Discretized Unimodal Mappings and Branching Processes

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Abstract

In computer simulations of smooth dynamical systems, the original phase space is replaced with the finite set of machine arithmetic. As a rule, the resulting spatially discretized dynamical systems do not inherit various functional properties of the original systems such as surjectivity and the existence of absolutely continuous invariant measures and this produces the phenomenon of computational collapse. This paper is concerned with the loss of such properties in spatial discretizations of dynamical systems described by symmetric unimodal mappings of the unit interval. The problem reduces to the study of set-valued negative semitrajectories of the discretized system. As the grid is refined, the asymptotic behaviour of the cardinality structure of the semitrajectories follows probabilistic laws corresponding to a branching process the transition probabilities of which are explicitly calculated. These results are illustrated by application to the discretized logistic mapping.

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1 Introduction

Discreteness and finiteness of machine arithmetic means that computer implementations of continuous dynamical systems are finite-state dynamical systems. In fixed-point arithmetic, these implementations can be modeled by dynamical systems on uniform grids which are obtained by composing the transition operator of the original system with the roundoff mapping. The resulting spatially discretized systems often do not inherit functional properties of the original systems as such surjectivity and existence of absolutely continuous invariant measures, which leads to the so-called computational collapse effect. The distortion of dynamics persists even with refinement of the grid and, as a rule, can be asymptotically described in probabilistic terms [2]–[12], [14, 15, 18]. Such persistent distortion implies that computational artifacts associated with discretization need not vanish, no matter how small computer epsilon is taken, and this analysis is confirmed by extensive computer experiments in the cited references.

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This paper considers discretization of smooth unimodal surjective mappings of the unit interval. This discretization induces a dynamical system, the transition operator of which is not surjective and similarly for its iterates. Study of this loss of surjectivity reduces to studying set-valued negative semitrajectories of the discretized system which are formed by the full preimages of points of the grid under the iterates of the discretized mapping.

It turns out that, as the grid is progressively refined, the asymptotic behaviour of cardinalities of the preimages follows probabilistic laws corresponding to an associated branching process, in the sense of *frequency measurability* of subsets of the grid. The frequency measurability of a set means the existence of an absolutely continuous asymptotic spatial distribution of its points, the density of which is described by a *frequency function*. Furthermore, the set of points of the grid, reachable by a fixed number of iterates of the discretized mapping, is frequency measurable, and its frequency function is closely related to the conditional probability of extinction of the associated branching process.

These results hold under an assumption similar to, but more stringent than the non-resonance condition used in [10]. The principal machinery uses ideas of the method of *quantization errors* and *compensating system* [18, 10, 12] which is based on the result on asymptotically uniform distribution of toral projections of smooth mappings [10] and involve weak convergence of measures [1].

Although here the analysis has been done for models of fixed point arithmetic, much the same effects are to be seen in many long iterations of chaotic dynamical systems in floating point arithmetic. This is to be expected, since floating point is a *locally* uniform grid although not *globally* so, in the sense that, although there is nonuniform clustering of points, for any fixed exponent the mantissa values are equally spaced. Global uniformity of the grid makes much of the theoretical computation of quantization errors and asymptotic probabilities more tractable than in the more general case.

The results of the paper can be applied to give rigorous justification of the heuristic and *ad hoc* models for computational collapse in chaotic systems [4]-[9], [13].

The paper is organized as follows. In Section 2, the class of smooth symmetric unimodal mappings and their spatial discretizations are defined. Section 3 describes the associated branching process. In Section 4, the notion of frequency measurability of subsets of the grid is defined. Section 5 formulates the main result of the paper, Theorem 1, which links the asymptotic distribution of cardinalities of preimages with the associated branching process. In Section 6, a recursion is derived for the frequency functions of the sets of reachable points. Section 7 illustrates the results above when applied to the discretized logistic mapping and states Lemma 2 which shows that the logistic mapping satisfies the conditions of the main theorem. Sections 8–13 contain the complete proof of the main theorem and provide additional insight into the structure of the preimages. The proof of Lemma 2 is given in Section 14.

2 Preliminaries

Denote by $\rho : \mathbb{R} \to \mathbb{Z}$ the *roundoff operator* which maps a point u to the nearest node of the integer lattice $\mathbb{Z}$,

$$\rho(u) = \lfloor u + 1/2 \rfloor \tag{1}$$

where $\lfloor \cdot \rfloor$ denotes the floor integer part of a real number. Clearly, the operator commutes with the additive group of translations of $\mathbb{Z}$, $\rho(u+z) = \rho(u) + z$ for all $u \in \mathbb{R}$, $z \in \mathbb{Z}$, and the full preimage of zero under the mapping is the half-open interval

$$V = \rho^{-1}(0) = [-1/2, 1/2). \tag{2}$$

For any positive integer $\nu \in \mathbb{N}$, define the uniform grid

$$X_\nu = \left(\nu^{-1}\mathbb{Z}\right) \bigcap [0,1], \tag{3}$$

and introduce the function $\rho_\nu : [0,1] \to X_\nu$ which maps a point x to the nearest node of the grid

$$\rho_\nu(x) = \nu^{-1}\rho(\nu x). \tag{4}$$

Let $f : [0,1] \to [0,1]$ be a continuously differentiable function interpreted as the transition operator of a smooth autonomous dynamical system with phase space $[0,1]$. Define its discretization $f_\nu : X_\nu \to X_\nu$ as the composition

$$f_\nu = \rho_\nu \circ f. \tag{5}$$

The mapping f_ν will be regarded as the transition operator of a spatially discrete autonomous dynamical system which models the computer implementation of the original system in fixed-point machine arithmetic with accuracy $1/\nu$.

For any $r \in \mathbb{N}$, consider the r-th full preimage of a point $x \in X_\nu$ under the mapping f_ν,

$$f_\nu^{-r}(x) = \{y \in X_\nu : \ f_\nu^r(y) = x\}. \tag{6}$$

The sequence $\{f_\nu^{-r}(x) : \ r \in \mathbb{N}\}$ can be regarded as a set-valued negative semitrajectory of the discretized system f_ν. Characterize the cardinality structure of the r-th initial fragment of the semitrajectory by the collection of nonnegative integers

$$\# f_\nu^{-1}(x), \ldots, \# f_\nu^{-r}(x) \tag{7}$$

where $\#(\cdot)$ denotes the cardinality of a finite set. For large ν, the dependence of these numbers on $x \in X_\nu$ displays stochastic properties which can be described asymptotically with the aid of an appropriate branching process for the special class of symmetric unimodal functions f satisfying the conditions

$$f(1-x) = f(x) \quad \text{for any } x \in [0,1], \tag{8}$$

$$f'(x) > 0 \quad \text{for all } x \in (0, 1/2), \tag{9}$$

$$f(0) = 0, \quad f(1/2) = 1. \tag{10}$$

3 Auxiliary branching process

Under the assumptions (8)–(10), the restriction $f|_{[0,1/2]}$ of the function f is invertible and the corresponding inverse function

$$g = \left(f|_{[0,1/2]}\right)^{-1} : [0,1] \to [0,1/2] \tag{11}$$

is continuous and strictly increasing on $[0,1]$, with $g(0) = 0$ and $g(1) = 1/2$. Moreover, the function g is continuously differentiable on $(0,1)$ and its derivative g' is positive on the interval.

A set $A \subset \mathbb{Z}$ is said to be a *discrete interval* if $A+V$ is a bounded interval in $\mathbb{R}$, or equivalently, if $A = [a,b] \bigcap \mathbb{Z}$ for some $a, b \in \mathbb{Z}$. Denote by $\mathcal{Z}$ the totality of discrete intervals including the empty set. The set $\mathcal{Z}$ is countable and is given the discrete topology.

Taking into account the properties of the inverse function (11), define a function $P : \mathcal{Z}^2 \times (0,1) \to [0,1]$ by

$$P(B \mid A, x) = \mathrm{mes}_1 \{v \in V : \ N(x, A, v) = B\}, \tag{12}$$

where $\mathrm{mes}_n(\,\cdot\,)$ denotes the n-dimensional Lebesgue measure of a set, and the mapping $N : (0,1) \times \mathcal{Z} \times V \to \mathcal{Z}$ is given by

$$N(x, A, v) = \mathbb{Z} \bigcap (g'(x)(A+V) + v). \tag{13}$$

Clearly, for any fixed $x \in (0,1)$ and $A \in \mathcal{Z}$, $\sum_{B \in \mathcal{Z}} P(B \mid A, x) = 1$ and hence, the function $P(\,\cdot \mid A, x)$ describes a probability distribution on the set $\mathcal{Z}$.

Consider the following homogeneous branching process of evolution of particles in discrete time $\mathbb{Z}_+$ and in phase space $(0,1) \times \mathcal{Z}$ with transition probabilities (12) (each such particle is a point in the phase space). For any $r \in \mathbb{Z}_+$, the r-th generation of particles consists of 2^r elements. At the $(r+1)$-th step of evolution, each particle of the r-th generation, independently of the previous generations and of the other particles in the current generation, produces two other particles by the following rule. A particle $(x, y) \in (0,1) \times \mathcal{Z}$ produces two particles $(g_0(x), z)$ and $(g_1(x), -z)$ where z is a random discrete interval with conditional probability distribution $P(\,\cdot \mid y, x)$, and the functions $g_0 : [0,1] \to [0,1/2]$ and $g_1 : [0,1] \to [1/2,1]$ are given by

$$g_k = k + (-1)^k g, \quad k = 0, 1. \tag{14}$$

Label the particles in the r-th generation by pairs (r,k), with $k \in \mathbb{Z} \bigcap [0, 2^r)$, so that the (r,k)-th particle is $(\gamma_{r,k}(x), \xi_{r,k})$ where the function $\gamma_{r,k} : [0,1] \to [0,1]$ is defined as

$$\gamma_{r,k} = g_{k_1} \circ \ldots \circ g_{k_r} \tag{15}$$

and the numbers $k_1, \ldots, k_r \in \{0,1\}$ are uniquely determined from the binary representation

$$k = \sum_{j=1}^{r} k_j \, 2^{r-j} \tag{16}$$

Conventionlly, take $\gamma_{0,0}(x) = x$. Note that each of the functions $\gamma_{r,k}$ is continuously differentiable and strictly monotonic on $(0,1)$, increasing for even $\sum_{j=1}^{r} k_j$, and decreasing otherwise. By definition of the branching process (see also Fig. 1) and from (14)–(16) it follows that for any $r \in \mathbb{N}$,

$$\gamma_{r,k+2^{r-1}} = 1 - \gamma_{r,k}, \quad \xi_{r,k+2^{r-1}} = -\xi_{r,k}, \quad 0 \le k < 2^{r-1}. \tag{17}$$

For what follows, another homogeneous branching process in a simpler phase space can be associated with the branching process described above. This will be done with the aid of the lemma below.

Lemma 1 *Let ω be a random variable, uniformly distributed on the interval V. Then for any bounded interval $\Delta \subset \mathbb{R}$, the probability distribution of the random variable $\eta_\Delta = \#\left(\mathbb{Z} \bigcap (\Delta + \omega)\right)$ is given by*

$$\begin{aligned} \mathbf{P}\left(\eta_\Delta = k\right) &= \operatorname{mes}_1 \left\{ v \in V : \ \#\left(\mathbb{Z} \bigcap (\Delta + v)\right) = k \right\} \\ &= \left(1 - \{\!\{L\}\!\}\right) \delta_{k,\lfloor L \rfloor} + \{\!\{L\}\!\} \, \delta_{k,\lceil L \rceil}, \end{aligned} \tag{18}$$

where $L = \operatorname{mes}_1 \Delta$ is the length of the interval, $\{\!\{v\}\!\} = v - \lfloor v \rfloor$ denotes the fractional part of a real number v, $\lceil v \rceil$ its ceiling integer part, and $\delta_{i,j}$ is the Kronecker symbol.

Proof. Since the function $\#\left(\mathbb{Z} \bigcap (\Delta + v)\right)$ is unit periodic in $v \in \mathbb{R}$, the distribution of the random variable η_Δ only depends on the length of the interval Δ. Therefore, without loss of generality it may be assumed that $\Delta = LV$. Clearly, if $L \in \mathbb{Z}_+$, then $\mathbf{P}\left(\eta_\Delta = L\right) = 1$ which is just (18). So suppose that $L \notin \mathbb{Z}_+$, in which case η_Δ takes the values $\lfloor L \rfloor$ or $\lceil L \rceil$. It is straightforward to verify that

$$\mathbf{P}\left(\eta_\Delta = \lfloor L \rfloor + k\right) = \mathbf{P}\left(\eta_{\{\!\{L\}\!\}V} = k\right) = \begin{cases} 1 - \{\!\{L\}\!\} & \text{for} \quad k = 0 \\ \{\!\{L\}\!\} & \text{for} \quad k = 1 \end{cases}$$

which, by the equality $\lfloor v \rfloor + 1 = \lceil v \rceil$ for any $v \notin \mathbb{Z}$, again is just (18), which completes the proof. ■

Note that for any $x \in (0,1)$ and $A \in \mathcal{Z}$, the length of the interval $g'(x)(A+V)$ is $g'(x)\#A$, and hence, applying Lemma 1 to this interval, from (12) and (13) obtain that for any $z \in \mathbb{Z}_+$,

$$\sum_{B \in \mathcal{Z}:\ \#B = z} P(B \mid A, x) = Q\left(z \mid \#A, x\right) \tag{19}$$

where the function $Q : \mathbb{Z}_+^2 \times (0,1) \to [0,1]$ is given by

$$Q\left(z \mid y,\, x\right) = \left(1 - \{\!\{g'(x)y\}\!\}\right) \delta_{z,\,\lfloor g'(x)y \rfloor} + \{\!\{g'(x)y\}\!\} \, \delta_{z,\,\lceil g'(x)y \rceil}. \tag{20}$$

Clearly, for any fixed $x \in (0,1)$ and $y \in \mathbb{Z}_+$, the function $Q\left(\cdot \mid y, x\right)$ describes a probability distribution on the set $\mathbb{Z}_+$ with mathematical expectation

$$\sum_{z=0}^{+\infty} z Q\left(z \mid y, x\right) = g'(x)y.$$

With the branching process described above, associate the random variables

$$\eta_{r,k} = \#\xi_{r,k}. \tag{21}$$

Since the conditional probability (19) depends only on the cardinality of the discrete interval A and not on its position in $\mathbb{Z}$, this implies that the random variables (21) inherit the Markov property of a branching process. More precisely, the pairs $(\gamma_{r,k}(x), \eta_{r,k})$, $r \in \mathbb{Z}_+$, $k \in \mathbb{Z} \bigcap [0, 2^r)$, form a branching process in the phase space $(0,1) \times \mathbb{Z}_+$ with transition probabilities (20). From (17), it follows that for any $r \in \mathbb{N}$,

$$\eta_{r,k+2^{r-1}} = \eta_{r,k}, \quad 0 \le k < 2^{r-1},$$

and hence the total cardinality of discrete intervals in the r-th generation of particles equals $2\zeta_r$, where

$$\zeta_r = \sum_{0 \le k < 2^{r-1}} \eta_{r,k}. \tag{22}$$

The conditional joint distribution of the random variables $\zeta_1, \ldots, \zeta_r$, given $\eta_{0,0}$, is described by functions $R_r : \mathbb{Z}_+^{r+1} \times (0,1) \to [0,1]$ of the form

$$R_r\left(z_1, \ldots, z_r \mid y, x\right) = \mathbf{P}_x\left(\zeta_1 = z_1, \ldots, \zeta_r = z_r \mid \eta_{0,0} = y\right), \tag{23}$$

where $\mathbf{P}_x(\,\cdot \mid \cdot\,)$ denotes the conditional probability measure in the sample space of the random variables (21). It is straightforward to verify that the functions (23) satisfy the recursion

$$R_{r+1}(z, w \mid y, x) = Q(z \mid y, x) \sum_{u,v \in \mathbb{Z}_+^r :\, u+v=w} R_r\left(u \mid z, g_0(x)\right) R_r\left(v \mid z, g_1(x)\right), \tag{24}$$

for all $r \in \mathbb{N}$, $x \in (0,1)$, $y, z \in \mathbb{Z}_+$ and $w \in \mathbb{Z}_+^r$, with initial condition

$$R_1(z \mid y, x) = Q(z \mid y, x). \tag{25}$$

4 Frequency measurability of sets

Recall that a set $G \subset \mathbb{R}$ is s said to be Jordan measurable if G is bounded and its boundary ∂G satisfies $\mathrm{mes}_1 \partial G = 0$. Following the ideas of [18], say that a family of subsets A_ν of the grid (3), parameterized by $\nu \in \mathbb{N}$, is *frequency measurable* with *frequency function* $p : (0,1) \to [0,1]$, if for any Jordan measurable set $G \subset (0,1)$,

$$\lim_{\nu \to +\infty} \left(\nu^{-1} \# \left(A_\nu \bigcap G\right)\right) = \int_G p(x)\, dx.$$

Frequency measurability is an asymptotic property of the whole family $\{A_\nu : \ \nu \in \mathbb{N}\}$ and not of any individual element. Roughly speaking, a frequency measurable subset of the grid possesses an absolutely continuous asymptotic spatial distribution of its points, and the frequency function is the density of this distribution. Clearly, the grid X_ν itself is a frequency measurable set with frequency function the identity.

Note that the totality of frequency measurable subsets of X_ν is closed under disjoint unions and complementation. That is, if $A_\nu, B_\nu \subset X_\nu$ are frequency measurable families, with frequency functions p and q, $A_\nu \bigcap B_\nu = \emptyset$ for all $\nu \in \mathbb{N}$, then the families $A_\nu \bigcup B_\nu$ and $X_\nu \setminus A_\nu$ are frequency measurable with frequency functions $p+q$ and $1-p$, respectively.

5 Asymptotic distribution of cardinalities of preimages

For a fixed $r \in I\!N$, from the first 2^{r-1} functions (15), construct a mapping $\Gamma_r : [0,1] \to [0,1/2]^{2^{r-1}}$ by

$$\Gamma_r(\cdot) = (\gamma_{r,0}(\cdot), \gamma_{r,1}(\cdot), \ldots, \gamma_{r,2^{r-1}}(\cdot))$$

and define the mapping $G_r : [0,1] \to [0,1/2]^{2^r-1}$ by

$$G_r = (\Gamma_j)_{1 \le j \le r}, \tag{26}$$

noting that $\sum_{j=1}^r 2^{j-1} = 2^r - 1$. By the properties of the function (11), the mapping G_r is continuously differentiable on $(0,1)$ and maps the interval into $(0,1/2)^{2^r-1}$. Say that a vector *resonates* if its entries are linearly dependent over the field of real rationals.

Theorem 1 *Let the mapping $f : [0,1] \to [0,1]$ satisfy the assumptions (8)–(10), and let*

$$\mathrm{mes}_1 \left\{ x \in (0,1) : \ \text{the vector} \ \begin{bmatrix} 1 \\ G_r'(x) \end{bmatrix} \in I\!R^{2^r} \ \text{resonates for some} \ r \in I\!N \right\} = 0. \tag{27}$$

Then for any fixed $r \in I\!N$ and $z_1, \ldots, z_r \in Z\!\!Z_+$, the family of sets

$$\left\{ x \in X_\nu : \ \# f_\nu^{-1}(x) = 2z_1, \ \ldots, \ \# f_\nu^{-r}(x) = 2z_r \right\} \tag{28}$$

is frequency measurable with frequency function $R_r(z_1, \ldots, z_r \mid 1, x)$, $x \in (0,1)$, defined in (23).

The complete proof of Theorem 1 is given in Sections 8–13. The theorem shows that, as $\nu \to +\infty$, the numbers (7) asymptotically behave, in the sense of frequency measurability, like the total cardinalities $2\zeta_1, \ldots, 2\zeta_r$ of discrete intervals in the r first generations of particles in the associated branching process described in Section 3 provided that $\xi_{0,0} = \{0\}$.

6 Frequency functions for sets of reachable points

In contrast to the original function f, its discretization f_ν is not surjective, and the set $f_\nu^r(X_\nu)$ of points of the grid X_ν reachable in r iterates of the function f_ν is nonincreasing in $r \in I\!N$.

Theorem 2 *Under the assumptions of Theorem 1, for any $r \in I\!N$, the family of sets $\{f_\nu^r(X_\nu)\}_{\nu \in I\!N}$ is frequency measurable with the frequency function*

$$\tau_r(x) = 1 - T_r(x, 1), \tag{29}$$

where the functions $T_r : (0,1) \times \mathbb{Z}_+ \to [0,1]$ *satisfy the recursion*

$$T_{r+1}(x,y) = \sum_{z \in \mathbb{Z}_+} T_r\left(g_0(x), z\right) T_r\left(g_1(x), z\right) Q(z \mid y, x) \tag{30}$$

for all $r \in \mathbb{N}$, $x \in (0,1)$ *and* $y \in \mathbb{Z}_+$, *with initial condition*

$$T_1(x,y) = Q(0 \mid y, x). \tag{31}$$

Proof. Since $f_\nu^r(X_\nu) = \{x \in X_\nu : \#f_\nu^{-r}(x) > 0\}$, from Theorem 1 it follows that the family of sets is frequency measurable with frequency function (29), where for any $x \in (0,1)$ and $y \in \mathbb{Z}_+$, $T_r(x,y)$ is the conditional probability of extinction of the branching process described in Section 3 during the r first steps of its evolution provided that the $(0,0)$-th particle is $(x, \xi_{0,0})$ with $\#\xi_{0,0} = \eta_{0,0} = y$. Since $Q(0 \mid 0, x) = 1$, the function $T_r : (0,1) \times \mathbb{Z}_+ \to [0,1]$ is expressed in terms of (23) as

$$T_r(x,y) = \sum_{z \in \mathbb{Z}_+^{r-1}} R_r\left(z, 0 \mid y, x\right). \tag{32}$$

In particular, by (25), the function T_1 is given by (31). Using (24), from (32) it is easy to obtain that

$$\begin{aligned} T_{r+1}(x,y) &= \sum_{z \in \mathbb{Z}_+} \sum_{w \in \mathbb{Z}_+^{r-1}} R_{r+1}(z, w, 0 \mid y, x) \\ &= \sum_{z \in \mathbb{Z}_+} Q(z \mid y, x) \sum_{u \in \mathbb{Z}_+^{r-1}} R_r\left(u, 0 \mid z, g_0(x)\right) \sum_{v \in \mathbb{Z}_+^{r-1}} R_r\left(v, 0 \mid z, g_1(x)\right) \\ &= \sum_{z \in \mathbb{Z}_+} Q(z \mid y, x) \, T_r\left(g_0(x), z\right) T_r\left(g_1(x), z\right). \end{aligned}$$

Therefore, the functions (32) can be equivalently defined by (30) and (31), and the theorem is proved. ■

In particular, from (20), it immediately follows that the function (31) takes the form $T_1(x,y) = \max(0,\, 1 - g'(x)y)$, and hence, by Theorem 2,

$$\lim_{\nu \to +\infty} \frac{\#f_\nu(X_\nu)}{\nu} = \int_0^1 \tau_1(x)\, dx = \int_0^1 \min\left(1, g'(x)\right) dx. \tag{33}$$

7 Application to logistic mapping

To illustrate the results above, consider the logistic mapping $f : [0,1] \to [0,1]$ defined by

$$f(x) = 4x(1-x). \tag{34}$$

Clearly, such a function f satisfies the conditions (8)–(10), and the corresponding function (11) takes the form

$$g(x) = \frac{1}{2}\left(1 - \sqrt{1-x}\right). \tag{35}$$

Lemma 2 *The logistic mapping (34) satisfies the condition (27).*

The proof of Lemma 2 is given in Section 14. This lemma shows that Theorems 1 and 2 apply to the discretized logistic mapping. In particular, since $g'(x) = \frac{1}{4\sqrt{1-x}}$ by (35), the limit (33) can be explicitly calculated,

$$\lim_{\nu \to +\infty} \frac{\# f_\nu(X_\nu)}{\nu} = \int_0^{15/16} \frac{dx}{4\sqrt{1-x}} + \frac{1}{16} = \frac{7}{16}.$$

The frequency functions (29) of the families of sets of reachable points $\{f_\nu^r(X_\nu)\}_{\nu \in I\!N}$, $1 \le r \le 5$, for the discretized logistic mapping, computed from the recursion (30)–(31), are graphed in Fig. 2. The table below contains the relative proportions of the reachable points for $\nu = 1000$, and their limit values

$$\lim_{\nu \to +\infty} \frac{\# f_\nu^r(X_\nu)}{\nu} = \int_0^1 \tau_r(x)\, dx$$

predicted by Theorem 2 and calculated by numerical integration of the frequency functions in MATLAB 5.0, along with the relative discrepancies between the empirical and theoretical results:

r	$\frac{\# f_{1000}^r(X_{1000})}{1000}$	$\int_0^1 \tau_r(x)\, dx$	rel. gap, %
1	0.4400	0.4375	0.5786
2	0.3290	0.3298	0.2386
3	0.2610	0.2609	0.0233
4	0.2140	0.2159	0.8671
5	0.1790	0.1835	2.4353

This comparison provides a strong experimental verification for the validity of Theorem 1 and its corollary Theorem 2 when applied to the discretized logistic mapping.

8 Definition of quantization errors

Using the function (4), define the discretizations $g_{\nu,0}, g_{\nu,1} : X_\nu \to X_\nu$ of the functions (14) as

$$g_{\nu,0} = \rho_\nu \circ g_0, \quad g_{\nu,1} = \rho_\nu \circ g_1 = 1 - g_{\nu,0}. \tag{36}$$

For any $r \in Z\!\!Z_+$ and any integer $k \in [0, 2^r)$, define the function $\gamma_{\nu,r,k} : X_\nu \to X_\nu$ by

$$\gamma_{\nu,r,k} = g_{\nu,k_1} \circ \ldots \circ g_{\nu,k_r} \tag{37}$$

where the numbers $k_1, \ldots, k_r \in \{0, 1\}$ are determined from the binary representation (16), and the convention $\gamma_{\nu,0,0}(x) = x$ is used. By the continuity of the function (11) on $[0,1]$, each of the functions (37) uniformly converges to the corresponding function $\gamma r, k$ of (15),

$$\lim_{\nu\to+\infty} \max_{x\in X_\nu} |\gamma_{\nu,r,k}(x) - \gamma_{r,k}(x)| = 0. \tag{38}$$

Consider the (r,k)*-th normalized deviation* function $d_{\nu,r,k} : X_\nu \to I\!R$ defined by

$$d_{\nu,r,k} = \nu \left(\gamma_{r,k} - \gamma_{\nu,r,k}\right). \tag{39}$$

Assembling these last functions for a fixed $r \in I\!N$, define the mapping $D_{\nu,r} : X_\nu \to I\!R^{2^r}$ as

$$D_{\nu,r} = \left(d_{\nu,r,k}\right)_{0\le k<2^r}. \tag{40}$$

Note that the $(1,0)$-th normalized deviation function (39) is easily expressed in terms of the functions (11) and (36) as

$$d_{\nu,1,0} = \nu \left(g - g_{\nu,0}\right) \tag{41}$$

and takes its values in the interval (2). Taking this into account, for any $r \in I\!N$ and any integer $k \in [0, 2^{r-1})$, define the (r,k)*-th quantization error* $e_{\nu,r,k} : X_\nu \to V$ by

$$e_{\nu,r,k} = d_{\nu,1,0} \circ \gamma_{\nu,r-1,k}. \tag{42}$$

Assembling the quantization errors for a fixed $r \in I\!N$, define the mapping $E_{\nu,r} : X_\nu \to V^{2^{r-1}}$ by

$$E_{\nu,r} = \left(e_{\nu,r,k}\right)_{0\le k<2^{r-1}}. \tag{43}$$

Finally, collecting together these last mappings, define the mapping $\mathcal{E}_{\nu,r} : X_\nu \to V^{2^r-1}$ as

$$\mathcal{E}_{\nu,r} = \left(E_{\nu,j}\right)_{1\le j\le r}, \tag{44}$$

where the dimensionality follows as for (26).

Lemma 3 *For any $r \in I\!N$, the mapping (40) can be represented in terms of the mapping (44) as*

$$D_{\nu,r}(x) = \Phi_{\nu,r}\left(x, \mathcal{E}_{\nu,r}(x)\right) \tag{45}$$

where the mapping $\Phi_{\nu,r} : [0,1] \times V^{2^r-1} \to I\!R^{2^r}$ converges to a continuous mapping Φ_r uniformly on the set $K \times V^{2^r-1}$ as $\nu \to +\infty$, with K an arbitrary closed subset of $(0,1)$.

Proof. The proof proceeds by induction on $r \in I\!N$. Since the function (4) satisfies the property $\rho_\nu(1-x) = 1 - \rho_\nu(x)$ for any $x \in X_\nu$, then the $(1,0)$- and $(1,1)$-th normalized deviation functions (39) are related to each other as

$$d_{\nu,1,1} = \nu \left(g_1 - g_{\nu,1}\right) = -d_{\nu,1,0}.$$

Hence, by (42),

$$d_{\nu,1,k} = (-1)^k\, e_{\nu,1,0}, \quad k = 0, 1.$$

Therefore, the representation (45) holds for $r = 1$ with the continuous mapping $\Phi_{\nu,1} : [0,1] \times V \to \mathbb{R}^2$ given by $\Phi_{\nu,1}(x,y) = (y, -y)$ which does not depend on ν.

Suppose that the statement of the lemma holds for some $r \in \mathbb{N}$. Denote by $\varphi_{\nu,r,k} : [0,1] \times V^{2^r-1} \to \mathbb{R}$ the coordinate functions of the mapping $\Phi_{\nu,r}$. To carry out the inductive step, define the function Δ_ν which, for any sufficiently large ν, maps a pair (x, y) of points $x \in (0,1)$ and $y \in \mathbb{R}$ to the number

$$\Delta_\nu(x,y) = \nu\left(g(x) - g\left(x - \nu^{-1}y\right)\right). \tag{46}$$

From the continuous differentiability of the function g on $(0,1)$, it follows that

$$\lim_{\nu \to +\infty} \Delta_\nu(x,y) = g'(x)\, y \tag{47}$$

uniformly in (x,y) over any compact subset of $(0,1) \times \mathbb{R}$. Fix an arbitrary integer $k \in [0, 2^{r+1})$ and represent it in the form

$$k = 2^r p + q \tag{48}$$

where the integers $p \in \{0,1\}$ and $q \in [0, 2^r)$ are given by

$$p = \lfloor 2^{-r} k \rfloor, \quad q = 2^r \{\!\{2^{-r} k\}\!\}. \tag{49}$$

Then, using (42) and (46), the $(r+1,k)$-th normalized deviation function (39) can be rewritten as

$$\begin{aligned}
d_{\nu,r+1,k} &= \nu\left(\gamma_{r+1,k} - \gamma_{\nu,r+1,k}\right) \\
&= \nu\left(\gamma_{r+1,k} - g_{\nu,p} \circ \gamma_{\nu,r,q}\right) \\
&= \nu\left(g_p \circ \gamma_{r,q} - g_p \circ \gamma_{\nu,r,q} + (-1)^p \nu^{-1} d_{\nu,1,0} \circ \gamma_{\nu,r,q}\right) \\
&= \nu\left(g_p \circ \gamma_{r,q} - g_p \circ \gamma_{\nu,r,q}\right) + (-1)^p e_{\nu,r+1,q} \\
&= \nu\left(g_p \circ \gamma_{r,q} - g_p \circ \left(\gamma_{r,q} - \nu^{-1} d_{\nu,r,q}\right)\right) + (-1)^p e_{\nu,r+1,q} \\
&= (-1)^p \left(\Delta_\nu\left(\gamma_{r,q}, d_{\nu,r,q}\right) + e_{\nu,r+1,q}\right).
\end{aligned} \tag{50}$$

Define the mapping $\Psi_{\nu,r} : [0,1] \times V^{2^r-1} \to \mathbb{R}^{2^r}$ by

$$\Psi_{\nu,r} = \left(\psi_{\nu,r,q}\right)_{0 \le q < 2^r}, \tag{51}$$

where the mappings $\psi_{\nu,r,q} : [0,1] \times V^{2^r-1} \to \mathbb{R}$ are given by

$$\psi_{\nu,r,q}(x,y) = \Delta_\nu\left(\gamma_{r,q}(x), \varphi_{\nu,r,q}(x,y)\right). \tag{52}$$

Then from (48)–(50) it is easy to obtain that the next of the mappings (40) can be written as

$$D_{\nu,r+1}(x) = \begin{bmatrix} I_{2^r} \\ -I_{2^r} \end{bmatrix} \left(\Psi_{\nu,r}\left(x, \mathcal{E}_{\nu,r}(x)\right) + E_{\nu,r+1}(x)\right), \tag{53}$$

where I_{2^r} denotes the identity matrix of order 2^r. Since $\mathcal{E}_{\nu,r+1} = (\mathcal{E}_{\nu,r}, E_{\nu,r+1})$, from (53) it follows that the representation (45) holds for $D_{\nu,r+1}$ with the mapping $\Phi_{\nu,r+1} : [0,1] \times V^{2^{r+1}-1} \to I\!R^{2^{r+1}}$ given by

$$\Phi_{\nu,r+1}(x,u,v) = \begin{bmatrix} I_{2^r} \\ -I_{2^r} \end{bmatrix} (\Psi_{\nu,r}(x,u) + v), \quad u \in V^{2^r-1}, \ v \in V^{2^r}. \tag{54}$$

By the uniform convergence (47) and by the assumption of the inductive step, each of the functions (52) converges to a continuous function $\psi_{r,q} : (0,1) \times V^{2^r-1} \to I\!R$,

$$\lim_{\nu\to+\infty} \psi_{\nu,r,q}(x,y) = g'(\gamma_{r,q}(x))\, \varphi_{r,q}(x,y) = \psi_{r,q}(x,y), \tag{55}$$

uniformly on the set $K \times V^{2^r-1}$ for any closed set $K \subset (0,1)$, where $\varphi_{r,q}(x,y) = \lim_{\nu\to+\infty} \varphi_{\nu,r,q}(x,y)$ is the corresponding coordinate function of the limit mapping Φ_r. Therefore, the mapping (51) converges to a continuous mapping $\Psi_r : (0,1) \times V^{2^r-1} \to I\!R^{2^r}$ in the same way which implies a similar convergence for the mapping (54) to the continuous mapping $\Phi_{r+1} : (0,1) \times V^{2^{r+1}-1} \to I\!R^{2^{r+1}}$ of form

$$\Phi_{r+1}(x,u,v) = \begin{bmatrix} I_{2^r} \\ -I_{2^r} \end{bmatrix} (\Psi_r(x,u) + v).$$

This completes the inductive step, thereby finishing the proof of the lemma. ■

From the proof of the lemma, it follows that each of the limit mappings $\Phi_r(x,y)$ is linear in $y \in V^{2^r-1}$ for any fixed $x \in (0,1)$.

9 Quasiperiodic representation of quantization errors

Lemma 4 **(a)** *For any $r \in I\!N$, the mapping (44) can be represented in terms of the mapping (26) as*

$$\mathcal{E}_{\nu,r}(x) = H_{\nu,r}(x, \nu G_r(x)), \tag{56}$$

where the mapping $H_{\nu,r} : [0,1] \times I\!R^{2^r-1} \to V^{2^r-1}$ satisfies the property that for any fixed $x \in [0,1]$, the mapping $H_{\nu,r}(x,\cdot) : I\!R^{2^r-1} \to V^{2^r-1}$ is unit periodic in its $2^r - 1$ variables;

(b) *the mapping $H_{\nu,r}$ converges to a mes_{2^r}-continuous mapping $H_r : (0,1) \times I\!R^{2^r-1} \to V^{2^r-1}$ in the sense that*

$$\lim_{(\nu,x,y)\to(+\infty,u,v)} H_{\nu,r}(x,y) = H_r(u,v) \quad \text{for all } (u,v) \in L_r, \tag{57}$$

where L_r is the continuity set of H_r on $(0,1) \times V^{2^r-1}$. Furthermore, for any fixed $x \in (0,1)$, the restriction $H_r(x,\cdot)|_{V^{2^r-1}}$ is a mes_{2^r-1}-preserving bijection of the set V^{2^r-1}.

Proof. The proof of both (a) and (b) proceeds by induction on $r \in \mathbb{N}$. Using the roundoff operator (1), define a function $h : \mathbb{R} \to V$ by

$$h(u) = u - \rho(u) = \{\!\!\{u + 1/2\}\!\!\} - 1/2.$$

Clearly, the function h is unit periodic and identically maps the interval V onto itself. From the definition of the quantization errors (42) and from (41), it easily follows that

$$\mathcal{E}_{\nu,1}(x) = e_{\nu,1,0}(x) = d_{\nu,1,0}(x) = h(\nu g(x)) = h\left(\nu G_1(x)\right). \tag{58}$$

Hence, the statement (a) of the lemma holds for $r = 1$ with $H_{\nu,1}(x, y) = h(y)$, which does not depend on ν. Clearly, the mapping $H_{\nu,1}$ converges to the mapping $H_1(x, y) = h(y)$ in the sense of (57), since the continuity set of H_1 is $L_1 = (0, 1) \times V$. This proves the statement (b) for $r = 1$.

Suppose that the statements of the lemma hold for some $r \in \mathbb{N}$. Using (58), (37), (39), (46), (15) and (52), for any integer $k \in [0, 2^r)$ rewrite the $(r+1, k)$-th quantization error as

$$\begin{aligned}
e_{\nu,r+1,k}(x) &= h\left(\nu g\left(\gamma_{\nu,r,k}(x)\right)\right)\\
&= h\left(\nu g\left(\gamma_{r,k}(x) - \nu^{-1} d_{\nu,r,k}(x)\right)\right)\\
&= h\left(\nu g\left(\gamma_{r,k}(x)\right) - \Delta_\nu\left(\gamma_{r,k}(x), d_{\nu,r,k}(x)\right)\right)\\
&= h\left(\nu \gamma_{r+1,k}(x) - \psi_{\nu,r,k}\left(x, \mathcal{E}_{\nu,r}(x)\right)\right)\\
&= h\left(\nu \gamma_{r+1,k}(x) - \psi_{\nu,r,k}\left(x, H_{\nu,r}\left(x, \nu G_r(x)\right)\right)\right).
\end{aligned}$$

Hence, since $G_{r+1} = (G_r, \Gamma_r)$, the next iteration of the mapping $E_{\nu,r}$ of (43) takes the form

$$E_{\nu,r+1}(x) = h_{\nu,r}\left(x, \nu G_{r+1}(x)\right),$$

where the mapping $h_{\nu,r} : [0, 1] \times \mathbb{R}^{2^{r+1}-1} \to V^{2^r}$ is given by

$$h_{\nu,r}\left(x, u, (v_k)_{0 \le k < 2^r}\right) = \left(h\left(v_k - \psi_{\nu,r,k}\left(x, H_{\nu,r}\left(x, u\right)\right)\right)\right)_{0 \le k < 2^r}. \tag{59}$$

Therefore, the representation (56) holds for the mapping $\mathcal{E}_{\nu,r+1}$ with the mapping $H_{\nu,r+1} : [0, 1] \times \mathbb{R}^{2^{r+1}-1} \to V^{2^{r+1}-1}$ given by

$$H_{\nu,r+1}(x, u, v) = \left(H_{\nu,r}(x, u), h_{\nu,r}(x, u, v)\right). \tag{60}$$

By the assumption of the inductive step and by the unit periodicity of the function h, from (59) it follows that for any fixed $x \in [0, 1]$ the mapping $h_{\nu,r}(x, \cdot) : \mathbb{R}^{2^{r+1}-1} \to V^{2^r}$ is unit periodic in its $2^{r+1} - 1$ variables. Hence, the mapping $H_{\nu,r+1}$ also satisfies the property, which completes the inductive step for the statement (a) of the lemma.

Using the uniform convergence (55), it is straightforward to verify that the mapping (60) converges in the sense of (57) to a mapping $H_{r+1} : (0, 1) \times \mathbb{R}^{2^{r+1}-1} \to V^{2^r}$ of the form

$$H_{r+1}(x, u, v) = \left(H_r(x, u), h_r(x, u, v)\right),$$

where the mapping $h_r : (0,1) \times \mathbb{R}^{2^{r+1}-1} \to V^{2^r}$ is defined by

$$h_r \left(x, u, (v_k)_{0 \le k < 2^r} \right) = \left(h \left(v_k - \psi_{r,k} \left(x, H_r \left(x, u \right) \right) \right) \right)_{0 \le k < 2^r} .$$

The continuity set L_{r+1} of the mapping H_{r+1} on $(0,1) \times V^{2^{r+1}-1}$ is given by

$$\begin{aligned} L_{r+1} &= \Big\{ (x,u,v) \in (0,1) \times V^{2^{r+1}-1} : \\ &\quad (x,u) \in L_r,\ v \in V^{2^r} \setminus \left(\Psi_r \left(x, H_r(x,u) \right) + N_r + \mathbb{Z}^{2^r} \right) \Big\}, \end{aligned}$$

where $N_r = \left\{ v = (v_k)_{0 \le k < 2^r} \in \mathbb{R}^{2^r} : \prod_{k=0}^{2^r} (v_k - 1/2) = 0 \right\}$. Since the set N_r is the union of a finite number of $(2^r - 1)$-dimensional hyperplanes in $\mathbb{R}^{2^r}$, it satisfies $\mathrm{mes}_{2^r} N_r = 0$. Therefore, the set L_{r+1} has full 2^{r+1}-dimensional Lebesgue measure in $(0,1) \times V^{2^{r+1}-1}$. Note that for any fixed $(x,u) \in (0,1) \times V^{2^r - 1}$, the restriction $h_r(x,u,\cdot)|_{V^{2^r}}$ is a mes_{2^r}-preserving bijection of the set V^{2^r}. Hence, for any fixed $x \in (0,1)$, the mapping $H_{r+1}(x,\cdot)|_{V^{2^{r+1}-1}}$ is a $\mathrm{mes}_{2^{r+1}-1}$-preserving bijection as the skew product [16] of the mes_{2^r-1}-preserving bijection $H_r(x,\cdot)|_{V^{2^r-1}}$ with the family of mes_{2^r}-preserving bijections $\left\{ h_r(x,u,\cdot)|_{V^{2^r}} :\ u \in V^{2^r-1} \right\}$. This completes the inductive step for the statement (b), and the proof is complete. ■

10 Asymptotically uniform distribution of toral projections

Identifying the m-th Cartesian power V^m of the interval $[-1/2, 1/2)$ with the m-dimensional torus, define the corresponding toral projection $W_m : \mathbb{R}^m \to V^m$ by

$$W_m \left((u_k)_{1 \le k \le m} \right) = \left(\{\!\{ u_k + 1/2 \}\!\} - 1/2 \right)_{1 \le k \le m} . \tag{61}$$

For a fixed mapping $G : [0,1] \to \mathbb{R}^m$ and for each $\nu \in \mathbb{N}$, define the countably additive measure μ_ν on Borel subsets $A \subset [0,1] \times V^m$ by

$$\mu_\nu(A) = \nu^{-1} \# \left\{ x \in X_\nu :\ (x,\ W_m(\nu G(x))) \in A \right\}. \tag{62}$$

The following lemma is a corollary from the general result on the asymptotically uniform distribution of toral projections of smooth mappings [10].

Lemma 5 *Let $G : [0,1] \to \mathbb{R}^m$ be a continuously differentiable mapping on $(0,1)$ satisfying*

$$\mathrm{mes}_1 \left\{ x \in (0,1) :\ \textit{the vector} \begin{bmatrix} 1 \\ G'(x) \end{bmatrix} \in \mathbb{R}^{m+1} \ \textit{resonates} \right\} = 0.$$

Then the measure μ_ν defined by (62) converges weakly to mes_{m+1} as $\nu \to +\infty$. Equivalently, for any continuous function $\varphi : [0,1] \times \mathbb{R}^m \to \mathbb{R}$ such that $\varphi(x,y)$ is unit periodic in the m coordinates of $y \in \mathbb{R}^m$ for any fixed $x \in [0,1]$,

$$\lim_{\nu \to +\infty} \left(\nu^{-1} \sum_{x \in X_\nu} \varphi\left(x,\ \nu G(x) \right) \right) = \int_{[0,1] \times V^m} \varphi(u)\, du.$$

11 Asymptotic independence and uniform distribution of quantization errors

Theorem 3 *Under the assumptions of Theorem 1, the quantization errors (42) are asymptotically uniformly distributed and independent random variables on the interval* $V = [-1/2, 1/2)$ *in the sense that for any* $r \in I\!N$ *and any bounded continuous function* $\varphi : [0,1] \times V^{2^r-1} \to I\!R$,

$$\lim_{\nu\to+\infty} \left(\nu^{-1} \sum_{x\in X_\nu} \varphi\left(x, \mathcal{E}_{\nu,r}(x)\right) \right) = \int_{[0,1]\times V^{2^r-1}} \varphi(u)\, du.$$

Proof. For any $\nu, r \in I\!N$, define a countably additive measure $\lambda_{\nu,r}$ on Borel subsets $B \subset (0,1) \times V^{2^r-1}$ by

$$\lambda_{\nu,r}(B) = \nu^{-1} \#\left\{x \in X_\nu :\ (x,\, \mathcal{E}_{\nu,r}(x)) \in B\right\}. \tag{63}$$

To prove the theorem, it suffices to show that the measure $\lambda_{\nu,r}$ converges weakly to mes_{2^r} as $\nu \to +\infty$. To do this, define another countably additive measure $\mu_{\nu,r}$ on Borel sets $B \subset (0,1) \times V^{2^r-1}$ by

$$\mu_{\nu,r}(B) = \nu^{-1} \#\left\{x \in X_\nu :\ (x,\, W_{2^r-1}(\nu G_r(x))) \in B\right\}, \tag{64}$$

recalling the notations (26) and (61). Using the statement (a) of Lemma 4, it is easy to obtain that

$$\lambda_{\nu,r} = \mu_{\nu,r} \circ S_{\nu,r}^{-1}, \tag{65}$$

where the mapping $S_{\nu,r} : (0,1) \times V^{2^r-1} \to (0,1) \times V^{2^r-1}$ is given by

$$S_{\nu,r}(x,y) = (x,\ H_{\nu,r}(x,y)).$$

From the statement (b) of Lemma 4, it follows that

$$\lim_{(\nu,w)\to(+\infty,u)} S_{\nu,r}(w) = S_r(u) \quad \text{for } \mathrm{mes}_{2^r}\text{--almost all } u \in (0,1) \times V^{2^r-1}, \tag{66}$$

where the mapping $S_r : (0,1) \times V^{2^r-1} \to (0,1) \times V^{2^r-1}$ is given by

$$S_r(x,y) = (x, H_r(x,y)) \tag{67}$$

and is a mes_{2^r}-continuous and mes_{2^r}-preserving bijection of the set $(0,1) \times V^{2^r-1}$. Since, by the assumptions of the theorem, the mapping G_r satisfies the conditions of Lemma 5, then the measure $\mu_{\nu,r}$ given by (64) weakly converges to mes_{2^r},

$$\mu_{\nu,r} \xrightarrow{w} \mathrm{mes}_{2^r} \quad \text{as } \nu \to +\infty. \tag{68}$$

By [1, Theorem 5.5, p. 34], the relations (65), (66) and (68) imply the weak convergence of the measure (63),

$$\lambda_{\nu,r} \xrightarrow{w} \mathrm{mes}_{2^r} \circ S_r^{-1} \quad \text{as } \nu \to +\infty.$$

The statement of the theorem now follows from the mes_{2^r}-preserving property of the mapping (67). ■

12 Representation of preimages in terms of quantization errors

Using the inverse function (11), for any $\nu \in I\!N$, define the set-valued mapping $\Omega_\nu : [0,1] \times \mathcal{Z} \to 2^{I\!R}$ by

$$\Omega_\nu(x, A) = \nu \left(g\left(x + \nu^{-1}(A + V)\right) - g(x) \right). \tag{69}$$

By the continuity of g, the set $\Omega_\nu(x, A)$ is a bounded interval in $I\!R$. Define the discrete interval-valued mapping $N_\nu : [0,1] \times \mathcal{Z} \times V \to \mathcal{Z}$ by

$$N_\nu(x, A, v) = Z\!\!Z \bigcap \left(\Omega_\nu\left(x, A\right) + v\right). \tag{70}$$

Using the functions (37) and the quantization errors (42), define the discrete interval-valued mappings $s_{\nu,r,k} : X_\nu \to \mathcal{Z}$ by the recursion

$$s_{\nu,r+1,k}(x) = N_\nu \left(\gamma_{\nu,r,k}(x),\, (-1)^{\lfloor k2^{1-r} \rfloor} s_{\nu,r,2^{r-1}\{\!\{k2^{1-r}\}\!\}}(x),\, e_{\nu,r+1,k}(x) \right), \tag{71}$$

for all $r \in Z\!\!Z_+$ and integers $k \in [0, 2^r)$, with initial condition

$$s_{\nu,0,0}(x) = \{0\}. \tag{72}$$

Theorem 4 *For any $r \in I\!N$, the r-th full preimage (6) of a point $x \in X_\nu$ by the discretized function f_ν of (5) can be represented as*

$$f_\nu^{-r}(x) = S\left(\bigcup_{0 \le k < 2^{r-1}} \left(\gamma_{\nu,r,k}(x) + \nu^{-1} s_{\nu,r,k}(x) \right) \right), \tag{73}$$

where the set-valued mapping $S : 2^{[0,1]} \to 2^{[0,1]}$ is defined by

$$S(G) = G \bigcup (1 - G). \tag{74}$$

Proof. Recalling (41), (42), (69) and (70), it is straightforward to verify that for any $\nu \in I\!N$, $x \in X_\nu$ and $A \in \mathcal{Z}$,

$$\begin{aligned} X_\nu \bigcap g\left(x + \nu^{-1}(A + V)\right) &= X_\nu \bigcap g\left(x + \nu^{-1}(A + V)\right) \\ &= X_\nu \bigcap \left(g_{\nu,0}(x) + \nu^{-1} e_{\nu,1,0}(x) \right. \\ &\quad \left. + \, g\left(x + \nu^{-1}(A + V)\right) - g(x) \right) \\ &= X_\nu \bigcap \left(g_{\nu,0}(x) + \nu^{-1} \left(\Omega_\nu(x, A) + e_{\nu,1,0}(x)\right) \right) \\ &= g_{\nu,0}(x) + \nu^{-1} \left(Z\!\!Z \bigcap \left(\Omega_\nu(x, A) + e_{\nu,1,0}(x)\right) \right) \\ &= g_{\nu,0}(x) + \nu^{-1} N_\nu\left(x, A, e_{\nu,1,0}(x)\right). \end{aligned}$$

Hence, for any $m \in I\!N$, $x_1, \ldots, x_m \in X_\nu$ and $A_1, \ldots, A_m \in \mathcal{Z}$,

$$
\begin{aligned}
f_\nu^{-1}\left(\bigcup_{k=1}^{m}\left(x_k + \nu^{-1}A_k\right)\right) &= S\left(X_\nu \bigcap \bigcup_{k=1}^{m} g\left(x_k + \nu^{-1}(A_k + V)\right)\right) \\
&= S\left(\bigcup_{k=1}^{m}\left(g_{\nu,0}(x) + \nu^{-1}N_\nu\left(x_k, A_k, e_{\nu,1,0}(x_k)\right)\right)\right). \quad (75)
\end{aligned}
$$

Now prove the representation (73) by induction on $r \in I\!N$. First, note that (75), (72) and (71) imply that for any $x \in X_\nu$,

$$
f_\nu^{-1}(x) = S\left(g_{\nu,0}(x) + \nu^{-1}N_\nu\left(x, \{0\}, e_{\nu,1,0}(x)\right)\right) = S\left(\gamma_{\nu,1,0}(x) + \nu^{-1}s_{\nu,1,0}(x)\right),
$$

which is just (73) for $r = 1$. To carry out the inductive step, suppose that (73) holds for some $r \in I\!N$. Taking (74) and (17) into account, obtain that for any $x \in X_\nu$,

$$
\begin{aligned}
f_\nu^{-(r+1)}(x) &= f_\nu^{-1}\left(S\left(\bigcup_{0 \le k < 2^{r-1}}\left(\gamma_{\nu,r,k}(x) + \nu^{-1}s_{\nu,r,k}(x)\right)\right)\right) \\
&= f_\nu^{-1}\left(\bigcup_{0 \le k < 2^{r}}\left(\gamma_{\nu,r,k}(x) + \nu^{-1}(-1)^{\lfloor k2^{1-r}\rfloor}s_{\nu,r,2^{r-1}\{\!\{k2^{1-r}\}\!\}}(x)\right)\right) \\
&= S\left(\bigcup_{0 \le k < 2^{r}}\left(g_{\nu,0}\left(\gamma_{\nu,r,k}(x)\right)\right.\right. \\
&\qquad \left.\left.+\nu^{-1}N_\nu\left(\gamma_{\nu,r,k}(x),\, (-1)^{\lfloor k2^{1-r}\rfloor}s_{\nu,r,2^{r-1}\{\!\{k2^{1-r}\}\!\}}(x),\, e_{\nu,1,0}\left(\gamma_{\nu,r,k}(x)\right)\right)\right)\right) \\
&= S\left(\bigcup_{0 \le k < 2^{r}}\left(\gamma_{\nu,r+1,k}(x) + \nu^{-1}s_{\nu,r+1,k}(x)\right)\right),
\end{aligned}
$$

which completes the inductive step, thereby finishing the proof of the theorem. ■

From (71) and (72) it follows that the mappings $s_{\nu,r,k}$ can be expressed in terms of the quantization errors. More precisely, for any $r \in \mathbb{Z}_+$, define the mapping $\theta_{\nu,r} : [0,1] \times \mathcal{Z}^{\max\left(1,2^{r-1}\right)} \times V^{2^r} \to \mathcal{Z}^{2^r}$ as

$$
\theta_{\nu,r} = \left(t_{\nu,r,k}\right)_{0 \le k < 2^r}, \tag{76}
$$

where the mappings $t_{\nu,r,k} : [0,1] \times \mathcal{Z}^{\max\left(1,2^{r-1}\right)} \times V^{2^r} \to \mathcal{Z}$ are given by

$$
t_{\nu,r,k}\left(x,\, (u_k)_{0 \le k < 2^{r-1}},\, (v_k)_{0 \le k < 2^r}\right) = N_\nu\left(\gamma_{\nu,r,k}(x),\, (-1)^{\lfloor k2^{1-r}\rfloor}u_{2^{r-1}\{\!\{k2^{1-r}\}\!\}},\, v_k\right). \tag{77}
$$

Taking (43) into account, the mappings $\sigma_{\nu,r} : X_\nu \to \mathcal{Z}^{\max\left(1,2^{r-1}\right)}$ defined by

$$
\sigma_{\nu,r} = \left(s_{\nu,r,k}\right)_{0 \le k < 2^{r-1}} \tag{78}
$$

satisfy the recursion

$$
\sigma_{\nu,r+1}(x) = \theta_{\nu,r}\left(x,\, \sigma_{\nu,r}(x),\, E_{\nu,r+1}(x)\right) \tag{79}
$$

for all $r \in \mathbb{Z}_+$ and $x \in X_\nu$, with initial condition

$$\sigma_{\nu,0}(x) = \{0\}. \tag{80}$$

From (79) and (80), it follows that each of the mappings (78) can be represented in terms of the quantization errors as

$$\sigma_{\nu,r}(x) = \Theta_{\nu,r}\left(x, \mathcal{E}_{\nu,r}(x)\right), \tag{81}$$

where the mappings $\Theta_{\nu,r} : [0,1] \times V^{2^r-1} \to \mathcal{Z}^{2^{r-1}}$ satisfy the recursion

$$\Theta_{\nu,r+1}(x,u,v) = \theta_{\nu,r}\left(x, \Theta_{\nu,r}(x,u), v\right) \tag{82}$$

for all $r \in \mathbb{Z}_+$, $x \in [0,1]$, $u \in V^{2^r-1}$ and $v \in V^{2^r}$, with initial condition

$$\Theta_{\nu,0}(x) = \{0\}. \tag{83}$$

Lemma 6 *For any $r \in \mathbb{Z}_+$, the mapping $\Theta_{\nu,r}$ in (81) converges to a mes_{2^r}-continuous mapping $\Theta_r : (0,1) \times V^{2^r-1} \to \mathcal{Z}^{2^{r-1}}$ in the sense that*

$$\lim_{(\nu,u)\to(+\infty,v)} \Theta_{\nu,r}(u) = \Theta_r(v) \quad \text{for all } v \in C_r, \tag{84}$$

where C_r is the continuity set of the mapping Θ_r.

Proof. The proof proceeds by induction on $r \in \mathbb{Z}_+$. The starting point for the induction immediately follows from (83), by which

$$\Theta_0(x) = \{0\} \tag{85}$$

and $C_0 = (0,1)$. Suppose that the statement of the lemma holds for some $r \in \mathbb{Z}_+$. By the continuous differentiability of the function g on $(0,1)$, for any $A \in \mathcal{Z}$, the bounded interval (69) converges to $g'(x)(A+V)$ in the sense that

$$\lim_{\nu\to+\infty} \mathrm{dist}\ \left(\Omega_\nu(x,A),\, g'(x)(A+V)\right) = 0$$

uniformly in x over any closed subset of $(0,1)$, where $\mathrm{dist}\,(\,\cdot\,,\,\cdot\,)$ denotes the Hausdorff distance between sets. Hence, writing an arbitrary discrete interval A as $A = \mathbb{Z} \bigcap [a,b]$, with $a, b \in \mathbb{Z}$, obtain that the set (70) converges to (13) in the discrete topology of $\mathcal{Z}$ in the sense that

$$\lim_{(\nu,y,u)\to(+\infty,x,v)} N_\nu(y,A,u) = N(x,A,v) \tag{86}$$

for any pair $(x,v) \in (0,1) \times V$ satisfying $v \notin \mathbb{Z} - g'(x)\{a-1/2, b+1/2\}$. The set of such pairs coincides with the continuity set of the limit mapping $N(\,\cdot, A, \cdot\,) : (0,1) \times V \to \mathcal{Z}$. By the uniform convergence (38) and by (86), from (77) it follows that each of the mappings (76) converges in the sense that for any fixed $u = (u_k)_{0\le k<2^{r-1}} \in \mathcal{Z}^{\max(1,2^{r-1})}$,

$$\lim_{(\nu,y,w)\to(+\infty,x,v)} \theta_{\nu,r}(y,u,w) = (t_{r,k}(x,u,v))_{0\le k<2^r} = \theta_r(x,u,v) \tag{87}$$

for any (x, v) in the continuity set of the limit mapping $\theta_r(\,\cdot, u, \cdot\,) : (0,1) \times V^{2^r} \to \mathcal{Z}^{2^r}$, where

$$t_{r,k}\left(x, u, (v_k)_{0 \le k < 2^r}\right) = N\left(\gamma_{r,k}(x),\, (-1)^{\lfloor k2^{1-r}\rfloor} u_{2^{r-1}\{\!\{k2^{1-r}\}\!\}},\, v_k\right). \tag{88}$$

Representing the discrete intervals u_k as $u_k = \mathbb{Z} \bigcap [a_k, b_k]$, with $a_k, b_k \in \mathbb{Z}$, it is easy to see that the last continuity set has the form $\{(x, v) : x \in (0,1),\ v \in L_r(x, u)\}$, where

$$\begin{aligned} &L_r(x, u) \\ &= \mathop{\mathsf{X}}_{0 \le k < 2^r} \left(V \setminus \left(\mathbb{Z} - (-1)^{\lfloor k2^{1-r}\rfloor} g'\left(\gamma_{r,k}(x)\right)\left\{a_{2^{r-1}\{\!\{k2^{1-r}\}\!\}} - \frac{1}{2},\, b_{2^{r-1}\{\!\{k2^{1-r}\}\!\}} + \frac{1}{2}\right\}\right)\right), \end{aligned}$$

where X denotes the Cartesian product. Clearly, the set $L_r(x, u)$ is of full 2^r-dimensional Lebesgue measure. From (82) and (87), it follows that the convergence (84) holds for the mapping $\Theta_{\nu,r+1}$ with limit

$$\Theta_{r+1}(x, u, v) = \theta_r\left(x, \Theta_r(x, u), v\right), \tag{89}$$

which has continuity set $C_{r+1} = \{(x, u, v) :\ (x, u) \in C_r,\ v \in L_r\left(x, \Theta_r(x, u)\right)\}$ of full 2^{r+1}-dimensional Lebesgue measure. This completes the inductive step, and the lemma is proved. ■

Theorem 5 *Under the assumptions of Theorem 1, the mappings $s_{\nu,r,k}$ defined by (71) and (72) are asymptotically distributed, in the sense of frequency measurability, as the random discrete intervals $\xi_{r,k}$ in the associated branching process of Section 3, provided that $\xi_{0,0} = \{0\}$. More precisely, for any $r \in \mathbb{N}$ and any collection of discrete intervals $a_{j,k} \in \mathcal{Z}$, $1 \le j \le r$, $0 \le k < 2^{j-1}$, the family of sets*

$$\left\{x \in X_\nu :\ s_{\nu,j,k}(x) = a_{j,k},\ 1 \le j \le r,\ 0 \le k < 2^{j-1}\right\}, \quad \nu \in \mathbb{N},$$

is frequency measurable with frequency function

$$\mathbf{P}_x\left(\xi_{j,k} = a_{j,k},\ 1 \le j \le r,\ 0 \le k < 2^{j-1} \mid \xi_{0,0} = \{0\}\right), \quad x \in (0,1).$$

Proof. Using the mappings defined by (82) and (83), for arbitrary $r \in \mathbb{N}$, define the mapping $\Upsilon_{\nu,r} : [0,1] \times V^{2^r - 1} \to [0,1] \times \mathcal{Z}^{2^r - 1}$ by

$$\Upsilon_{\nu,r}\left(x, v_1, \ldots, v_r\right) = \left(x, \Theta_{\nu,1}(x, v_1), \ldots, \Theta_{\nu,r}\left(x, v_1, \ldots, v_r\right)\right), \tag{90}$$

where $v_k \in V^{2^{k-1}}$. From Lemma 6, it is easy to obtain that the mapping $\Upsilon_{\nu,r}$ converges to a mes_{2^r}-continuous mapping $\Upsilon_r : (0,1) \times V^{2^r - 1} \to (0,1) \times \mathcal{Z}^{2^r - 1}$ of the form

$$\Upsilon_r\left(x, v_1, \ldots, v_r\right) = \left(x, \Theta_1(x, v_1), \ldots, \Theta_r\left(x, v_1, \ldots, v_r\right)\right), \tag{91}$$

in the sense that

$$\lim_{(\nu,u)\to(+\infty,v)} \Upsilon_{\nu,r}(u) = \Upsilon_r(v) \tag{92}$$

for any continuity point v of the limit mapping Υ_r. As was shown in the proof of Theorem 3, under the assumptions of Theorem 1, the countably additive measure $\lambda_{\nu,r}$

defined on Borel subsets of $(0,1)\times V^{2^r-1}$ by (63) converges weakly to mes_{2^r} as $\nu\to+\infty$. Therefore, by [1, Theorem 5.5, p. 34], from (92) it follows that the countably additive measure $\mu_{\nu,r}$ defined on Borel subsets of $(0,1)\times\mathcal{Z}^{2^r-1}$ by

$$\mu_{\nu,r}=\lambda_{\nu,r}\circ\Upsilon_{\nu,r}^{-1} \tag{93}$$

converges weakly to a measure μ_r of the form

$$\mu_{\nu,r}\xrightarrow{w}\mathrm{mes}_{2^r}\circ\Upsilon_r^{-1}=\mu_r \quad \text{as } \nu\to+\infty. \tag{94}$$

For the collection of discrete intervals $a_{j,k}$ described in the enunciation of the theorem, write

$$A_j=(a_{j,k})_{0\le k<2^{j-1}}\in\mathcal{Z}^{\max(1,2^{j-1})},\quad 0\le j\le r,$$

with $A_0=a_{0,0}=\{0\}$. For an arbitrary Borel set $G\subset(0,1)$, consider the set

$$B=G\times(A_1,\ldots,A_r)\subset(0,1)\times\mathcal{Z}^{2^r-1}. \tag{95}$$

Using (91), (89), (85), (87), (88), (13) and (12), it is straightforward to verify that the value of the limit measure μ_r on the set B is

$$\begin{aligned}
\mu_r(B) &= \int\limits_G\prod_{j=0}^{r-1}\mathrm{mes}_{2^j}\left\{v\in V^{2^j}:\ \theta_j(x,A_j,v)=A_{j+1}\right\}dx\\
&= \int\limits_G\prod_{j=0}^{r-1}\prod_{0\le k<2^j}\mathrm{mes}_1\{v\in V:\\
&\quad N\left(\gamma_{j,k}(x),\,(-1)^{\lfloor k2^{1-j}\rfloor}a_{j,2^{j-1}\{\!\{k2^{1-j}\}\!\}},\,v\right)=a_{j+1,k}\right\}dx\\
&= \int\limits_G\prod_{j=0}^{r-1}\prod_{0\le k<2^j}P\left(a_{j+1,k}\mid(-1)^{\lfloor k2^{1-j}\rfloor}a_{j,2^{j-1}\{\!\{k2^{1-j}\}\!\}},\,\gamma_{j,k}(x)\right)dx\\
&= \int\limits_G\mathbf{P}_x\left(\xi_{j,k}=a_{j,k},\ 1\le j\le r,\ 0\le k<2^{j-1}\mid\xi_{0,0}=\{0\}\right)dx.
\end{aligned} \tag{96}$$

Therefore, if the set G is Jordan measurable, then the set (95) satisfies

$$\mu_r(\partial B)=\mu_r\left((\partial G)\times(A_1,\ldots,A_r)\right)=0$$

which, by the well-known criterion of weak convergence of measures applied to (94), yields

$$\lim_{\nu\to+\infty}\mu_{\nu,r}(B)=\mu_r(B). \tag{97}$$

Now compute the value of the measure (93) on the set (95). From (90), (81), (78) and (63), it follows that

$$\begin{aligned}
\mu_{\nu,r}(B) &= \nu^{-1}\#\left\{x\in G\bigcap X_\nu:\ \sigma_{\nu,j}(x)=A_j,\ 1\le j\le r\right\}\\
&= \nu^{-1}\#\left\{x\in G\bigcap X_\nu:\ s_{\nu,j,k}(x)=a_{j,k},\ 1\le j\le r,\ 0\le k<2^{j-1}\right\}.
\end{aligned} \tag{98}$$

By (97), for any Jordan measurable set $G\subset(0,1)$, the right hand side of (98) converges to that of (96), which immediately implies the statement of the theorem. ■

13 Proof of Theorem 1

From (13), it is straightforward to obtain that for any $x \in (0,1)$, $A \in \mathcal{Z}$ and $v \in V$,

$$\mathrm{dist}\,(N(x,A,v),\{0\}) \le g'(x)\,\mathrm{dist}\,(A,\{0\}) + \frac{g'(x)+1}{2}.$$

Hence, using the proof of Lemma 6, it is easy to show inductively that the mappings $s_{\nu,r,k}$ defined by (71) and (72) are asymptotically bounded in the sense that for any $r \in I\!N$, any integer $k \in [0, 2^{r-1})$ and any closed set $K \subset (0,1)$,

$$\limsup_{\nu\to+\infty} \max_{x\in K\bigcap X_\nu} \mathrm{dist}\,(s_{\nu,r,k}(x),\,\{0\}) < +\infty.$$

Therefore, for any $r \in I\!N$ and any closed set $K \subset (0,1)$, there exists $\nu_* \in I\!N$ so that for any $\nu \ge \nu_*$ and any $x \in K \bigcap X_\nu$, the sets

$$\gamma_{\nu,r,k}(x) + \nu^{-1} s_{\nu,r,k}(x), \quad 0 \le k < 2^{r-1},$$

are all pairwise disjoint and lie in $(0,1/2)$. By (73), this implies that

$$\# f_\nu^{-r}(x) = 2 \sum_{0\le k<2^{r-1}} \# s_{\nu,r,k}(x).$$

By Theorem 5, from this last equation it follows that the family (28) is frequency measurable with frequency function

$$\mathbf{P}_x\,(\zeta_1 = z_1,\ldots,\zeta_r = z_r \mid \xi_{0,0} = \{0\}) = R_r\,(z_1,\ldots,z_r \mid 1, x)\,, \quad x \in (0,1),$$

which describes the conditional joint distribution of the random variables $\zeta_1,\ldots,\zeta_r$ defined in (22), and Theorem 1 is proved. ■

14 Proof of Lemma 2

It is well-known that the logistic mapping (34) has a unique absolutely continuous invariant probability measure. The distribution function $D : [0,1] \to [0,1]$ for this measure is given by

$$D(u) = \sin^2 \frac{\pi u}{2}. \tag{99}$$

Using (35) and forming congruent functions to (14) by D, obtain functions $\tilde{g}_0 : [0,1] \to [0,1/2]$ and $\tilde{g}_1 : [0,1] \to [1/2,1]$ of the form

$$\tilde{g}_k(u) = D^{-1}\,(g_k(D(u))) = k + (-1)^k u/2, \quad k = 0,1.$$

Hence, for any $r \in \mathbb{Z}_+$ and any integer $k \in [0, 2^r)$, the corresponding congruence transformation of the function (15) yields the function

$$\tilde{\gamma}_{r,k} = \tilde{g}_{k_1} \circ \ldots \circ \tilde{g}_{k_r} = D^{-1} \circ \gamma_{r,k} \circ D, \tag{100}$$

which is given by

$$\tilde{\gamma}_{r,k}(u) = (-1)^{k_1+\ldots+k_r}\left(p_{r,k} + 2^{-r}\,u\right), \tag{101}$$

with $\tilde{\gamma}_{0,0}(u) = u$, where

$$p_{r,k} = \sum_{j=1}^{r} 2^{1-j}\,(-1)^{k_j+\ldots+k_r}\,k_j,$$

and the integers $k_1, \ldots, k_r \in \{0,1\}$ are defined by (16). It is straightforward to verify by induction on $r \in \mathbb{Z}_+$ that

$$M_r = \left\{p_{r,k} :\ 0 \le k < 2^{r-1}\right\} = V \bigcap \left(2^{1-r}\mathbb{Z}\right). \tag{102}$$

In particular, $\# M_r = 2^{r-1}$ and consequently the numbers $p_{r,k}$, $0 \le k < 2^{r-1}$, are all pairwise distinct for any $r \in \mathbb{N}$.

Now take a fixed but otherwise arbitrary $r \in \mathbb{N}$ and let $\alpha_{j,k}$, $0 \le j \le r$, $0 \le k < 2^{j-1}$, be a collection of 2^r real rationals, not all zero. Using the functions (100), for any $x \in (0,1)$, write the corresponding linear combination of the entries of the vector $(1, G_r'(x))$ in (27) as

$$\sum_{j=0}^{r}\sum_{0\le k<2^{j-1}} \alpha_{j,k}\gamma_{j,k}'(x) = \sum_{j=0}^{r}\sum_{0\le k<2^{j-1}} \alpha_{j,k}\tilde{\gamma}_{j,k}'(D^{-1}(x))D'\left(\tilde{\gamma}_{j,k}(D^{-1}(x))\right)(D^{-1})'(x).$$

Since $(D^{-1})'(x) > 0$, the last equality implies that

$$\begin{aligned}&\left\{x \in (0,1) :\ \sum_{j=0}^{r}\sum_{0\le k<2^{j-1}} \alpha_{j,k}\gamma_{j,k}'(x) = 0\right\}\\ &= D\left(\left\{u \in (0,1) :\ \sum_{j=0}^{r}\sum_{0\le k<2^{j-1}} \alpha_{j,k}\tilde{\gamma}_{j,k}'(u)D'\left(\tilde{\gamma}_{j,k}(u)\right) = 0\right\}\right).\end{aligned} \tag{103}$$

From (101) it follows that each of the functions $\tilde{\gamma}_{j,k}'(u)$ is identically constant and equals $\pm 2^{-j}$. Since, by (99), $D'(u) = \frac{\pi}{2}\sin(\pi u)$, then using (102),

$$\begin{aligned}&\left\{u \in (0,1) :\ \sum_{j=0}^{r}\sum_{0\le k<2^{j-1}} \alpha_{j,k}\tilde{\gamma}_{j,k}'(u)D'\left(\tilde{\gamma}_{j,k}(u)\right) = 0\right\}\\ &= \left\{u \in (0,1) :\ \sum_{j=0}^{r}\sum_{p\in M_j} \beta_{j,p}\sin\left(\pi\left(p + 2^{-j}u\right)\right) = 0\right\},\end{aligned} \tag{104}$$

where $\beta_{j,p}$, $0 \le j \le r$, $p \in M_j$, is a collection of 2^r real rationals, obtained from $\alpha_{j,k}$ by a nonsingular linear transformation, and hence not all zero. Now the set (104) is finite. To see this, from (102) again, it is easy to see that

$$\sum_{j=0}^{r}\sum_{p\in M_j} \beta_{j,p}\sin\left(\pi\left(p + 2^{-j}u\right)\right) = \operatorname{Im}\left(\beta_{0,0}\,\mathrm{e}^{i\pi u} + \beta_{1,0}\,\mathrm{e}^{i\pi u/2} + \sum_{j=2}^{r}\sigma_j\,\mathrm{e}^{i\pi u/2^j}\right), \tag{105}$$

where $\mathrm{Im}(\cdot)$ denotes the imaginary part of a complex number and

$$\sigma_j = \sum_{p\in M_j} \beta_{j,p}\, e^{i\pi p} = -i \sum_{0\le q<2^{j-1}} \beta_{j,-1/2+q2^{1-j}} \left(e^{i\pi 2^{1-j}}\right)^q. \tag{106}$$

For an arbitrary fixed integer j satisfying $2 \le j \le r$, put $n = 2^{j-1}$ and consider the complex number $\lambda = e^{i\pi/n}$ which is a root of the polynomial

$$u^n + 1 = (u-1)^n + \sum_{k=1}^{n-1} C_n^k (u-1)^k + 2, \tag{107}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$ are the binomial coefficients. Since the number n is even, so also are C_n^k for all $1 \le k \le n-1$. The coefficient at $(u-1)^n$ in (107) is odd, and the free term 2 is even, but not evenly even. Hence, by Eisenstein's Theorem [17, Vol. 1, p. 95], the polynomial $u^n + 1$ is irreducible in the algebra of polynomials over the field of real rationals, which implies that the number λ cannot be a root of any nontrivial polynomial of degree $n-1$ with real rational coefficients. Therefore, in (106), $\sigma_j = 0$ if and only if the real rationals $\beta_{j,p}$, $p \in M_j$, are all zero. Hence, at least one of the numbers $\beta_{0,0}, \beta_{1,0}, \sigma_2, \ldots, \sigma_r$ is nonzero, which implies that (105) is a nontrivial linear combination of linearly independent functions $e^{i\pi u 2^{-j}}$, $0 \le j \le r$. Therefore, the set (104) is finite as it is formed by the roots of a nonzero trigonometric polynomial. This implies finiteness of the set (103). Since in the considerations above, $r \in \mathbb{N}$ was arbitrary, the set

$$\left\{ x \in (0,1) : \text{ the vector } \begin{bmatrix} 1 \\ G_r'(x) \end{bmatrix} \in \mathbb{R}^{2^r} \text{ resonates for some } r \in \mathbb{N} \right\}$$

is only countable, the condition (27) holds and the lemma is proved. ■

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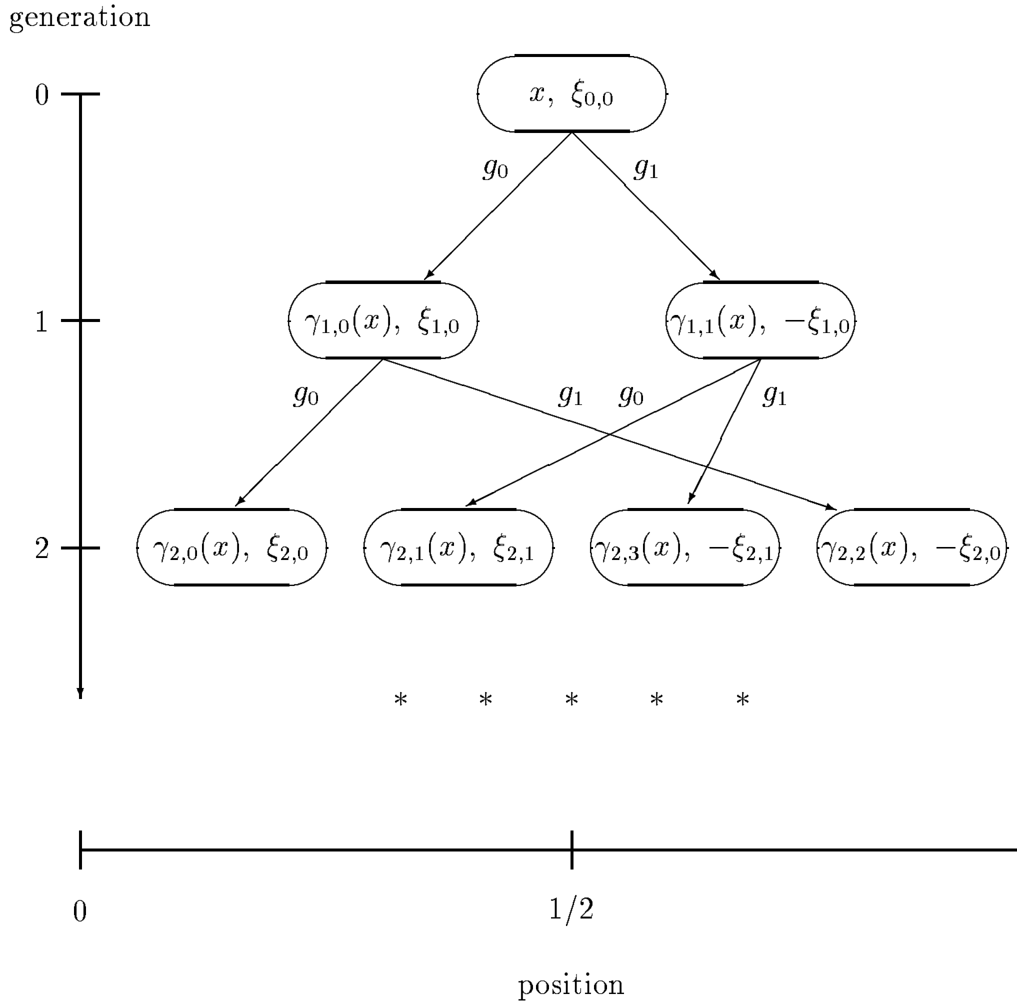

Figure 1: The structure of particles of the zeroth, first and second generations in the associated branching process.

Figure 2: The frequency functions $\tau_r(x)$ of the sets of reachable points $f_\nu^r(X_\nu)$, $1 \leq r \leq 5$, for the discretized logistic mapping.