Higher Order Terms in Asymptotic Expansion for Information Loss in Quantized Random Processes

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ABSTRACT

Uniform quantization of random vectors onto ε-grids $\varepsilon\mathbb{Z}^n$ is considered. Higher order terms in asymptotic expansions for the entropy of the ε-quantized random vector and for the loss of the mutual information between two random vectors under such quantization as $\varepsilon \to 0+$ are obtained. The coefficients in these asymptotics are explicitly calculated for Gaussian distributed vectors. Taken for initial segments of stationary Gaussian sequences, these factors have limit average values per unit of time. For such sequences governed by state-space equations, computation of these average values is reduced to solution of algebraic matrix Riccati and Lyapunov equations.

Index Terms – Quantization, asymptotic entropy and loss of information, stationary Gaussian sequences, Riccati and Lyapunov equations.

EDICS Number: SP 2.1.6 (Quantization Effects).

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1 Introduction

Digital based measurements of physical quantities and computer simulations of dynamical systems are always accompanied by some degree of distortion caused by the spatial discretization arising from finiteness of computer arithmetic. Development of quantitative information theoretical techniques for estimating the quality of representing continuum signals by their discrete counterparts has been an area of intensive research (see [3] and references therein, as well as [1, 2, 6, 7]).

The present paper is concerned with asymptotic analysis of information loss under analog-to-digital conversion of multi-dimensional random data into fixed-point machine arithmetic. More precisely, an n-dimensional random vector ξ with absolutely continuous probability distribution is rounded off onto a uniform cubic grid, the stepsize ε of which plays the role of a small parameter. The entropy [10] of the resulting quantized vector ξ_ε coincides with the information contained by ξ_ε about the original vector ξ. As is well known, the entropy affords asymptotic approximation $-n\ln\varepsilon + h(\xi)$ as $\varepsilon \to 0+$, where $h(\xi)$ is the differential entropy (for the scalar case, this fact dates back to [12]).

We show that if the probability density function (pdf) of ξ satisfies appropriate regularity conditions, then the higher order terms in the asymptotic expansion for the entropy of ξ_ε are $\mathbf{D}(\xi)\varepsilon^2 + O(\varepsilon^4)$, where $\mathbf{D}(\xi)$ is a positive coefficient called an *entropy shifting factor*.

Further, we consider discretization of random vector pairs (ξ, η), with absolutely continuous joint distribution. The mutual information between ξ_ε and η_ε does not exceed that between the original vectors [11]. It turns out that if the joint and marginal pdf's satisfy the same regularity conditions as in the first case, then the principal term of the asymptotic expansion for the corresponding loss of information is $\mathbf{L}(\xi, \eta)\varepsilon^2$ where $\mathbf{L}(\xi, \eta)$ is a nonnegative *information loss factor*.

An important class of distributions satisfying the regularity conditions is provided by Gaussian distributions with non-singular covariance matrices. For this case, the entropy shifting and information loss factors afford an explicit calculation. Furthermore, taken for initial segments of stationary Gaussian sequences, these factors have limit average values per unit of time. For such sequences governed by state space equations, computing the limit values reduces to solution of algebraic matrix Riccati and Lyapunov equations.

As one of the principal tools for obtaining these results, the notion of supersummability of a function is introduced. This is a stronger property than Lebesgue integrability and, technically, allows correct replacement of sums by integrals. Another auxiliary result used in the paper is the analytic parameter dependence and differentiation of the stabilizing solution of an algebraic matrix Riccati equation [8], with the derivative satisfying an algebraic matrix Lyapunov equation.

The paper is organized as follows. In Section 2, quantization on uniform grids is described. Section 3 studies the asymptotic behaviour of entropy of a quantized random vector. Section 4 provides an asymptotic expression for loss of the mutual information between two quantized random vectors. In Section 5, these results are applied to quantized stationary Gaussian sequences. Sections 6 and 7 develop properties of supersummable functions on which the replacement of sums by integrals is based. These

properties along with the differentiability of stabilizing solutions to Riccati equations are used in the final two sections to prove the asymptotic expressions.

2 Uniform quantization

For any $\varepsilon > 0$, denote by $R_\varepsilon : \mathbb{R}^n \to \varepsilon\mathbb{Z}^n$ the roundoff operator which maps a vector $u = (u_k)_{1\le k\le n}$ to the nearest node of the cubic grid $\varepsilon\mathbb{Z}^n$ with stepsize ε,

$$R_\varepsilon(u) = \varepsilon(\lfloor u_k/\varepsilon + 1/2\rfloor)_{1\le k\le n} = \varepsilon R_1(u/\varepsilon).$$

Here, $\mathbb{Z}^n$ is the n-dimensional integer lattice, $\lfloor\,\cdot\,\rfloor$ denotes the floor function, and the convention to round up midpoints is used. Clearly, R_1 commutes with the additive group of translations of $\mathbb{Z}^n$, that is,

$$R_1(u+z) = R_1(u) + z \quad \text{for all } u \in \mathbb{R}^n,\ z \in \mathbb{Z}^n,$$

and the preimage of the zero vector under the mapping is the half-open cube

$$R_1^{-1}(0) = V = [-1/2, 1/2)^n. \tag{1}$$

The roundoff operator R_ε will be interpreted as a model for the computer discretization procedure in fixed-point arithmetic with accuracy ε or for the uniform quantizer with cellwidth ε (we do not take the overflow or overload distortion effects into account).

3 Entropy of quantized random vectors

Let ξ be a random vector with values in $\mathbb{R}^n$. Suppose that its distribution P is absolutely continuous with pdf $p : \mathbb{R}^n \to \mathbb{R}_+$ with respect to n-dimensional Lebesgue measure. Then the ε-quantization $\xi_\varepsilon = R_\varepsilon(\xi)$ of the vector is distributed on the ε-grid $\varepsilon\mathbb{Z}^n$ with probability mass function

$$p_\varepsilon(x) = P(R_\varepsilon^{-1}(x)) = \int\limits_{x+\varepsilon V} p(y)\,dy = \varepsilon^n\,\mathbf{E}\,p(x+\varepsilon\theta), \quad x \in \varepsilon\mathbb{Z}^n, \tag{2}$$

where θ is a random vector, distributed uniformly on the cube V, and $\mathbf{E}\,(\,\cdot\,)$ denotes mathematical expectation. The entropy of the vector ξ_ε is defined in a standard way as

$$H(\xi_\varepsilon) = -\sum_{x\in\varepsilon\mathbb{Z}^n} p_\varepsilon(x)\ln p_\varepsilon(x). \tag{3}$$

Since ξ_ε is obtained from ξ by a deterministic operator with countable range, the entropy $H(\xi_\varepsilon)$ coincides with the mutual information $I(\xi_\varepsilon;\xi)$ between ξ_ε and ξ. From (2) and (3), it follows that

$$H(\xi_\varepsilon) = -n\ln\varepsilon - \varepsilon^n \sum_{x\in\varepsilon\mathbb{Z}^n} f_\varepsilon(x), \tag{4}$$

where

$$f_\varepsilon(x) = g_\varepsilon(x) \ln g_\varepsilon(x) \tag{5}$$

and

$$g_\varepsilon(x) = \mathbf{E}\, p(x + \varepsilon\theta). \tag{6}$$

By Jensen's inequality and by the convexity of the function $u \ln u$ in $u \geq 0$, from (5) and (6) it easily follows that

$$f_\varepsilon(x) \leq \mathbf{E}\,(p(x + \varepsilon\theta) \ln p(x + \varepsilon\theta)),$$

and consequently, from (4),

$$H(\xi_\varepsilon) \geq -n \ln \varepsilon - \varepsilon^n \sum_{x \in \varepsilon\mathbb{Z}^n} \mathbf{E}\,(p(x + \varepsilon\theta) \ln p(x + \varepsilon\theta)) = -n \ln \varepsilon + h(\xi), \tag{7}$$

where

$$h(\xi) = -\int\limits_{\mathbb{R}^n} p(x) \ln p(x)\, dx \tag{8}$$

is the differential entropy [10] of the vector ξ. As was mentioned in Introduction, the discrepancy between the left and right hand sides of the inequality (7) vanishes as $\varepsilon \to 0+$. In what follows, we shall obtain higher order terms in the asymptotic expansion of $H(\xi_\varepsilon)$. The result is given by the theorem below which requires some additional definitions.

Define the operator U_ε which maps a function $f : \mathbb{R}^n \to \mathbb{R}$ to the function $U_\varepsilon(f) : \mathbb{R}^n \to \mathbb{R}_+$ given by

$$U_\varepsilon(f)(x) = \sup_{y \in V} |f(x + \varepsilon y)|. \tag{9}$$

Say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *supersummable* if $\int_{\mathbb{R}^n} U_\varepsilon(f)(x)dx < +\infty$ for some $\varepsilon > 0$. Clearly, the property is equivalent to integrability of $U_\varepsilon(f)$ for all sufficiently small ε since $U_\varepsilon(f)(x)$ is nondecreasing in $\varepsilon > 0$ for any fixed f and x.

Note that supersummability of a function is a stronger property than its Lebesgue integrability. For example, the function $f : \mathbb{R} \to \mathbb{R}_+$ (see Figure 1) described by

$$f(x) = \sum_{r \geq 0} g(2^r(|x| - \sum_{k=1}^{r} k^{-1})), \quad g(x) = \max(0,\, 1 - 2|x|),$$

is Lebesgue integrable but not supersummable. Indeed,

$$\int_{\mathbb{R}} f(x)dx = (1 + 2\sum_{k \geq 1} 2^{-k}) \int_{-1/2}^{1/2} g(x)dx = 3/2,$$

although $U_{1/r}(f)(x) = 1$ for all $r \in \mathbb{N}$, $|x| > \sum_{k=1}^{r} k^{-1}$ and hence,

$$\int_{\mathbb{R}} U_\varepsilon(f)(x)dx = +\infty \quad \text{for any } \varepsilon > 0.$$

For any $N \in \mathbb{Z}_+$, denote by F_N the class of N times continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ which, together with all their partial derivatives up to order N, are Lebesgue integrable and moreover such that the N-th order partial derivatives are all supersummable. In particular, F_0 is the class of continuous supersummable functions.

Definition 1 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be regularly supersummable if the following conditions are satisfied.*

S1 *f is strictly positive and four times continuously differentiable everywhere in $\mathbb{R}^n$;*

S2 *$f \ln f \in F_4$;*

S3 *$(1 + \ln f)\triangle f \in F_2$ where $\triangle$ is the Laplacian;*

S4 *the functions $U_\varepsilon(1+\ln f)\, U_\varepsilon\left(\frac{\partial^4 f}{\partial x_i \partial x_j \partial x_k \partial x_l}\right)$ and $U_\varepsilon\left(\frac{1}{f}\right) U_\varepsilon\left(\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2\right)$ are Lebesgue integrable for some $\varepsilon > 0$ and any $1 \le i, j, k, l \le n$.*

Theorem 1 *Suppose that the pdf p of the random vector ξ is regularly supersummable. Then the entropy $H(\xi_\varepsilon)$ is asymptotically related to the differential entropy $h(\xi)$ by*

$$H(\xi_\varepsilon) = -n \ln \varepsilon + h(\xi) + \mathbf{D}(\xi)\varepsilon^2 + O(\varepsilon^4) \quad \text{as } \varepsilon \to 0+, \tag{10}$$

where

$$\mathbf{D}(\xi) = \frac{1}{24} \int\limits_{\mathbb{R}^n} \frac{|\nabla p(x)|^2}{p(x)} dx = \frac{1}{24}\, \mathbf{E}\, |\nabla \ln p(\xi)|^2, \tag{11}$$

with ∇ denoting the gradient operator.

The proof of Theorem 1 will be given in Section 8. The quantity $\mathbf{D}(\xi)$ will be called the *entropy shifting factor*.

In particular, suppose that the random vector ξ is Gaussian distributed with expectation μ and nonsingular covariance matrix Γ. That is, its pdf is given by

$$p(x) = (2\pi)^{-n/2} (\det \Gamma)^{-1/2} \exp(-(x-\mu)^T \Gamma^{-1} (x-\mu)/2). \tag{12}$$

For such p, any partial derivative $\frac{\partial^N p(x)}{\partial x_{i_1} \ldots \partial x_{i_N}}$ is the product of $p(x)$ and an algebraic polynomial in the coordinates of x and so p satisfies the conditions (S1)–(S4) of Definition 1. Furthermore, (12) implies that $\nabla \ln p(x) = -\Gamma^{-1}(x-\mu)$ and hence, the entropy shifting factor (11) is just

$$\mathbf{D}(\xi) = \frac{1}{24}\, \mathbf{E}\, |\Gamma^{-1}(\xi - \mu)|^2 = \frac{1}{24} \operatorname{Tr} \Gamma^{-1} \tag{13}$$

and does not depend on μ.

4 Information in quantized random vectors

Let ξ and η be two random vectors of dimension n_1 and n_2, respectively, and denote $n = n_1 + n_2$. Suppose that the n-dimensional vector

$$\zeta = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{14}$$

is absolutely continuously distributed with pdf $p : I\!R^n \to I\!R_+$. Then the marginal pdf's $p_1 : I\!R^{n_1} \to I\!R_+$ and $p_2 : I\!R^{n_2} \to I\!R_+$ of the vectors ξ and η are given by

$$p_1(x) = \int\limits_{I\!R^{n_2}} p(x,y)dy, \quad p_2(y) = \int\limits_{I\!R^{n_1}} p(x,y)dx. \tag{15}$$

For any $\varepsilon > 0$, denote by ξ_ε and η_ε the quantizations of the vectors ξ and η onto the ε-grids $\varepsilon Z\!\!Z^{n_1}$ and $\varepsilon Z\!\!Z^{n_2}$ so that

$$\zeta_\varepsilon = R_\varepsilon(\zeta) = \left[\begin{array}{c} \xi_\varepsilon \\ \eta_\varepsilon \end{array} \right].$$

The information $I(\xi_\varepsilon;\eta_\varepsilon)$ in ξ_ε about η_ε is no greater than the information $I(\xi;\eta)$ in ξ about η. The loss of mutual information between ξ and η under the quantization is measured by the non-negative quantity

$$L_\varepsilon(\xi,\eta) = I(\xi;\eta) - I(\xi_\varepsilon;\eta_\varepsilon). \tag{16}$$

Corollary 1 *Let the pdf p of $\zeta = [\xi^T, \eta^T]^T$ and the marginal p.d.f's p_1, p_2 of the subvectors ξ and η all be regularly supersummable. Then the loss of information (16) asymptotically satisfies*

$$L_\varepsilon(\xi,\eta) = \mathbf{L}(\xi,\eta)\,\varepsilon^2 + O(\varepsilon^4) \quad \textit{as } \varepsilon \to 0+,$$

where

$$\begin{aligned} \mathbf{L}(\xi,\eta) &= \mathbf{D}(\zeta) - \mathbf{D}(\xi) - \mathbf{D}(\eta) \\ &= \frac{1}{24}\,\mathbf{E}\,(|\nabla_{x,y} \ln p(\zeta)|^2 - |\nabla_x \ln p_1(\xi)|^2 - |\nabla_y \ln p_2(\eta)|^2), \end{aligned} \tag{17}$$

with $\nabla_x = \frac{\partial}{\partial x}$, $\nabla_y = \frac{\partial}{\partial y}$ and $\nabla_{x,y} = \left[\begin{array}{c} \nabla_x \\ \nabla_y \end{array} \right]$ denoting the appropriate gradient operators.

Proof. Note that the mutual informations $I(\xi;\eta)$ and $I(\xi_\varepsilon;\eta_\varepsilon)$ can be expressed in terms of the differential entropy (8) and the entropy (3) as

$$I(\xi;\eta) = h(\xi) + h(\eta) - h(\zeta), \tag{18}$$

$$I(\xi_\varepsilon;\eta_\varepsilon) = H(\xi_\varepsilon) + H(\eta_\varepsilon) - H(\zeta_\varepsilon). \tag{19}$$

Applying Theorem 1, from (16), (18) and (19) obtain that

$$\begin{aligned} L_\varepsilon(\xi,\eta) &= (n_1 + n_2 - n)\ln\varepsilon + (\mathbf{D}(\zeta) - \mathbf{D}(\xi) - \mathbf{D}(\eta))\,\varepsilon^2 + O(\varepsilon^4) \\ &= \mathbf{L}(\xi,\eta)\,\varepsilon^2 + O(\varepsilon^4) \quad \text{as } \varepsilon \to 0+, \end{aligned}$$

which completes the proof of the corollary. ■

The non-negative quantity $\mathbf{L}(\xi,\eta)$ will be called the *information loss factor* and can be computed as follows. Note that the entropy shifting factor of the vector (14) can be represented as

$$\mathbf{D}(\zeta) = K(\xi \mid \eta) + K(\eta \mid \xi), \tag{20}$$

where

$$K(\xi \mid \eta) = \frac{1}{24}\,\mathbf{E}\,|\nabla_x \ln p_{1|2}(\xi \mid \eta)|^2, \quad K(\eta \mid \xi) = \frac{1}{24}\,\mathbf{E}\,|\nabla_y \ln p_{2|1}(\eta \mid \xi)|^2, \tag{21}$$

and $p_{1|2}$, $p_{2|1}$ are the conditional pdf's of the vectors ξ and η given by

$$p_{1|2}(x \mid y) = \frac{p(x,y)}{p_2(y)}, \quad p_{2|1}(y \mid x) = \frac{p(x,y)}{p_1(x)}.$$

In particular, suppose that the vector (14) is Gaussian distributed with zero mean and nonsingular covariance matrix

$$\Sigma = \begin{bmatrix} \Phi & \Pi \\ \Pi^T & \Psi \end{bmatrix}, \tag{22}$$

where

$$\Phi = \mathbf{E}\,(\xi\xi^T), \quad \Psi = \mathbf{E}\,(\eta\eta^T), \quad \Pi = \mathbf{E}\,(\xi\eta^T).$$

Since Σ is positive definite, so also are the covariance matrices Φ and Ψ, and the matrix

$$G = \Phi^{-1/2}\Pi\Psi^{-1/2} \tag{23}$$

is contracting which implies the nonsingularity of $I_{n_1} - GG^T$ and $I_{n_2} - G^TG$, where I_r denotes the identity matrix of order r.

Theorem 2 *For the Gaussian distributed vector (14) with nonsingular covariance matrix (22), the information loss factor (17) is*

$$\mathbf{L}(\xi,\eta) = \frac{1}{24}\,(\mathrm{Tr}\,(\Phi^{-1}GH_2G^T) + \mathrm{Tr}\,(\Psi^{-1}G^TH_1G)), \tag{24}$$

where

$$H_1 = (I_{n_1} - GG^T)^{-1}, \quad H_2 = (I_{n_2} - G^TG)^{-1}. \tag{25}$$

Proof. By the Normal Correlation Theorem [13, pp. 303–304], the conditional pdf $p_{1|2}(\,\cdot \mid \eta)$ of ξ, given η, is Gaussian with non-singular covariance matrix

$$\Phi - \Pi\Psi^{-1}\Pi^T = \Phi^{1/2}(I_{n_1} - GG^T)\Phi^{1/2}, \tag{26}$$

where the matrix G is defined in (23). Hence, the first of the quantities (21) coincides with the entropy shifting factor of a Gaussian distributed vector with covariance matrix (26) and, by (13), is given by

$$\begin{aligned} K(\xi \mid \eta) &= \frac{1}{24}\,\mathbf{E}\,\mathbf{E}\,(|\nabla_x \ln p_{1|2}(\xi \mid \eta)|^2 \,|\, \eta) \\ &= \frac{1}{24}\,\mathrm{Tr}\,(\Phi - \Pi\Psi^{-1}\Pi^T)^{-1} = \frac{1}{24}\,\mathrm{Tr}\,(\Phi^{-1}H_1), \end{aligned} \tag{27}$$

where $\mathbf{E}(\cdot \mid \cdot)$ denotes conditional mathematical expectation. Similarly,

$$K(\eta \mid \xi) = \frac{1}{24} \operatorname{Tr} (\Psi - \Pi^T \Phi^{-1} \Pi)^{-1} = \frac{1}{24} \operatorname{Tr} (\Psi^{-1} H_2). \tag{28}$$

Using (20), from (27) and (28) obtain that the information loss factor (17) is given by

$$\begin{aligned} \mathbf{L}(\xi, \eta) &= K(\xi \mid \eta) + K(\eta \mid \xi) - \mathbf{D}(\xi) - \mathbf{D}(\eta) \\ &= \frac{1}{24} (\operatorname{Tr} (\Phi^{-1}(H_1 - I_{n_1})) + \operatorname{Tr} (\Psi^{-1}(H_2 - I_{n_2}))). \end{aligned} \tag{29}$$

By the identities

$$H_1 - I_{n_1} = G H_2 G^T, \quad H_2 - I_{n_2} = G^T H_1 G,$$

the equality (29) easily implies (24), thereby completing the proof of the theorem. ■

In particular, if ξ and η are scalar Gaussian random variables with variances $\Phi, \Psi > 0$ and correlation coefficient $G \in (-1, 1)$, the formulas (24), (25) yield

$$\mathbf{L}(\xi, \eta) = \frac{1}{24} \left(\frac{1}{\Phi} + \frac{1}{\Psi} \right) \frac{G^2}{1 - G^2}.$$

5 Entropy and information for quantized stationary Gaussian sequences

Let $Z = (\zeta_k)_{-\infty < k < +\infty}$ be an n-dimensional stationary Gaussian sequence with zero mean and covariance function

$$\sigma_k = \mathbf{E} (\zeta_k \zeta_0^T), \quad k \in \mathbb{Z}. \tag{30}$$

For any $N \in \mathbb{N}$, the initial segment

$$Z_N = (\zeta_k)_{1 \le k \le N} \tag{31}$$

of the sequence is a Gaussian distributed vector with block Toeplitz covariance matrix

$$\Gamma_N = \operatorname*{block}_{1 \le j,k \le N} (\sigma_{j-k}). \tag{32}$$

Note that if the matrix spectral density function $S : [-\pi, \pi] \to \mathbb{C}^{n \times n}$ of the sequence Z associated with its covariance function σ_k by

$$S(\omega) = \sum_{k=-\infty}^{+\infty} \sigma_k \, \mathrm{e}^{ik\omega} \tag{33}$$

is continuous and positive definite for any ω, then by the well-known property of Toeplitz forms [4], the matrices Γ_N satisfy

$$\lim_{N \to +\infty} (\operatorname{Tr} \Gamma_N^{-1} / N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Tr} (S(\omega))^{-1} \, d\omega.$$

Hence, the average value of the entropy shifting factor $\mathbf{D}(Z_N) = \frac{1}{24}\operatorname{Tr}\Gamma_N^{-1}$ of the segment Z_N per unit of time exists and is given by

$$\mathcal{D}(Z) = \lim_{N\to+\infty}(\mathbf{D}(Z_N)/N) = \frac{1}{48\pi}\int_{-\pi}^{\pi}\operatorname{Tr}(S(\omega))^{-1}\,d\omega. \tag{34}$$

Let the Gaussian sequence Z be described by the state-space equations

$$s_{k+1} = As_k + Bv_k, \tag{35}$$
$$\zeta_k = Cs_k + Dv_k, \tag{36}$$

where A, B, C, D are appropriately dimensioned constant matrices, with A asymptotically stable and DD^T nonsingular. Here, $(v_k)_{-\infty<k<+\infty}$ is a sequence of independent identically distributed Gaussian vectors with zero mean and identity covariance matrix.

Associate with the state space formulation (35), (36) the matrix algebraic Riccati equation in Q,

$$Q = AQA^T + BB^T - \Lambda\Theta\Lambda^T, \tag{37}$$
$$\Lambda = (AQC^T + BD^T)\,\Theta^{-1}, \tag{38}$$
$$\Theta = CQC^T + DD^T. \tag{39}$$

A solution Q of this equation is called *stabilizing* if the matrix is symmetric and positive semidefinite, and $A - \Lambda C$ is asymptotically stable. As is well-known [8], the stabilizing solution of the Riccati equation (37)–(39) is unique.

Theorem 3 *For the Gaussian sequence Z governed by (35), (36), the average entropy shifting factor (34) is*

$$\mathcal{D}(Z) = \frac{1}{24}\operatorname{Tr}(\Theta^{-1}(CMC^T + I_n)), \tag{40}$$

where the matrix M is the solution of the Lyapunov equation

$$M = (A - \Lambda C)M(A - \Lambda C)^T + \Lambda\Lambda^T, \tag{41}$$

and the matrices Λ and Θ are expressed in terms of the stabilizing solution Q of the Riccati equation (37)–(39).

The theorem will be proved in Section 9. Note that the quantity $\mathcal{D}(Z)$ defined by the equations (37)–(41) is a function F of the matrices A, B, C, D from (35), (36). Hence, if the elements of the sequence Z are partitioned into subvectors ξ_k and η_k of dimension n_1 and n_2,

$$\zeta_k = \begin{bmatrix} \xi_k \\ \eta_k \end{bmatrix},$$

then the average entropy shifting factors of the stationary Gaussian sequences $X = (\xi_k)_{-\infty<k<+\infty}$ and $Y = (\eta_k)_{-\infty<k<+\infty}$ can be computed as

$$\mathcal{D}(X) = F(A, B, C_1, D_1), \quad \mathcal{D}(Y) = F(A, B, C_2, D_2)$$

where C_i and D_i are the respective blocks of the matrices C and D consisting of n_i rows,

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$

Then it is possible to compute the average information loss factor of the sequences X and Y per unit of time as

$$\begin{aligned} \mathcal{L}(X,Y) &= \mathcal{D}(Z) - \mathcal{D}(X) - \mathcal{D}(Y) \\ &= F(A,B,C,D) - F(A,B,C_1,D_1) - F(A,B,C_2,D_2). \end{aligned}$$

6 Properties of supersummable functions

Note that the operator U_ε defined by (9) is positively homogeneous, convex and submultiplicative with respect to the cone of non-negative functions. That is, for any $\lambda \in I\!R$ and any functions $f, g : I\!R^n \to I\!R$,

$$U_\varepsilon(\lambda f) = |\lambda|\, U_\varepsilon(f),$$

$$U_\varepsilon(f+g) \le U_\varepsilon(f) + U_\varepsilon(g),$$

$$U_\varepsilon(fg) \le U_\varepsilon(f)\, U_\varepsilon(g).$$

Furthermore, the operators U_ε, indexed over all $\varepsilon > 0$, satisfy the semigroup property

$$U_\varepsilon \circ U_\delta = U_{\varepsilon+\delta}, \quad \varepsilon,\ \delta > 0.$$

Define real linear functionals Σ_ε and J mapping a function $f : I\!R^n \to I\!R$ to the quantities

$$\Sigma_\varepsilon(f) = \varepsilon^n \sum_{x \in \varepsilon Z\!\!Z^n} f(x), \tag{42}$$

$$J(f) = \int_{I\!R^n} f(x)dx. \tag{43}$$

Lemma 1 *For any $\varepsilon > 0$ and any function $f : I\!R^n \to I\!R$,*

$$\Sigma_\varepsilon(|f|) \le J(U_\varepsilon(f)).$$

Proof. From (9), it follows that for any $x \in I\!R^n$ and $y \in (-1/2, 1/2)^n$,

$$|f(x)| \le \sup_{z \in V} |f(x + \varepsilon(y+z))| = U_\varepsilon(f)(x + \varepsilon y).$$

Therefore,

$$|f(x)| \le \mathbf{E}\, U_\varepsilon(f)(x + \varepsilon\theta),$$

where θ is a random vector uniformly distributed on the cube V, and hence

$$\Sigma_\varepsilon(|f|) \le \varepsilon^n \sum_{x \in \varepsilon Z\!\!Z^n} \mathbf{E}\, U_\varepsilon(f)(x + \varepsilon\theta) = J(U_\varepsilon(f)). \quad \blacksquare$$

By Lemma 1, the supersummability of a function f implies absolute convergence of the series $\Sigma_\varepsilon(f)$ for all sufficiently small $\varepsilon > 0$.

Lemma 2 *The functional classes F_N satisfy the inclusion $F_{N+1} \subset F_N$ for any $N \in \mathbb{Z}_+$.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. From the mean value theorem, it follows that for any $\varepsilon > 0$, $x \in \mathbb{R}^n$ and $y = (y_k)_{1 \le k \le n} \in V$,

$$|f(x+\varepsilon y) - f(x)| \le \varepsilon \sup_{0<\lambda<\varepsilon} \left| \sum_{k=1}^{n} \frac{\partial f(x+\lambda y)}{\partial x_k} y_k \right| \le \frac{\varepsilon}{2} \sum_{k=1}^{n} U_\varepsilon \left(\frac{\partial f}{\partial x_k} \right)(x).$$

This immediately implies

$$U_\varepsilon(f) \le |f| + \frac{\varepsilon}{2} \sum_{k=1}^{n} U_\varepsilon \left(\frac{\partial f}{\partial x_k} \right)$$

and hence, if $f \in F_1$ then $J(U_\varepsilon(f)) < +\infty$ for all sufficiently small $\varepsilon > 0$ which means the supersummability of f and the inclusion $f \in F_0$. The statement of the lemma now follows by induction. ■

7 Integrals replace sums as $\to 0+$

Lemma 3 *The functionals Σ_ε and J defined in (42) and (43) are asymptotically related as follows:*

(a) *for any function $f \in F_2$,*

$$\Sigma_\varepsilon(f) = J(f) + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0+;$$

(b) *for any function $f \in F_4$,*

$$\Sigma_\varepsilon(f) = J(f) - \frac{\varepsilon^2}{24} J(\triangle f) + O(\varepsilon^4) \quad \text{as } \varepsilon \to 0+ .$$

Proof. For any $\varepsilon > 0$ and any $N \in \mathbb{N}$, define the operator $Q_{\varepsilon,N}$ mapping an N times continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ to $Q_{\varepsilon,N}(f) : \mathbb{R}^n \to \mathbb{R}$ given by

$$Q_{\varepsilon,N}(f)(x) = \mathbf{E} \left(f(x+\varepsilon\theta) - \sum_{k=0}^{N-1} \frac{\varepsilon^k}{k!} \frac{\partial^k f(x)}{\partial x^k}[\theta] \right), \tag{44}$$

where θ is a random vector, distributed uniformly on the cube (1). Here,

$$\frac{\partial^k f(x)}{\partial x^k} = \left(\frac{\partial^k f(x)}{\partial x_{i_1} \dots \partial x_{i_k}} \right)_{1 \le i_1, \dots, i_k \le n}$$

denotes the tensor of order k formed by the partial derivatives of f in the coordinates of x. Any such tensor $A = (a_{i_1,\dots,i_k})_{1 \le i_1,\dots,i_k \le n}$ maps a vector $y = (y_i)_{1 \le i \le n}$ to

$$A[y] = \sum_{1 \le i_1, \dots, i_k \le n} a_{i_1,\dots,i_k}\, y_{i_1} \dots y_{i_k}.$$

By the mean value theorem, from (44) it follows that

$$|Q_{\varepsilon,N}(f)| \le \frac{(\varepsilon/2)^N}{N!} \sum_{1\le i_1,\dots,i_N\le n} U_\varepsilon\left(\frac{\partial^N f}{\partial x_{i_1}\dots\partial x_{i_N}}\right) \tag{45}$$

and hence, by Lemma 1 and by the positive homogeneity, convexity and semigroup properties of the operator U_ε,

$$|\Sigma_\varepsilon(Q_{\varepsilon,N}(f))| \le \frac{(\varepsilon/2)^N}{N!} \sum_{1\le i_1,\dots,i_N\le n} J\left(U_{2\varepsilon}\left(\frac{\partial^N f}{\partial x_{i_1}\dots\partial x_{i_N}}\right)\right). \tag{46}$$

By the central symmetry of the cube $(-1/2,1/2)^n$, for any tensor A of odd order, $\int\limits_V A[y]dy = 0$. Therefore, using (44) with $N=2$, obtain that for any function $f\in F_2$,

$$J(f) = \varepsilon^n \sum_{x\in\varepsilon\mathbb{Z}^n} \mathbf{E}\,(f(x+\varepsilon\theta)) = \Sigma_\varepsilon(f) + \Sigma_\varepsilon(Q_{\varepsilon,2}(f)),$$

that is,

$$\Sigma_\varepsilon(f) = J(f) - \Sigma_\varepsilon(Q_{\varepsilon,2}(f)). \tag{47}$$

Since from (46) it follows that

$$|\Sigma_\varepsilon(Q_{\varepsilon,2}(f))| \le \frac{\varepsilon^2}{8} \sum_{1\le i,j\le n} J\left(U_{2\varepsilon}\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)\right) = O(\varepsilon^2) \quad \text{as } \varepsilon\to 0+, \tag{48}$$

then (47) implies the statement (a) of the lemma.

Now prove (b). By regarding a matrix as a second-order tensor, it is easy to verify that

$$\mathbf{E}\,A[\theta] = \mathbf{E}\,(\theta^T A\theta) = \operatorname{Tr} A \int\limits_{-1/2}^{1/2} u^2\,du = \frac{1}{12}\operatorname{Tr} A, \quad A\in I\!R^{n\times n}. \tag{49}$$

Let $f\in F_4$. Using (49) and the identity $\triangle = \operatorname{Tr}\frac{\partial^2}{\partial x^2}$, rewrite (44) for $N=4$ to obtain

$$\Sigma_\varepsilon(f) = J(f) - \frac{\varepsilon^2}{24}\Sigma_\varepsilon(\triangle f) - \Sigma_\varepsilon(Q_{\varepsilon,4}(f)). \tag{50}$$

Since the inclusion $f\in F_4$ implies that $\triangle f\in F_2$, the relation (47) is applicable to the function $\triangle f$. Therefore, from (50) it follows that

$$\Sigma_\varepsilon(f) = J(f) - \frac{\varepsilon^2}{24}J(\triangle f) + T_\varepsilon(f) \tag{51}$$

where

$$T_\varepsilon(f) = \frac{\varepsilon^2}{24}\Sigma_\varepsilon(Q_{\varepsilon,2}(\triangle f)) - \Sigma_\varepsilon(Q_{\varepsilon,4}(f)). \tag{52}$$

Applying (48) to the function $\triangle f$, and (46) with $N=4$ to the function f, from (52) it is straightforward to obtain that

$$|T_\varepsilon(f)| \le \frac{\varepsilon^4}{128} \sum_{1\le i,j,k,l\le n} J\left(U_{2\varepsilon}\left(\frac{\partial^4 f}{\partial x_i\partial x_j\partial x_k\partial x_l}\right)\right) = O(\varepsilon^4) \quad \text{as } \varepsilon\to 0+$$

which together with (51) implies the statement (b) and completes the proof of the lemma. ■

8 Proof of Theorem 1

Following the line of proof of Lemma 3 and using the operators $Q_{\varepsilon,N}$ defined in (44), the function $g_\varepsilon : I\!R^n \to I\!R_+$ given by (6) can be represented in the form

$$g_\varepsilon = p + Q_{\varepsilon,2}(p) = p + \frac{\varepsilon^2}{24}\triangle p + Q_{\varepsilon,4}(p). \tag{53}$$

Applying the Taylor formula with remainder in Lagrange form to the function $u \ln u$ and using (53), rewrite the function f_ε defined in (5) as

$$f_\varepsilon = p \ln p + (1 + \ln p)\, Q_{\varepsilon,2}(p) + \frac{(Q_{\varepsilon,2}(p))^2}{2(p + \theta_\varepsilon Q_{\varepsilon,2}(p))}, \tag{54}$$

where θ_ε is a function on $I\!R^n$ with values in $(0,1)$. Suppose that the pdf p is regularly supersummable. Then, by the condition (S1), the function $\ln p$ is four times continuously differentiable, and by (S2), the statement (b) of Lemma 3 is applicable to the function $p \ln p$ which yields

$$\begin{aligned}
\Sigma_\varepsilon(p \ln p) &= J(p \ln p) - \frac{\varepsilon^2}{24} J(\triangle(p \ln p)) + O(\varepsilon^4) \\
&= J(p \ln p) - \frac{\varepsilon^2}{24} J\left((1 + \ln p)\triangle p + \frac{|\nabla p|^2}{p}\right) + O(\varepsilon^4) \quad \text{as } \varepsilon \to 0+ .
\end{aligned} \tag{55}$$

By the statement (a) of Lemma 3, the condition (S3) implies that

$$\Sigma_\varepsilon((1 + \ln p)\triangle p) = J((1 + \ln p)\triangle p) + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0+ . \tag{56}$$

By the condition (S4) and by Lemma 1,

$$\begin{aligned}
|\Sigma_\varepsilon((1 + \ln p)Q_{\varepsilon,4}(p))| &\le \frac{\varepsilon^4}{384} \sum_{1 \le i,j,k,l \le n} J\left(U_\varepsilon(1 + \ln p) U_{2\varepsilon}\left(\frac{\partial^4 p}{\partial x_i \partial x_j \partial x_k \partial x_l}\right)\right) \\
&= O(\varepsilon^4) \quad \text{as } \varepsilon \to 0+ .
\end{aligned} \tag{57}$$

From (53), (56) and (57), it is easy to obtain that the second term on the right of (54) satisfies

$$\begin{aligned}
\Sigma_\varepsilon((1 + \ln p)Q_{\varepsilon,2}(p)) &= \frac{\varepsilon^2}{24}\Sigma_\varepsilon((1 + \ln p)\triangle p) + \Sigma_\varepsilon((1 + \ln p)Q_{\varepsilon,4}(p)) \\
&= \frac{\varepsilon^2}{24} J((1 + \ln p)\triangle p) + O(\varepsilon^4) \quad \text{as } \varepsilon \to 0+ .
\end{aligned} \tag{58}$$

Consider the third term on the right of (54). By the inequality (45), the numerator of the fraction can be estimated as

$$(Q_{\varepsilon,2}(p))^2 \le \frac{\varepsilon^4}{64}\left(\sum_{1 \le i,j \le n} U_\varepsilon\left(\frac{\partial^2 p}{\partial x_i \partial x_j}\right)\right)^2 \le \frac{\varepsilon^4 n^2}{64} \sum_{1 \le i,j \le n} U_\varepsilon\left(\left(\frac{\partial^2 p}{\partial x_i \partial x_j}\right)^2\right). \tag{59}$$

Since $0 < \theta_\varepsilon < 1$, from (6) and (53) it follows that the denominator of the fraction satisfies

$$\frac{1}{p + \theta_\varepsilon Q_{\varepsilon,2}(p)} \le U_\varepsilon\left(\frac{1}{p}\right). \tag{60}$$

By Lemma 1 and by the condition (S4), the inequalities (59) and (60) imply that

$$\begin{aligned}\Sigma_\varepsilon\left(\frac{(Q_{\varepsilon,2}(p))^2}{p + \theta_\varepsilon Q_{\varepsilon,2}(p)}\right) &\le \frac{\varepsilon^4 n^2}{64} \sum_{1\le i,j\le n} J\left(U_{2\varepsilon}\left(\frac{1}{p}\right) U_{2\varepsilon}\left(\left(\frac{\partial^2 p}{\partial x_i \partial x_j}\right)^2\right)\right) \\ &= O(\varepsilon^4) \quad \text{as } \varepsilon \to 0+. \end{aligned} \tag{61}$$

Assembling (55), (58) and (61), from (54) it is straightforward to obtain

$$\Sigma_\varepsilon(f_\varepsilon) = J(p\ln p) - \frac{\varepsilon^2}{24} J\left(\frac{|\nabla p|^2}{p}\right) + O(\varepsilon^4) \quad \text{as } \varepsilon \to 0+$$

which together with (4), (8) and (11) implies (10). This completes the proof of the theorem. ■

9 Proof of Theorem 3

For a fixed but otherwise arbitrary $\lambda \ge 0$, associate with (35)–(36) a stationary Gaussian sequence $Z_\lambda = (\zeta_k^\lambda)_{-\infty<k<+\infty}$ governed by the equations

$$s_{k+1}^\lambda = As_k^\lambda + Bv_k, \tag{62}$$

$$\zeta_k^\lambda = Cs_k^\lambda + Dv_k + \sqrt{\lambda}\, w_k, \tag{63}$$

where $(w_k)_{-\infty<k<+\infty}$ is a sequence of mutually independent n-dimensional Gaussian distributed vectors with zero mean and identity covariance matrix, independent of the sequence $(v_k)_{-\infty<k<+\infty}$. Hence, Z_λ has matrix spectral density function $S_\lambda : [-\pi, \pi] \to \mathbb{C}^{n\times n}$ of kind

$$S_\lambda = \lambda I_n + S \tag{64}$$

where S is the matrix spectral density function of the sequence Z given by (33). Note that by the Szegö-Kolmogorov formula [5],

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det S_\lambda(\omega)\, d\omega = \ln\det\Theta_\lambda, \tag{65}$$

where the matrix Θ_λ corresponds to the stabilizing solution Q_λ of the filtering algebraic Riccati equation [9]

$$Q_\lambda = \mathbf{E}\left((s_0^\lambda - \widehat{s}_0^\lambda)(s_0^\lambda - \widehat{s}_0^\lambda)^T\right) = AQ_\lambda A^T + BB^T - \Lambda_\lambda\Theta_\lambda\Lambda_\lambda^T, \tag{66}$$

$$\Lambda_\lambda = \mathbf{E}\left((s_1^\lambda - \widehat{s}_1^\lambda)(\zeta_1^\lambda - \widehat{\zeta}_1^\lambda)^T\right)\Theta_\lambda^{-1} = (AQ_\lambda C^T + BD^T)\Theta_\lambda^{-1}, \tag{67}$$

$$\Theta_\lambda = \mathbf{E}\left((\zeta_1^\lambda - \widehat{\zeta}_1^\lambda)(\zeta_1^\lambda - \widehat{\zeta}_1^\lambda)^T\right) = CQ_\lambda C^T + DD^T + \lambda I_n, \tag{68}$$

where

$$\widehat{s}_0^\lambda = \mathbf{E}\,(s_0^\lambda|(\zeta_k^\lambda)_{k\le 0}), \quad \widehat{s}_1^\lambda = \mathbf{E}\,(s_1^\lambda|(\zeta_k^\lambda)_{k\le 0}), \quad \widehat{\zeta}_1^\lambda = \mathbf{E}\,(\zeta_1^\lambda|(\zeta_k^\lambda)_{k\le 0}).$$

Clearly, the matrix Q_0 coincides with the stabilizing solution Q of the Riccati equation (37)–(39), with

$$\Lambda_0 = \Lambda, \quad \Theta_0 = \Theta.$$

Moreover, if such a solution exists, the matrix Q_λ is differentiable with respect to λ in a neighbourhood of 0, and its derivative at 0 satisfies the equation

$$\begin{aligned} M &= \left.\frac{dQ_\lambda}{d\lambda}\right|_{\lambda=0} \\ &= AMA^T + \Lambda \left.\frac{d\Theta_\lambda}{d\lambda}\right|_{\lambda=0} \Lambda^T - \left.\frac{d}{d\lambda}(\Lambda_\lambda\Theta_\lambda)\right|_{\lambda=0} \Lambda^T - \Lambda \left.\frac{d}{d\lambda}(\Theta_\lambda\Lambda_\lambda^T)\right|_{\lambda=0} \\ &= AMA^T + \Lambda(CMC^T + I_n)\Lambda^T - AMC^T\Lambda^T - \Lambda CMA^T \\ &= (A-\Lambda C)M(A-\Lambda C)^T + \Lambda\Lambda^T, \end{aligned} \tag{69}$$

which immediately follows from (66)–(68) and is the expression in (41). Using the differentiation rule $(\ln|\det X|)' = \mathrm{Tr}\,(X^{-1}X')$, from (64)–(69) it is straightforward to obtain that

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Tr}\,(S(\omega))^{-1}\,d\omega &= \frac{1}{2\pi}\frac{d}{d\lambda}\int_{-\pi}^{\pi} \ln\det S_\lambda(\omega)\,d\omega\,\Big|_{\lambda=0} \\ &= \mathrm{Tr}\left(\Theta^{-1}\left.\frac{d\Theta_\lambda}{d\lambda}\right|_{\lambda=0}\right) = \mathrm{Tr}\,(\Theta^{-1}(CMC^T + I_n)). \end{aligned} \tag{70}$$

The relations (34) and (70) easily yield (40), thereby completing the proof of the theorem. ■

Acknowledgements

The work is supported by the Australian Research Council grant A 4970 2246. The authors thank Victor Kozyakin for helpful discussions of the results of this paper.

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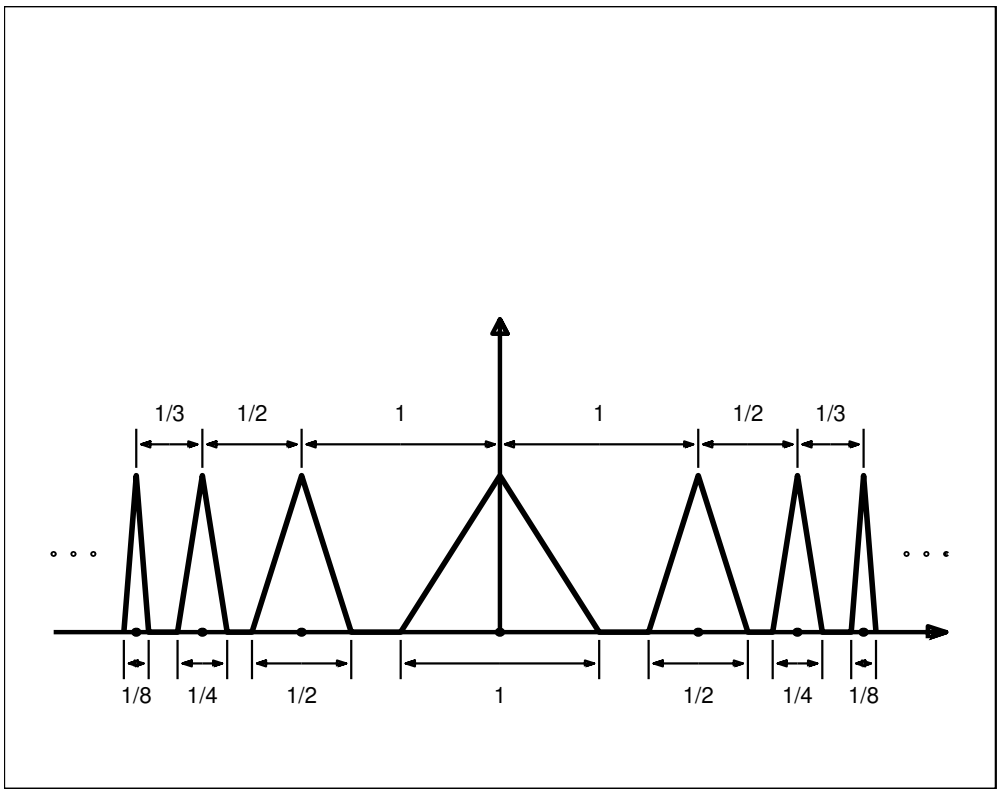

Figure 1: The graph of an integrable but not supersummable function in one variable.