REVIEW OF "THE INTEGRAL: AN EASY APPROACH AFTER KURZWEIL AND HENSTOCK"

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An embarrassing feature of Lebesgue's theory of integration is that a real valued function defined on an interval can have a continuous primitive defined everywhere, but the function itself is not integrable. For the most part, analysts today are willing to live with this deficiency in exchange for the fruitful properties of spaces of integrable functions.

The book under review starts with an enlightening historical discussion on this point. If $f : (a, b) \to \mathbb{R}$ is a function which has a *primitive*, that is, a continuous function $F : [a, b] \to \mathbb{R}$ such that F is differentiable on the open interval (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$, then we might define $\int_a^b f$ as the number $F(b) - F(a)$ and call $\int_a^b f$ the definite *Newton's integral* of f on $[a, b]$: this is just what we need to calculate integrals in elementary calculus courses. The continuous function $F : [0, 1] \to \mathbb{R}$ defined by $F(x) = x^2 \cos(\pi/x^2)$ for every $x \in (0, 1]$, is the primitive of $f = F'$, so the Newton integral of f on $[0, 1]$ exists and $\int_0^1 f = -1$. However, f is not Lebesgue integrable on $[0, 1]$ because $\int_0^1 |f| = +\infty$.

The significance of the integral for the calculation of areas and volumes, for example, comes from its interpretation as a limit of sums and the Newton integral often facilitates the easy calculation of these limits. As long as one is only interested in integrating bounded and continuous functions defined on an interval, then Riemann's definition of the integral is enough to establish the connection between the integral as a limit of sums and primitive functions. The (first) Fundamental Theorem of Calculus can be paraphrased by saying that the Newton and Riemann definite integrals of a bounded and continuous function $f : (a, b) \to \mathbb{R}$ both exist and agree in value.

In 1957 J. Kurzweil [6] and independently, R. Henstock [2] in 1961, showed that by modifying Riemann's definition of the integral, it is possible to obtain an integral whose domain of definition contains the class of Newton integrable functions and the class of Lebesgue

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integrable functions and the values of the integrals agree on their common domains. After Lebesgue, other approaches to integration were proposed, see [9] for example. The Kurzweil-Henstock definition (the KH-integral) coincides with the integral introduced by O. Perron in 1914 by a different method. As with the KH-integral, all these integrals have the characteristic property that the absolute value $|f|$ of a function f need not be integrable, even if f is integrable.

The appeal of the Kurzweil-Henstock integral is that it is simple enough to be the first definition of the integral that a motivated undergraduate student is exposed to. Notice the word *motivated*: for most undergraduate mathematics students, we are overjoyed (and pass them) if they can calculate simple primitive functions, let alone have a cogent understanding of why they are doing it.

Despite the ever-diminishing intended audience, the authors do an excellent job of presenting their material. The book is written with the same clarity and enthusiasm that the reviewer remembers from his first exposure to university mathematics by one of the authors (R.V.). Chapter 1 starts with a resumé of the Riemann-Darboux definition of the integral. A number of carefully worked examples illustrate the limitations of the theory. The KH-integral is presented in Chapter 2 and invites comparison with the Riemann integral of Chapter 1. Here the basic properties of integral calculus for the KH-integral are proved. Chapter 3 establishes the connection with the Perron integral and the Lebesgue integral (for which f and $|f|$ are KH-integrable) and proofs of the monotone and dominated convergence theorems for the KH-integral are given. Along the way, the classical function properties of bounded variation and absolute continuity are examined. The connection with McShane's approach to the Lebesgue integral is made. Chapters 4 and 5 are concerned with more technical extensions of the convergence theory. Integration in several dimensions—Fubini's theorem, change of variables—is tackled in Chapter 6. Applications to line integrals, the Cauchy Theorem, Dirichlet's problem and Fourier series are given in Chapter 7.

The book works on several levels. Parts of Chapters 1 and 2 could be used for a first course on integration. Chapters 3, 6 and parts of Chapter 7 could be used in place of a first course in Lebesgue theory. Because the Lebesgue theory is touched on in Sections 3.11-3.13, there would be no gap in a potential student's knowledge, only a different emphasis. Chapters

4 and 5 require more mathematical sophistication and could be used for a course at the beginning graduate level. A collection of signposted exercises flesh out the material.

As far as a course in (absolute) integration goes, the book deals with Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^n$ where intervals and rectangles form a distinguished family of sets in the definition of the KH-integral. A supplementary book would be needed to treat abstract measures and probability theory for use in, say, the current growth area of financial mathematics. Some work has been done on treating integrals with respect to Wiener measure and the Feynman integral from the Kurzweil-Henstock viewpoint [8]. In addition to the books on Kurzweil-Henstock integration mentioned in the bibliography, [11] has appeared and R. Bartle has promised a monograph, see [1].

I cannot resist mentioning another elementary approach to integration, with an Australian connection, inspired not by Riemann, but by Archimedes' calculation of the area of a section of the parabola [3],[4]. Textbooks along these lines have appeared [7],[10] and [5] shows how to treat some forms of divergent integrals and sums.

References

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