested in studying further properties of the KH integral.

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