STABILITY OF A CLASS OF DIFFERENTIAL INCLUSIONS

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ABSTRACT. Stability of differential inclusions $\dot{x} \in \mathbb{F}(x(t))$ is studied by using minorant and majorant mappings F^- and F^+, $F^-(x) \leq \mathbb{F}(x) \leq F^+(x)$. Properties of F^-, F^+ are developed in terms of partial orderings, with the condition that F^-, F^+ are either heterotone or pseudoconcave. The main results concern asymptotically stable absorbing sets, including the case of a single equilibrium point, and are illustrated by examples of control systems.
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1. Introduction

Complete description of processes involved in continuous time systems is frequently unavailable in control design. Instead, models are sought which incorporate uncertainties in the dynamical system. One class of such models are differential inclusions of the form

$$\dot{x} \in \mathbb{F}(x), \qquad x \in \mathbb{R}^n, \tag{1}$$

where $\mathbb{F}$ is a setvalued mapping [1, 2, 3, 5]. A solution is an absolutely continuous function $x(t)$ satisfying (1) almost everywhere.

Commonly, $\mathbb{F}$ is described by a set of selections $F(\cdot, \lambda)$,

$$\mathbb{F}(\cdot) = \{F(\cdot, \lambda) : \ \lambda \in \Lambda\}. \tag{2}$$

Thus every trajectory of (1) is defined by a differential equation

$$\dot{x} = F(x(t), \lambda(t)), \tag{3}$$

for some function $\lambda(t) \in \Lambda$. This is a well-known description of many problems in control theory, see for example [3] and the references therein.

The stability of such systems has been widely studied: see for example [1, 2, 3, 4, 7, 8, 14, 15]. The principal emphasis of this paper is the exploitation of properties expressed in terms of partial orderings of the state space. In particular, we study systems (1) for which there exist a minorant F^- and a majorant F^+, both either heterotone or pseudoconcave (cf. Section 3), such that $F^-(x) \leq \mathbb{F}(x) \leq F^+(x)$. The stability of (1) is shown to depend on properties of differential equations involving

F^- and F^+. Definitions and notation are given in the next section and the main results on asymptotic stability of fixed points and absorbing sets are developed in Sections 3 and 4, and are illustrated by control examples in Section 5.

2. Preliminaries

Partial ordering. Partially order $\mathbb{R}^n$ by the nonnegative orthant $\mathbb{R}^n_+$. That is, $x \geq y$ if and only if $x - y \in \mathbb{R}^n_+$, and write, equivalently, $y \leq x$. So $x \geq y$ is equivalent to the n coordinate inequalities, $x_i \geq y_i$. If all $x_i > y_i$, write $x > y$.

A conic interval $\langle v, w \rangle$ is the set of elements x such that $v \leq x \leq w$. An element z is said to be an upper bound for a set $M \subset \mathbb{R}^n$ if $z \geq x$ for all $x \in M$. An upper bound $z_0 = \sup M$ is called the least upper bound if any other upper bound z satisfies $z_0 \leq z$. The notions of lower and greatest lower bound, $z_0 = \inf M$, are defined in a similar way.

***H* and *H*$^+$-mappings.** A mapping F from $\mathbb{R}^n$ to itself is said to be, respectively

- *heterotone, or an H-mapping* ([13]), on a set $M \subset \mathbb{R}^n$ if there exists an associated mapping $\widetilde{F} : M \times M \to M$ such that $\widetilde{F}(x, x) = F(x)$ for all $x \in M$ and $\widetilde{F}(\cdot, w)$ is monotone increasing for every fixed $w \in M$ and $\widetilde{F}(v, \cdot)$ is monotone decreasing for every fixed $v \in M$.
- *an H^+-mapping,* on a set $M \subset \mathbb{R}^n$ if there exists an associated mapping $\widehat{F} : M \times M \times M \to M$ such that $\widehat{F}(x, x; x) = F(x)$ for all $x \in M$ and the i-th component function, $i = 1, 2, \ldots, n$, $\widehat{F}_i(v, w; u) = \widehat{F}_i(v, w; u_i)$ of $\widehat{F}(v, w; u)$ depends in the third argument only on the i-th component of the vector u, and is monotone increasing in v for any fixed w, $u_i \in M$ and monotone decreasing in w for any fixed v, $u_i \in M$.

A mapping that is heterotone on M will simply be called heterotone where no ambiguity can arise.

An H-mapping F is said to be *unified* if the equations

$$\widetilde{F}(v, w) = v, \quad \widetilde{F}(w, v) = w$$

have no solution such that $v \leq w$, $v \neq w$.

An H^+-mapping F is said to be, respectively,

- *positively reflective* if all component functions of the associated mapping $\widehat{F}$ satisfy the inequalities $\widehat{F}_i(v, w; 0) > 0$ for any $v, w \geq 0$.
- *unified* if the system of equations

$$\widehat{F}(v, w; v) = 0, \quad \widehat{F}(w, v; w) = 0$$

have no solution such that $v \leq w,\ v \neq w$.

- *solvable* on M if for any $v, w \in M$ there exists $u \in M$ such that $\widehat{F}(v, w; u) = 0$

A conic interval $\langle v, w\rangle$ is said to be *strongly invariant* with respect to a heterotone mapping F if $\widetilde{F}(v, w) \geq v$ and $\widetilde{F}(w, v) \leq w$. Strong invariance of $\langle v, w\rangle$ implies invariance of $\langle v, w\rangle$, since

$$v \leq \widetilde{F}(v, w) \leq \widetilde{F}(x, x) = F(x) \leq \widetilde{F}(w, v) \leq w$$

for any $x \in \langle v, w\rangle$. Similarly, a conic interval $\langle v, w\rangle$ is said to be strongly invariant with respect to an H^+-mapping F if both $\widehat{F}(v, w; v) \geq 0$ and $\widehat{F}(w, v; w) \leq 0$.

If F is heterotone on R^n_+, it is said to be *pseudoconcave*, (respectively, ν*-pseudoconcave*) if $\widetilde{F}(v, w) > 0$ for all nonzero $v, w \geq 0$ and for all $v, w > 0$ and any $\tau \in (0, 1)$,

$$\widetilde{F}\left(\tau v,\, \frac{1}{\tau} w\right) > \tau \widetilde{F}(v, w), \tag{4}$$

(respectively,

$$\widetilde{F}\left(\tau v,\, \frac{1}{\tau} w\right) > \tau^{\nu} \widetilde{F}(v, w), \quad \nu \in (0, 1), \tag{5}$$

for ν-pseudoconcavity).

Let an H^+-mapping be positively reflective and solvable on $\operatorname{int} \mathbb{R}^n_+$. Let $a,\ b \in \mathbb{R}^n$ be solutions of

$$\widehat{F}(v, w; a) = 0, \quad \widehat{F}(\tau v, \frac{1}{\tau} w; b) = 0. \tag{6}$$

If for any $v, w > 0$, $\tau \in (0, 1)$, it follows from (6) that $b > \tau a$, F is said to be H^+*-pseudoconcave.* If, on the other hand, $b > \tau^{\nu} a$, $\nu \in (0, 1)$, F is *strongly* ν*-pseudoconcave.*

Note that if F is heterotone, then $G(x) = F(x) - x$ is an H^+-mapping, while if F is pseudoconcave, then G is H^+-pseudoconcave.

The theory of heterotone and pseudoconcave mappings has been developed in [13], and for self-containment the following are restated:

Lemma 1. *Suppose that mappings F and G are heterotone (respectively, pseudoconcave, ν-pseudoconcave). Then so also are the sum $F + G$ and the composition FG.*

Define the Birkhoff metric [11] in $\operatorname{int} \mathbb{R}^n_+$ by

$$\rho(x, y) = \min\{\alpha :\ e^{-\alpha} y \leq x \leq e^{\alpha} y\}, \tag{7}$$

for all $x, y > 0$.

Lemma 2. *A ν-pseudoconcave H-mapping is a contraction mapping with constant ν in the Birkhoff metric.*

Note that $\left(\operatorname{int}\mathbb{R}^n_+, \rho\right)$ is a complete metric space.

Stability and absorption. In all that follows, it is assumed that the vector fields of differential equations satisfy conditions sufficient for existence of solutions, and for extension of solutions to all of $[0,\infty)$ (see for example [9]). That is, viable solutions [2] exist on $[0,\infty)$.

The set

$$S = \{z :\ \exists\lambda \in \Lambda :\ F(z,\lambda) = 0\}$$

is called a *set of equilibrium points* of the family (2). Note that S may depend on the particular representation (2). The point x^* is said to be a *common equilibrium point* of the family (2) if $F(x^*,\lambda) = 0$ for all $\lambda \in \Lambda$.

With respect to the inclusion (1), a bounded invariant set X is said to be:

- *Absorbing* if any trajectory of (1) converges to X, that is $\operatorname{dist}(x(t),X) \to 0$ as $t \to \infty$, where $\operatorname{dist}(x,X) = \inf_{\xi\in X}\|x-\xi\|$.
- *Stable* if for any neighborhood W of X there exists a neighborhood V of X such that $x(0) \in V$ implies that $x(t) \in W$ for any $t \geq 0$.
- *Asymptotically absorbing* if it is stable and absorbing.

When X is a singleton set $\{x^*\}$, these describe convergence of trajectories to an equilibrium, and stability and asymptotic stability of an equilibrium, respectively.

3. Dynamical systems with H^+-mappings

If F is an H^+-mapping, many properties of the differential equation

$$\dot{x} = F(x), \tag{8}$$

can be found by considering an associated system

$$\dot{v} = \widehat{F}(v,w;v), \quad \dot{w} = \widehat{F}(w,v;w). \tag{9}$$

Under natural conditions, H-mappings have fixed points without requiring continuity [13]. There is an analog for H^+-mappings.

Lemma 3. *Let F be a unified H^+-mapping with a strongly invariant conic interval $\langle v^0,w^0\rangle$. Suppose the associated mapping $\widehat{F}$ is ℓ-Lipschitz on $\langle v^0,w^0\rangle$. Then the equation $F(x)=0$ has a solution $x^* \in \langle v^0,w^0\rangle$, giving an equilibrium of (8).*

Proof. Write Ω for the set of pairs (v, w) such that $v, w \in \langle v^0, w^0 \rangle$, $v \leq w$ and $\widehat{F}(v, w; v) \geq 0$, $\widehat{F}(w, v; w) \leq 0$. Since $\langle v^0, w^0 \rangle$ is strongly invariant, the set Ω is nonempty. Let Ω^- denote the set of points $v \in \langle v^0, w^0 \rangle$ such that there exists $w \in \langle v^0, w^0 \rangle$ such that $(v, w) \in \Omega$. Similarly, let Ω^+ denote the set of points $w \in \langle v^0, w^0 \rangle$ such that there exists $v \in \langle v^0, w^0 \rangle$ such that $(v, w) \in \Omega$. Then

$$(v^*, w^*) = (\sup \Omega^-, \inf \Omega^+), \tag{10}$$

is a solution of the equation $F(x) = 0$. To see this, first note that the mapping $x + \alpha F(x)$ is heterotone if $\alpha > 0$ satisfies $\alpha \ell < 1$, whence for any $(v, w) \in \Omega$ it follows that

$$v^* + \alpha \widehat{F}(v^*, w^*; v^*) \geq v + \alpha \widehat{F}(v, w; v) \geq v.$$

Thus, $v^* + \alpha \widehat{F}(v^*, w^*; v^*)$ is an upper bound for Ω^-. Therefore $v^* + \alpha \widehat{F}(v^*, w^*; v^*) \geq v^*$. Consequently, $\widehat{F}(v^*, w^*; v^*) \geq 0$. In similar fashion, $\widehat{F}(w^*, v^*; w^*) \leq 0$. On the other hand, setting $G(\xi, \eta) = \xi + \alpha \widehat{F}(\xi, \eta; \xi)$, since the mapping $x + \alpha F(x)$ is heterotone, it follows that G is nondecreasing in ξ and nonincreasing in η. Hence,

$$v = G(v^*, w^*) \leq G(w^*, w^*) \leq G(w^*, v^*) = w,$$

$$G(v, w) \geq G(v^*, w^*) = v, \quad G(w, v) \leq G(w^*, v^*) = w,$$

and $\langle v, w \rangle$ is strongly invariant. Consequently, $(v, w) \in \Omega$, and $v \geq v^*$, $w \leq w^*$ contradicts the extremal properties of v^*, w^*. So it follows that $\widehat{F}(v^*, w^*; v^*) = 0$ and $\widehat{F}(w^*, v^*; w^*) = 0$. Since F is unified, $v^* = w^*$ and so $0 = \widehat{F}(v^*, v^*; v^*) = F(v^*)$, take $x^* = v^*$ and the proof is complete. $\square$

In everything that follows below, all mappings are assumed to be continuous.

Lemma 4. [9]. *Suppose that to each $x_0 \in \mathbb{R}^n_+$ there corresponds a unique solution $x(t)$, $t \geq 0$, of the initial value problem*

$$\frac{dx_i}{dt} = h_i(t, x), \quad x_i(0) = x_0, \quad i = 1, \dots, n,$$

where the vector field (h_i) is continuous and satisfies

$$h_i(t, x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \geq 0,$$

for all $i = 1, \dots, n$, $t \geq 0$ and all nonnegative x_j. Then each solution $x(t)$ with initial value $x_0 \in \mathbb{R}^n_+$ is nonnegative, that is, $x(t) \in \mathbb{R}^n_+$ for all $t \geq 0$.

Lemma 5. *Suppose that a continuous H^+-mapping F has a strongly invariant conic interval $\langle v^0, w^0 \rangle$. Then F is solvable on $\langle v^0, w^0 \rangle$.*

Proof. For any $v, w \in \langle v^0, w^0\rangle$,

$$\begin{aligned}\widehat{F}(v,w;w^0) &\le \widehat{F}(w^0,v^0;w^0) \le 0,\\ \widehat{F}(v,w;v^0) &\ge \widehat{F}(v^0,w^0;v^0) \ge 0.\end{aligned}$$

Thus every component function $\widehat{F}_i(v,w;u_i)$ changes sign on $v_i^0 \le u_i \le w_i^0$ and is zero at some u_i^* in the interval, by the intermediate value theorem, giving $u^* = (u_i^*)$ as a zero of $\widehat{F}$. □

Theorem 1. *Suppose that a continuously differentiable H^+-mapping F is unified and has a strongly invariant conic interval $\langle v^0, w^0\rangle$. Then (8) has a unique equilibrium, which is asymptotically stable on $\langle v^0, w^0\rangle$.*

Proof. Existence of an equilibrium follows from Lemma 3. Write $x(t)$ for the solution of (8) with initial value $x(0) \in \langle v^0, w^0\rangle$ and $v(t), w(t)$ for the solution of (9) with initial values $v(0) = v^0$, $w(0) = w^0$. Let

$$\alpha(t) = x(t) - v(t), \quad \beta(t) = w(t) - x(t).$$

Clearly,

$$\begin{aligned}\dot{\alpha} &= \widehat{F}(v+\alpha, w-\beta; v+\alpha) - \widehat{F}(v,w;v),\\ \dot{\beta} &= \widehat{F}(w,v;w) - \widehat{F}(w-\beta, v+\alpha; w-\beta),\end{aligned} \tag{11}$$

which satisfies the requirements of Lemma 4. Therefore $\alpha(t) \ge 0$, $\beta(t) \ge 0$, whence

$$v(t) \le x(t) \le w(t) \quad \text{for all} \quad t \ge 0. \tag{12}$$

Now, letting $y = \widehat{F}(v,w;v)$, $z = -\widehat{F}(w,v;w)$, gives

$$\begin{aligned}\dot{y}_i &= \sum_{j=1}^n \frac{\partial \widehat{F}_i(v,w;v)}{\partial v_j} y_j - \sum_{j=1}^n \frac{\partial \widehat{F}_i(v,w;v)}{\partial w_j} z_j,\\ \dot{z}_i &= \sum_{j=1}^n \frac{\partial \widehat{F}_i(w,v;w)}{\partial w_j} z_j - \sum_{j=1}^n \frac{\partial \widehat{F}_i(w,v;w)}{\partial v_j} y_j.\end{aligned}$$

Applying Lemma 4 to the above system, obtain

$$\dot{v} = y(t) \ge 0, \quad \dot{w} = -z(t) \le 0.$$

Consequently, $v(t)$ is monotone increasing, $w(t)$ is monotone decreasing. Since both $v(t)$ and $w(t)$ are bounded, $v(t) \to v^*$, $w(t) \to w^*$. Now, F is unified, so $v^* = w^* = x^*$ and (12) gives the result. □

The H^+-mappings F and G are said to be *equivalent* on M if

$$\operatorname{sgn} \widehat{F}(v,w;v) = \operatorname{sgn} \widehat{G}(v,w;v), \tag{13}$$

for all $v, w \in M$, where $\operatorname{sgn} x$ denotes the vector $(\operatorname{sgn} x_1, \ldots, \operatorname{sgn} x_n)$. This induces equivalence classes on the set of all H^+-mappings. Let $\mathcal{C}_F$

denote the set of H^+-mappings equivalent to F, such that solutions of $\dot{x} = G(x)$, $G \in \mathcal{C}_F$, exist and can be extended on $(0, \infty)$.

Theorem 2. *Suppose that the setvalued mapping $\mathbb{F}$ has a representation (2), where Λ is compact, and that all selections $F(x, \lambda)$ lie in an equivalence class $\mathcal{C}_F$ of continuous H^+-mappings on $\langle v^0, w^0 \rangle$. Suppose there exists a unified $F(x, \lambda) \in \mathcal{C}_F$, with strongly invariant conic interval $\langle v^0, w^0 \rangle$. Then (1) has a unique common equilibrium, which is asymptotically stable in $\langle v^0, w^0 \rangle$.*

Proof. The proof follows along similar lines to that of Theorem 1 and will only be sketched. Take a trajectory $x(t)$ of (1), satisfying $\dot{x} = F(x(t), \lambda(t))$ for some $\lambda(t) \in \Lambda$, that is, $\dot{x} = F(x, t)$, where $F(\cdot, t) \in \mathcal{C}_F$ for any $t \geq 0$. Following the technique of Theorem 1, instead of (9) consider

$$\dot{v} = \widehat{F}(v, w; v, t) \quad \dot{w} = \widehat{F}(w, v; w, t).$$

Then replacing (11) is the nonautonomous system

$$\begin{aligned} \dot{\alpha} &= \widehat{F}(v + \alpha, w - \beta; v + \alpha, t) - \widehat{F}(v, w; v, t), \\ \dot{\beta} &= \widehat{F}(w, v; w, t) - \widehat{F}(w - \beta, v + \alpha; w - \beta, t), \end{aligned}$$

which satisfies Lemma 4. Proceeding in much the same way as in Theorem 1, obtain $\dot{v}(t) \geq 0, \quad \dot{w}(t) \leq 0$. It is not difficult to see that $v(t) \to v^*$, $w(t) \to w^*$, and so $v^* = w^* = x^*$ is a common equilibrium of (2), since Λ is compact and all selections $F(x, \lambda)$ are continuous. $\square$

4. Dynamical systems with pseudoconcave mappings

A straightforward verification gives the following lemma.

Lemma 6. *Suppose that an H^+-mapping F is H^+-pseudoconcave (respectively, strongly ν-pseudoconcave). Then on $\operatorname{int} \mathbb{R}^n_+$, F is equivalent to $G(x) - x$, where G is heterotone and pseudoconcave (resp. ν-pseudoconcave).*

Theorem 3. *Suppose that a continuous H^+-mapping F is H^+-pseudoconcave and that $F(x^*) = 0$ for some $x^* > 0$. Then x^* is a unique equilibrium in $\operatorname{int} \mathbb{R}^n_+$ for (8) and is asymptotically stable on $\operatorname{int} \mathbb{R}^n_+$. Moreover, if $\mathcal{X}_F \subseteq \mathcal{C}_F$ is compact, x^* is asymptotically stable on $\operatorname{int} \mathbb{R}^n_+$ for the differential inclusion $\dot{x} \in \mathcal{X}_F$.*

Proof. Note that for any $\tau \in (0, 1)$, if

$$\widehat{F}\left(\tau x^*, \frac{1}{\tau} x^*; u\right) = 0$$

then $u > \tau x^*$ from the definition of H^+-pseudoconcavity and so

$$\widehat{F}\left(\tau x^*, \frac{1}{\tau}x^*; \tau x^*\right) \geq 0.$$

Similarly,

$$\widehat{F}\left(\frac{1}{\tau}x^*, \tau x^*; \frac{1}{\tau}x^*\right) \leq 0.$$

Thus any conic interval $\langle \tau x^*, \frac{1}{\tau}x^* \rangle$, $\tau \in (0,1)$, is strongly invariant. To complete the proof it only remains to use Theorem 1 for (8) and Theorem 2 for $\dot{x} \in \mathfrak{X}_F$. $\square$

Note. When F is strongly ν-pseudoconcave, Lemma 6 and Lemma 2 give the existence of an equilibrium $x^* > 0$ and this assumption can thus be dropped for such mappings.

Example 1. Consider the system

$$\begin{aligned} \dot{x_1} &= \frac{x_1 + x_2}{x_1^2 x_2} - x_1, \\ \dot{x_2} &= \frac{(x_2+1) - x_2^5(x_1+3)}{x_2(7+\sin x_2)}, \end{aligned} \tag{14}$$

On $\operatorname{int}\mathbb{R}^2_+$, it can be shown by Lemma 6 that this is equivalent to the system

$$\begin{aligned} \dot{x_1} &= \sqrt[3]{\frac{x_1+x_2}{x_2}} - x_1, \\ \dot{x_2} &= \sqrt[5]{\frac{x_2+1}{x_1+3}} - x_2, \end{aligned}$$

which is of the form $\dot{x} = G(x) - x$, where $G(x)$ is ν-pseudoconcave, $\nu \in (2/3, 1)$. An appropriate choice of an associated mapping for $G(x) = (g_1(x), g_2(x))$ is $\widetilde{G}(u,v) = (\widetilde{g}_1, \widetilde{g}_2)$, where

$$\widetilde{g}_1(v,w) = \sqrt[3]{\frac{v_1+v_2}{w_2}}, \quad \widetilde{g}_2(v,w) = \sqrt[5]{\frac{v_2+1}{w_1+3}}.$$

From Theorem 3, the original system (14) has a unique asymptotically stable equilibrium on $\operatorname{int}\mathbb{R}^2_+$.

5. Asymptotically stable absorbing sets

The mappings F^- and F^+ are said to be a *minorant* and a *majorant* of $\mathbb{F}(x)$ on M, respectively, if

$$F^-(x) \leq \mathbb{F}(x) \leq F^+(x) \tag{15}$$

for all $x \in M$, where $M \subset \mathbb{R}^n$.

When F^- and F^+ are H^+-mappings or H^+-pseudoconcave, study of the inclusion (1) can be done via the system

$$\dot v = \widehat{F}^-(v,w;v), \quad \dot w = \widehat{F}^+(w,v;w). \tag{16}$$

Under natural conditions, any solution $x(t)$ of (1) satisfies $v(t) \le x(t) \le w(t)$, where $(v(t), w(t))$ is the solution to (16).

A conic interval $\langle v, w\rangle$ is said to be *strongly invariant*

- with respect to a pair of H-mappings (F^-, F^+) if

$$\widetilde{F}^-(v,w) \ge v, \quad \widetilde{F}^+(w,v) \le w. \tag{17}$$

- with respect to a pair of H^+-mappings (F^-, F^+) if

$$\widehat{F}^-(v,w;v) \ge 0, \quad \widehat{F}^+(w,v;w) \le 0. \tag{18}$$

A pair of H^+-mappings (F^-, F^+) is said to be *unified* on M if the system of equations

$$\widehat{F}^-(v,w;v) = 0, \quad \widehat{F}^+(w,v;w) = 0, \tag{19}$$

has no solution $v, w \in M$ such that $v \le w,\ v \ne w$.

Theorem 4. *Suppose that a pair of continuous H^+-mappings (F^-, F^+) is a minorant-majorant of the setvalued mapping $\mathbb{F}$. Let the conic interval $\langle v^0, w^0\rangle$ be strongly invariant with respect to (F^-, F^+). Then the solution $(v(t), w(t))$ of (16), with initial values $v(0) = v^0$, $w(0) = w^0$, converges $v(t) \to v^*$, $\quad w(t) \to w^*$ $\quad$ as $\quad t \to \infty$, where v^*, w^* is a solution of (19) on $\langle v^0, w^0\rangle$ with the following property: for any other solution v', w', $v^* \le v'$ and $w^* \ge w'$. Furthermore, any solution $x(t)$ of (1), with initial value $x(0) \in \langle v^0, w^0\rangle$, satisfies*

$$v(t) \le x(t) \le w(t), \quad t \ge 0, \tag{20}$$

and $\langle v^*, w^*\rangle$ is an asymptotically absorbing set on $\langle v^0, w^0\rangle$ for (1).

Proof. Let $x(t)$ be a solution of (1), that is

$$\dot x(t) = F(x,t), \quad F(x,t) \in \mathbb{F}(x),$$

with initial condition $x(0) \in \langle v^0, w^0\rangle$. Writing $\alpha(t) = x(t) - v(t)$, $\quad \beta(t) = w(t) - x(t)$, obtain

$$\begin{aligned}
\dot\alpha &= F(x,t) - \widehat{F}^-(x,x;x) + \widehat{F}^-(x,x;x) - \widehat{F}^-(x-\alpha, x+\beta; x-\alpha),\\
\dot\beta &= \widehat{F}^+(x+\beta, x-\alpha; x+\beta) - \widehat{F}^+(x,x;x) + \widehat{F}^+(x,x;x) - F(x,t),
\end{aligned}$$

which satisfies the conditions of Lemma 4, since (15) holds. Thus $\alpha(t) \ge 0$, $\beta(t) \ge 0$, and (20) follows. As in the proof of Theorem 1, let

$y = \widehat{F}^-(v, w; v)$, $z = -\widehat{F}^+(w, v; w)$ and observe that

$$\dot{y}_i = \sum_{j=1}^{n} \frac{\partial \widehat{F}_i^-(v, w; v)}{\partial v_j} y_j - \sum_{j=1}^{n} \frac{\partial \widehat{F}_i^-(v, w; v)}{\partial w_j} z_j,$$

$$\dot{z}_i = \sum_{j=1}^{n} \frac{\partial \widehat{F}_i^+(w, v; w)}{\partial w_j} z_j - \sum_{j=1}^{n} \frac{\partial \widehat{F}_i^+(w, v; w)}{\partial v_j} y_j.$$

Applying Lemma 4 to this system, it follows that $\dot{v} = y(t) \geq 0, \quad \dot{w} = -z(t) \leq 0$. Therefore $v(t)$ is monotone increasing and $w(t)$ monotone decreasing. Since $v(t)$ and $w(t)$ are bounded there exist v^*, w^* such that $v(t) \to v^*$, $w(t) \to w^*$ as $t \to \infty$. □

Corollary 1. *If, in addition to the conditions of Theorem 4, the pair (F^-, F^+) is unified, then there exists a unique equilibrium x^* of (1), which is asymptotically stable on $\langle v^0, w^0 \rangle$.*

Lemma 7. *Let (F^-, F^+) be a pair of continuous, ν-pseudoconcave H-mappings. Then the mapping $\mathcal{F} :\ E \times E \to E \times E$,*

$$\mathcal{F}(v, w) = (\widetilde{F}^-(v, w),\ \widetilde{F}^+(w, v)),$$

contracts on $\operatorname{int} \mathbb{R}_+^{2n}$*, with constant ν, in the metric*

$$r[(v, w),\ (v', w')] = \max\{\rho(v, v'),\ \rho(w, w')\}, \tag{21}$$

where ρ is the Birkhoff metric (7). If (F^-, F^+) are H^+-pseudoconcave, then on $\operatorname{int} \mathbb{R}_+^{2n}$

$$r[\mathcal{F}(v, w),\ \mathcal{F}(v', w')] < r[(v, w),\ (v', w')], \quad (v, w) \neq (v', w'). \tag{22}$$

Proof. Let F^- and F^+ be ν-pseudoconcave. Take two pairs of elements, (v, w) and (v', w'). Clearly,

$$\begin{aligned} \widetilde{F}^-(v, w) &\geq \widetilde{F}^-(e^{-\rho(v,v')} v', e^{\rho(w,w')} w) \\ &\geq \min\{e^{-\nu\rho(v,v')}, e^{-\nu\rho(w,w')}\} \widehat{F}^-(v', w') \\ &= e^{-\nu r'} \widetilde{F}^-(v', w'), \end{aligned}$$

where $r' = r[(v, w),\ (v', w')]$. Similarly,

$$\widetilde{F}^-(v', w') \geq e^{-\nu r'} \widetilde{F}^-(v, w),$$

along with two similar inequalities involving F^+. From the definitions (7) and (21), the conclusion follows. If the pair is H^+-pseudoconcave, the proof is very similar. □

Theorem 5. *Suppose that a minorant F^- and a majorant F^+ of $\mathbb{F}$ are continuous H^+-pseudoconcave mappings with a common equilibrium point $x^* > 0$. Then the pair (F^-, F^+) is unified and the equilibrium x^* of (1) is unique and asymptotically stable on* $\operatorname{int} \mathbb{R}_+^n$.

Proof. By Lemma 6, F^- is equivalent to the mapping $G^-(x) - x$ on $\operatorname{int}\mathbb{R}^n_+$, where G^- is heterotone and pseudoconcave. There is a similar equivalence between $G^+(x) - x$ and F^+. By Lemma 7, the mapping $\mathcal{G} = \{\widetilde{G}^-, \widetilde{G}^+\}$ satisfies (22) on $\operatorname{int}\mathbb{R}^{2n}_+$. Hence, the equations

$$\widetilde{G}^-(v, w) = v, \quad \widetilde{G}^+(w, v) = w,$$

have a unique solution on $\operatorname{int}\mathbb{R}^{2n}_+$. Consequently, the system (19) has no more than one solution on $\operatorname{int}\mathbb{R}^{2n}_+$, that is, the pair (F^-, F^+) is unified on $\operatorname{int}\mathbb{R}^{2n}_+$. Now the Corollary 1 to Theorem 4 completes the proof. $\square$

Theorem 6. *Let a minorant F^- and majorant F^+ of $\mathbb{F}$ be continuous ν-pseudoconcave H^+-mappings. Then (1) has an asymptotically absorbing set $\langle v^*, w^*\rangle \subset \operatorname{int}\mathbb{R}^n_+$, where (v^*, w^*) is the unique solution of (19) on $\operatorname{int}\mathbb{R}^n_+$. Moreover, all solutions of the system (16), with positive initial conditions, approach (v^*, w^*) as $t \to \infty$.*

Proof. From Lemma 6, F^- is equivalent to a mapping $G^-(x) - x$ on $\operatorname{int}\mathbb{R}^n_+$, where G^- is heterotone and ν-pseudoconcave. Similarly $G^+(x) - x$ is equivalent to F^+. Lemma 2 guarantees that each mapping G^-, G^+ has a fixed point on $\operatorname{int}\mathbb{R}^n_+$, that is,

$$G^-(x^*_-) = x^*_-, \quad G^+(x^*_+) = x^*_+,$$

for some $x^*_-, x^*_+ > 0$. Lemma 7, Lemma 6 and the contraction mapping principle imply that (19) has a unique solution (v^*, w^*) on $\operatorname{int}\mathbb{R}^n_+$. Now, $v^* \leq w^*$. To see this take $x^0 = (x^*_- + x^*_+)/2$ and choose $\gamma > 0$ such that $G^-(x^0) \geq \gamma x^0$. Obviously,

$$\widetilde{G}^-\left(\tau x^0, \frac{1}{\tau}x^0\right) \geq \tau^\nu \widetilde{G}^-(x^0, x^0) \geq \tau^\nu \gamma x^0 \geq \tau x^0,$$

if $\tau^{1-\nu} \leq \gamma$. Similarly, choose $\xi > 0$ such that $G^+(x^0) \leq \xi x^0$ to obtain

$$\widetilde{G}^+\left(\frac{1}{\tau}x^0, \tau x^0\right) \leq \tau^{-\nu} \xi x^0 \leq \frac{1}{\tau}x^0,$$

if $\tau^{1-\nu} \leq \xi^{-1}$. Thus,the conic interval $\langle \tau x^0, \frac{1}{\tau}x^0\rangle$ is strongly invariant for (G^-, G^+), as well as for (F^-, F^+), if $\tau^{1-\nu} \leq \min(\gamma, \xi^{-1})$. Thus, the system (16), with initial conditions $v(0) = \tau x^0$, $w(0) = \frac{1}{\tau}x^0$, converges (see the proof of Theorem 4) to the solution (v^*, w^*), $v^* \leq w^*$, of (19). By Theorem 4, the conic interval $\langle v^*, w^*\rangle$ is an asymptotically absorbing set on $\langle \tau x^0, \frac{1}{\tau}x^0\rangle$ for any sufficiently small $\tau > 0$, and is thus so for $\operatorname{int}\mathbb{R}^n_+$. $\square$

The following useful result is simply proved.

Lemma 8. *Suppose that $\mathbb{F}$ has a representation (2) and that each $F(x,\lambda)$ is an H^+-mapping (respectively an H^+-pseudoconcave mapping). Then so are*

$$F_-(x) = \inf\{F(x,\lambda) : \lambda \in \Lambda\}, \ F_+(x) = \sup\{F(x,\lambda) : \lambda \in \Lambda\}, \quad (23)$$

if the infinum and supremum in (23) exist.

Clearly, the mappings (23) are natural candidates for a minorant and a majorant.

6. Theoretical control examples

This section considers examples in the following framework. Each system is described by a differential inclusion $\dot{x} \in \mathbb{F}(x,u)$, where the setvalued mapping has a representation

$$\mathbb{F}(x,u) = \bigcup_{\lambda\in\Lambda} \{F(x,u,\lambda)\}.$$

That is, each trajectory satisfies $\dot{x} = F(x(t),u(t),\lambda(t))$, $x \in \mathbb{R}^n$, $\lambda(t) \in \Lambda$ is a perturbation and the control vector $u(\cdot)$ lies in some admissable class of controls U. The control goal is asymptotic stabilizability of an equilibrium x^* or to have an asymptotically absorbing neighbourhood of x^*.

Example 2. Let $F = (f_1, f_2, f_3)$ be defined by

$$\begin{aligned}
f_1(x,u,\lambda) &= \frac{x_1 + x_3 + \lambda_2}{x_1 x_2 (1+\lambda_1)} - u_1 x_1 + u_3,\\
f_2(x,u,\lambda) &= \frac{1+2\lambda_1}{x_3 \ln(1+x_2)} - u_3 x_2,\\
f_3(x,u,\lambda) &= u_2^{-1} \max\left(\frac{\sqrt{x_1+\lambda_1\lambda_2}}{x_3}, \sqrt{\frac{x_2+x_3+\lambda_1}{x_3}} \right) - 1,
\end{aligned}$$

where $\lambda_1, \lambda_2 \in [\varepsilon, N] \subset (0,\infty)$, and let $x^* = (1,1,1)$.

For x^* to be an equilibrium, that is $F(x^*, u(\lambda), \lambda) = 0$, a simple calculation shows that the control signal must satisfy

$$\begin{aligned}
u_1 &= \frac{1+2\lambda_1}{\ln 2} + \frac{2+\lambda_2}{1+\lambda_1},\\
u_2 &= \max(\sqrt{1+\lambda_1\lambda_2}, \sqrt{2+\lambda_1}),\\
u_3 &= \frac{1+2\lambda_1}{\ln 2}.
\end{aligned}$$

For this control there is an equivalent mapping $G(x) - x$, where G is heterotone and has an associated mapping $\widetilde{G}$ with component functions

$$\begin{aligned}
\widetilde{g}_1(v,w,u,\lambda) &= \sqrt{\frac{v_1+v_3+\lambda_2}{u_1w_2(1+\lambda_1)} + \frac{u_3}{u_1}v_1}, \\
\widetilde{g}_2(v,w,u,\lambda) &= \varphi^{-1}\left(\frac{1+2\lambda_1}{u_3w_3}\right), \\
\widetilde{g}_3(v,w,u,\lambda) &= u_2^{-1}\max\left(\sqrt{v_1+\lambda_1\lambda_2}\,,\,\sqrt{v_3(v_2+v_3+\lambda_1)}\right).
\end{aligned}$$

Here, $\varphi(y) = y\ln(1+y)$. Clearly, G is pseudoconcave for any $\lambda \in \Lambda$. By Lemmas 8 and 6, the minorant and majorant given by (23) are also pseudoconcave. Theorem 5 implies that the equilibrium $x^* = (1,1,1)$ is unique and asymptotically stable on $\operatorname{int}\mathbb{R}^3_+$.

Example 3. Suppose that, after determination of a control law, the equivalent system $G(x) - x$ has the form

$$\begin{aligned}
\dot{x}_1 &= \sqrt[3]{\frac{(2+\lambda_2)x_1+\cos x_1}{3x_2+\lambda_1-\sin x_2}} - x_1, \\
\dot{x}_2 &= \sqrt[4]{\frac{(x_1+\lambda_1)(x_2+\lambda_2)}{x_1+\lambda_2}} - x_2,
\end{aligned}$$

where $\lambda_1(t), \lambda_2(t) \in [\varepsilon, N] \subset (0,\infty)$. Clearly, any selection $G(x,\lambda)$ is heterotone and ν-pseudoconcave. This can be seen from the associated mapping $\widetilde{G} = (\widetilde{g}_1, \widetilde{g}_2)$, where

$$\begin{aligned}
\widetilde{g}_1(v,w,\lambda) &= \sqrt[3]{\frac{(2+\lambda_2)v_1+\cos v_1}{3w_2+\lambda_1-\sin w_2}}, \\
\widetilde{g}_2(v,w,\lambda) &= \sqrt[4]{\frac{(v_1+\lambda_1)(v_2+\lambda_2)}{w_1+\lambda_2}},
\end{aligned}$$

Now, Lemma 8 and Theorem 6 guarantee that the system has an asymptotically absorbing set $\langle v^*, w^*\rangle \subset \operatorname{int}\mathbb{R}^2_+$, where all solutions of (16), with positive initial conditions, approach (v^*, w^*) as $t \to \infty$. Alternatively, by Lemma 7 the iterative scheme

$$v^{k+1} = \widetilde{G}^-(v^k, w^k), \quad w^{k+1} = \widetilde{G}^+(w^k, v^k), \tag{24}$$

where $\widetilde{G}^-$ and $\widetilde{G}^+$ may be constructed as in (23), converges to (v^*, w^*) from any initial conditions $v^0, w^0 > 0$.

If $\lambda_1(t), \lambda_2(t) \in [1,3]$, take as minorant

$$\widetilde{G}^-(v,w) = \left(\sqrt[3]{\frac{3v_1 + \cos v_1}{3w_2 + 3 - \sin w_2}}, \quad \sqrt[4]{\frac{(v_1+1)(v_2+1)}{w_1+3}} \right),$$

and as majorant

$$\widetilde{G}^+(v,w) = \left(\sqrt[3]{\frac{5v_1 + \cos v_1}{3w_2 + 1 - \sin w_2}}, \quad \sqrt[4]{\frac{(v_1+3)(v_2+3)}{w_1+1}} \right).$$

An easy calculation gives $v^* = (0.754, 0.966)$, $w^* = (0.967, 1.659)$.

Example 4. Consider the system

$$\dot{x}_1 = \frac{3x_2 + \lambda}{x_1^3 x_2} - u, \quad \dot{x}_2 = \frac{x_1 + 3\lambda}{x_2^3 x_1} - u, \quad \lambda(t) \in [1,2].$$

For a region containing $x^* = (1,1)$ to be absorbing, let $u = 2 + 2\lambda$. From Theorem 6 there exists a conical interval $\langle v^*, w^* \rangle \subset \operatorname{int} \mathbb{R}^2_+$ of x^* which is an asymptotically absorbing set. Using the iteration (24), with conservative minorant and majorant,

$$v_1^{k+1} = \sqrt[3]{\frac{3v_2^k + 1}{6w_2^k}}, \qquad v_2^{k+1} = \sqrt[3]{\frac{v_1^k + 3}{6w_1^k}},$$
$$w_1^{k+1} = \sqrt[3]{\frac{3w_2^k + 2}{4v_2^k}}, \qquad w_2^{k+1} = \sqrt[3]{\frac{w_1^k + 6}{4v_1^k}},$$

gives $v^* = (0.764, 0.446)$, $w^* = (1.041, 1.321)$.

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