Loss of Invertibility of Smooth Diffeomorphisms Under Spatial Discretization

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ABSTRACT

Suppose that f_ε is a computer realization, with computer accuracy ε, of a smooth diffeomorphism f on $\mathbb{R}^n$. Since f_ε is the restriction of f to a finite arithmetic grid, the concomitant roundoff effects lead to some loss of both injectivity and surjectivity. This paper shows that as computer accuracy $\varepsilon \to +0$, there exist asymptotic spatial distributions of injectivity and surjectivity points of the discretization f_ε. The distributions are absolutely continuous and the probability densities are explicitly calculated in terms of the Jacobian of the original mapping f. The results are applied to asymptotic analysis of discretized conformal mappings of the complex plane.

1 Introduction

When continuous dynamical systems are simulated, the continuum state space is replaced by the finite set of computer arithmetic and this introduces roundoff effects. These can produce marked differences between the behaviour of the computer realization and that of the original system, especially in longterm computer runs [2, 3, 4, 9]. Such effects are linked both to the exponential propagation of very small errors by chaotic systems and to the finiteness of the computer state space. A striking feature of these artifacts is that they can persist no matter how much the computer accuracy ε is refined and have a well defined asymptotic statistical structure as $\varepsilon \to +0$ [5, 10].

A related question is the degree to which functional properties, as distinct from the evolution of system behaviour, are affected by discretization arising from computer realization. With a smooth diffeomorphism f, it is clear that both surjectivity and injectivity will be affected to some degree. This paper quantifies the severity of these artifacts locally in terms of the Jacobian f' and shows that, far from disappearing with increasing computer accuracy ε, the effects persist and that there exist asymptotic statistical distributions of injectivity and surjectivity points as $\varepsilon \to +0$.

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The paper is organized as follows. In Section 2, the spatial discretization of a smooth diffeomorphism is specified and injectivity and surjectivity points are defined. In Section 3, two main theorems are formulated about the asymptotic distributions of the points. Section 4 applies these results to two-dimensional diffeomorphisms described by nonlinear conformal mappings on the complex plane. In Sections 5–8, the complete proofs of the main theorems are given, proceeding through a number of lemmas.

2 Preliminaries

Let $f : X \to Y$ be a smooth diffeomorphism of two bounded open subsets $X,\ Y \subset I\!R^n$. For arbitrary $\varepsilon > 0$, denote by

$$X_\varepsilon = (\varepsilon \mathbb{Z}^n) \bigcap X\,, \quad Y_\varepsilon = (\varepsilon \mathbb{Z}^n) \bigcap Y$$

the *ε-grids* of X and Y, where $\mathbb{Z}^n$ is the n-dimensional integer lattice in $I\!R^n$. Consider the corresponding *ε-discretization* $f_\varepsilon : X_\varepsilon \to \varepsilon\mathbb{Z}^n$ of the diffeomorphism f defined as the composition

$$f_\varepsilon = \varepsilon\, R \circ \varepsilon^{-1} \circ f$$

where $R : I\!R^n \to \mathbb{Z}^n$ is the roundoff operator which maps a vector to the nearest node of $\mathbb{Z}^n$,

$$R\left((u_k)_{1\le k\le n}\right) = (\lfloor u_k + 1/2 \rfloor)_{1\le k\le n}$$

where $\lfloor\cdot\rfloor$ denotes the floor integer part of a number. Clearly, R commutes with the additive group of translations of $\mathbb{Z}^n$,

$$R(u + z) = R(u) + z \quad \text{for all } u \in I\!R^n,\ z \in \mathbb{Z}^n\,,$$

and the preimage of the zero vector under the operator is the half-open cube

$$V = \{u \in I\!R^n :\ R(u) = 0\} = [-1/2, 1/2)^n\,.$$

The basic property that the diffeomorphism f is a bijection is, in general, lost when the mapping is replaced by its discretization f_ε. This loss of invertibility can be formalized as follows. Denoting

$$f_\varepsilon^{-1}(y) = \{x \in X_\varepsilon :\ f_\varepsilon(x) = y\}$$

as the preimage of a point $y \in \varepsilon\mathbb{Z}^n$ under the mapping f_ε, introduce the sets

$$X_\varepsilon^{\rm inj} = \left\{x \in X_\varepsilon :\ \#\left(f_\varepsilon^{-1}\left(f_\varepsilon(x)\right)\right) = 1\right\}\,, \tag{1}$$

$$Y_\varepsilon^{\rm sur} = Y_\varepsilon \bigcap f_\varepsilon\left(X_\varepsilon\right) \tag{2}$$

consisting respectively of *injectivity* and *surjectivity points* of f_ε. Here, $\#(\cdot)$ stands for the cardinality of a finite set. Since f_ε is not bijective in general, $X_\varepsilon^{\rm inj}$ and $Y_\varepsilon^{\rm sur}$ are distinct from X_ε and Y_ε. But it is an interesting question as to whether the limits

$$\lim_{\varepsilon\to+0} \frac{\#X_\varepsilon^{\rm inj}}{\#X_\varepsilon}\,, \quad \lim_{\varepsilon\to+0} \frac{\#Y_\varepsilon^{\rm sur}}{\#Y_\varepsilon} \tag{3}$$

exist. In a sense, these characterize the degree to which the discretization f_ε inherits injectivity and surjectivity of the original diffeomorphism as computer accuracy $\varepsilon \to 0$. We shall show that the limits (3) exist and depend explicitly upon the Jacobian of the diffeomorphism f.

3 Asymptotic distributions of injectivity and surjectivity points

Let
$$K(y) = \left(f^{-1}\right)'(y) = \left(f'\left(f^{-1}(y)\right)\right)^{-1}$$
denote the Jacobian matrix of the inverse diffeomorphism $f^{-1} : Y \to X$ at a point $y \in Y$ and define the function $H : Y \times \mathbb{R}^n \to \mathbb{Z}_+$ by

$$H(y,u) = \#\left(\mathbb{Z}^n \bigcap (u + K(y)V)\right) . \tag{4}$$

Associate with H the sequence of functions $q_k : Y \to [0,1]$ defined by

$$q_k(y) = \mathrm{mes}_n \{u \in [0,1]^n : \; H(y,u) = k\}$$

where $\mathrm{mes}_n(\,\cdot\,)$ stands for the n-dimensional Lebesgue measure of a set. From this definition it follows that
$$\sum_{k \ge 0} q_k(y) = 1$$
and, consequently, for a fixed $y \in Y \subset \mathbb{R}^n$, $\{q_k(y)\}$ describes a probability distribution on the set of nonnegative integers $\mathbb{Z}_+$. The mathematical expectation of this distribution is given by

$$\sum_{k \ge 0} k\, q_k(y) = \int\limits_{[0,1]^n} H(y,u)\, du = |\det K(y)| \,. \tag{5}$$

To see this, note that for any bounded measurable set $G \subset \mathbb{R}^n$,

$$\int\limits_{[0,1]^n} \#\left(\mathbb{Z}^n \bigcap (u + G)\right) du = \mathrm{mes}_n G \,.$$

Since $\mathrm{mes}_n V = 1$, (5) is immediate. By the Chebyshev inequality, from (5) it follows that
$$1 - q_0(y) \le |\det K(y)| \,.$$
For any $\varepsilon > 0$, introduce the sets

$$Y_{\varepsilon,k} = \left\{y \in Y_\varepsilon : \; \# f_\varepsilon^{-1}(y) = k\right\} \tag{6}$$

which, when taken for all $k \in \mathbb{Z}_+$, form a partition of the ε-grid Y_ε.

Theorem 1 *Let the diffeomorphism $f : X \to Y$ satisfy the condition that the rows of the matrix*

$$\begin{bmatrix} I_n \\ f'(x) \end{bmatrix} \in \mathbb{R}^{2n \times n}$$

are rationally independent for almost all $x \in X$, which may be written as

$$\operatorname{mes}_n \left\{ x \in X : \ (f'(x))^T z \in \mathbb{Z}^n \right\} = 0 \quad \textit{for all } z \in \mathbb{Z}^n \setminus \{0\} .$$

Then, for any $k \in \mathbb{Z}_+$ and any Jordan measurable set $B \subset Y$,

$$\lim_{\varepsilon \to +0} \left(\varepsilon^n \# \left(B \bigcap Y_{\varepsilon,k} \right) \right) = \int_B q_k(y)\, dy .$$

Theorem 1 (which will be proved in Sections 5–8 together with Theorem 2) shows that each of the sets $Y_{\varepsilon,k}$ possesses an asymptotic spatial distribution which is absolutely continuous with density q_k.

Theorem 2 *Under the assumption of Theorem 1, for any Jordan measurable sets $A \subset X$ and $B \subset Y$,*

$$\lim_{\varepsilon \to +0} \left(\varepsilon^n \# \left(A \bigcap X_\varepsilon^{\mathrm{inj}} \right) \right) = \int_A p_{\mathrm{inj}}(x)\, dx , \tag{7}$$

$$\lim_{\varepsilon \to +0} \left(\varepsilon^n \# \left(B \bigcap Y_\varepsilon^{\mathrm{sur}} \right) \right) = \int_B p_{\mathrm{sur}}(y)\, dy , \tag{8}$$

where the functions $p_{\mathrm{inj}} : X \to \mathbb{R}_+$ and $p_{\mathrm{sur}} : Y \to \mathbb{R}_+$ are given by

$$p_{\mathrm{inj}}(x) = q_1(f(x)) \, |\det f'(x)| , \tag{9}$$

$$p_{\mathrm{sur}}(y) = 1 - q_0(y) . \tag{10}$$

By Theorem 2, there exist absolutely continuous asymptotic spatial distributions for injectivity and surjectivity points of the discretization f_ε with the densities (9) and (10) respectively. In particular, the limits (3) are computed as

$$\lim_{\varepsilon \to +0} \frac{\# X_\varepsilon^{\mathrm{inj}}}{\# X_\varepsilon} = \frac{1}{\operatorname{mes}_n X} \int_Y q_1(y)\, dy , \tag{11}$$

$$\lim_{\varepsilon \to +0} \frac{\# Y_\varepsilon^{\mathrm{sur}}}{\# Y_\varepsilon} = \frac{1}{\operatorname{mes}_n Y} \int_Y (1 - q_0(y))\, dy . \tag{12}$$

4 Application: discretized conformal mappings

Let $g : X \to Y$ be a conformal mapping of two bounded domains X and Y in the complex plane $\mathbb{C}$, and let $g^{-1} : Y \to X$ denote the corresponding inverse mapping. In view of the natural bijection between $x = (x_1, x_2) \in \mathbb{R}^2$ and $w = x_1 + ix_2 \in \mathbb{C}$, g can be identified with the two-dimensional diffeomorphism f by

$$f(x) = (\operatorname{Re} g(w), \operatorname{Im} g(w)) \,. \tag{13}$$

Since the Jacobian matrix for such a diffeomorphism is given by

$$f'(x) = |g'(w)| \begin{bmatrix} \cos\arg g'(w) & -\sin\arg g'(w) \\ \sin\arg g'(w) & \cos\arg g'(w) \end{bmatrix} \tag{14}$$

(with arg standing for the argument of a complex number), the rational dependence of the rows of the matrix

$$\begin{bmatrix} I_2 \\ f'(x) \end{bmatrix} \in \mathbb{R}^{4\times 2}$$

is equivalent to $g'(w)$ being a complex rational. Therefore, the diffeomorphism (13) *does not* obey the condition of Theorem 1 if and only if there exists a complex rational c such that

$$\operatorname{mes}_2 \{w \in X : \; g'(w) = c\} > 0 \,.$$

But since g is analytic, this last implies the linearity of the mapping g

$$g(w) = cw + d$$

where d is a complex constant. Hence, as soon as the conformal mapping g is nonlinear, the corresponding diffeomorphism (13) satisfies the assumption of Theorem 1.

Taking this into account and using (14), it is straightforward to obtain that the asymptotic densities (9) and (10) of injectivity and surjectivity points for the discretized nonlinear conformal mapping g at points $w \in X$ and $z \in Y$ are

$$p_{\mathrm{inj}}(w) = P\left(\frac{1}{g'(w)}\right) |g'(w)|^2 \,, \tag{15}$$

$$p_{\mathrm{sur}}(z) = 1 - Q\left(\frac{1}{g'\left(g^{-1}(z)\right)}\right) \,, \tag{16}$$

where the functions P, $Q : \mathbb{C} \to [0,1]$ are defined by the following equalities which can be verified with the aid of Fig. 1–4:

$$\begin{aligned} P(v) &= \operatorname{mes}_2 \left\{ u \in [0,1)^2 : \; \# \left(\mathbb{Z}^2 \bigcap \left(u + \begin{bmatrix} \operatorname{Re} v & -\operatorname{Im} v \\ \operatorname{Im} v & \operatorname{Re} v \end{bmatrix} [-1/2, 1/2)^2 \right) \right) = 1 \right\} \\ &= \begin{cases} |v|^2 & \text{for } \; |v|^2 \le \|v\|_\infty \\ -3|v|^2 + 4\|v\|_1 - 4|\operatorname{Re} v \operatorname{Im} v|/|v|^2 & \text{for } \; \|v\|_\infty < |v|^2 \le \|v\|_1 \\ (2\|v\|_\infty/|v| - |v|)^2 & \text{for } \; \|v\|_1 < |v|^2 \le 2\|v\|_\infty \\ 0 & \text{otherwise} \end{cases} \,, \end{aligned} \tag{17}$$

$$
\begin{aligned}
Q(v) &= \operatorname{mes}_2\left\{u\in[0,1)^2:\ \#\left(\mathbb{Z}^2\bigcap\left(u+\begin{bmatrix}\operatorname{Re}v & -\operatorname{Im}v\\ \operatorname{Im}v & \operatorname{Re}v\end{bmatrix}[-1/2,1/2)^2\right)\right)=0\right\}\\
&= \begin{cases} 1-|v|^2 & \text{for } |v|^2\le\|v\|_\infty\\ (\|v\|_1/|v|-|v|)^2 & \text{for } \|v\|_\infty<|v|^2\le\|v\|_1\\ 0 & \text{otherwise}\end{cases},
\end{aligned}
\tag{18}
$$

where

$$
\|v\|_\infty=\max\left(|\operatorname{Re}v|,\,|\operatorname{Im}v|\right),\qquad \|v\|_1=|\operatorname{Re}v|+|\operatorname{Im}v|
$$

denote the l_∞- and l_1-norms of a complex number v.

As an illustrative example, consider the mapping

$$
g(w)=w^2 \tag{19}
$$

which conformally transforms the quarter-disc

$$
X=\{w\in\mathbb{C}:\ \operatorname{Re}w,\,\operatorname{Im}w>0,\ |w|<1\},
$$

onto the half-disc

$$
Y=\{z\in\mathbb{C}:\ \operatorname{Im}z>0,\ |z|<1\}.
$$

Using (17) and (18), it is straightforward to obtain the following expressions for the asymptotic densities (15) and (16)

$$
\begin{aligned}
p_{\mathrm{inj}}(w) &= 4|w|^2\,P\left(\frac{1}{2w}\right)\\
&= \begin{cases} 1 & \text{for } \|w\|_\infty\ge\frac12\\ -3+8\|w\|_1-16|\operatorname{Re}w\operatorname{Im}w| & \text{for } \|w\|_\infty<\frac12\le\|w\|_1\\ (4\|w\|_\infty-1)^2 & \text{for } \|w\|_1<\frac12\le 2\|w\|_\infty\\ 0 & \text{for } \|w\|_\infty<\frac14\end{cases},
\end{aligned}
$$

$$
\begin{aligned}
p_{\mathrm{sur}}(z) &= 1-Q\left(\frac{1}{2\sqrt{z}}\right)\\
&= \begin{cases} \frac{1}{4|z|} & \text{for } \|\sqrt{z}\|_\infty\ge\frac12\\ 1-\left(\|\sqrt{z}\|_1-\frac12\right)^2/|z| & \text{for } \|\sqrt{z}\|_\infty<\frac12\le\|\sqrt{z}\|_1\\ 1 & \text{for } \|\sqrt{z}\|_1<\frac12\end{cases}.
\end{aligned}
$$

The graphs of the densities are shown in Fig. 5 and 6. It is easy to see their qualitative agreement with the experimental results provided by Fig. 7 and 8 where the injectivity and surjectivity points of the 0.01-discretized mapping (19) are plotted. Furthermore, using the identities

$$
\|\sqrt{z}\|_\infty=\sqrt{\frac{|z|+|\operatorname{Re}z|}{2}},\qquad \|\sqrt{z}\|_1=\|\sqrt{z}\|_\infty+\sqrt{\frac{|z|-|\operatorname{Re}z|}{2}},
$$

and computing the integrals on the right hand sides of (11) and (12), obtain

$$\begin{aligned}
\lim_{\varepsilon\to+0} \frac{\#X_\varepsilon^{\rm inj}}{\#X_\varepsilon} &= 1-\frac{5}{8\pi} = 0.8011\ldots\,, \\
\lim_{\varepsilon\to+0} \frac{\#Y_\varepsilon^{\rm sur}}{\#Y_\varepsilon} &= 0.4337\ldots\,.
\end{aligned}$$

A comparison of these limit values with the empirical fractions of injectivity and surjectivity points of $g_{0.01}$

$$\frac{\#X_{0.01}^{\rm inj}}{\#X_{0.01}} = \frac{6248}{7750} = 0.8062\ldots\,, \qquad \frac{\#Y_{0.01}^{\rm sur}}{\#Y_{0.01}} = \frac{6753}{15599} = 0.4329\ldots$$

gives strong experimental evidence of the validity of Theorem 2 when applied to spatial discretizations of nonlinear conformal mappings.

5 Periodicity in cardinality of preimages

Define $H_\varepsilon : Y \times I\!R^n \to Z\!\!\!Z_+$ by

$$H_\varepsilon(y,u) = \#\left(Z\!\!\!Z^n \bigcap (u + G_\varepsilon(y))\right)\,,$$

and the set-valued mapping $G_\varepsilon : Y \to 2^{I\!R^n}$ by

$$G_\varepsilon(y) = \left(f^{-1}\left((y+\varepsilon V)\bigcap Y\right) - f^{-1}(y)\right)/\varepsilon\,.$$

Then the cardinality of the f_ε preimage set of a point $y \in Y_\varepsilon$ can be rewritten as

$$\#f_\varepsilon^{-1}(y) = H_\varepsilon\left(y,\, \varepsilon^{-1} f^{-1}(y)\right)\,. \tag{20}$$

Clearly, $H_\varepsilon(y,u)$ is unit periodic in the n coordinates of $u \in I\!R^n$ for any fixed $y \in Y$. Taking this *conditional periodicity* into account, consider the asymptotic behaviour of the function H_ε on the set $Y \times [0,1)^n$ as $\varepsilon \to +0$.

Lemma 1 *The function H_ε converges to the function H in (4) in the sense that*

$$\lim_{\varepsilon\to+0,\ w\to v} H_\varepsilon(w) = H(v) \quad \textit{for almost all } v \in Y \times [0,1)^n\,.$$

Proof. Note that H_ε is piecewise constant and its restriction to the set $Y \times [0,1)^n$ is continuous on

$$L_\varepsilon = \{(y,u) :\ y \in Y \text{ and } u \in [0,1)^n \setminus (-\partial G_\varepsilon(y) + Z\!\!\!Z^n)\}\,, \tag{21}$$

where $\partial(\cdot)$ denotes the boundary of a set. Moreover, for any $0 < \varepsilon_1 < \varepsilon_2$, $H_\varepsilon(w)$ is constant for $\varepsilon \in (\varepsilon_1, \varepsilon_2)$ and $w \in \bigcap_{\varepsilon\in(\varepsilon_1,\varepsilon_2)} L_\varepsilon$. Similarly, using the central symmetry

$\partial V = -\partial V$, the restriction of the piecewise constant function H to $Y \times [0,1)^n$ is continuous on the open set

$$L = \{(y,u): \; y \in Y \text{ and } u \in (0,1)^n \setminus (K(y)\,\partial V + \mathbb{Z}^n)\} \,. \tag{22}$$

By the continuous differentiability of f^{-1},

$$\lim_{\varepsilon \to +0} \text{dist}\,(\partial G_\varepsilon(y),\, K(y)\partial V) = 0 \tag{23}$$

uniformly in y over any given compact subset of Y, where dist$\,(\cdot,\cdot)$ is the Hausdorff distance between sets. From (21)–(23) it follows that for any $v \in L$ there exist an open neighbourhood $M \subset L$ of v and $\delta > 0$ such that

$$M \subset \bigcap_{0<\varepsilon<\delta} L_\varepsilon \,.$$

Hence,

$$H_\varepsilon(w) = H(v) \quad \text{for all } \varepsilon \in (0,\delta),\; w \in M \,.$$

This last relation implies that

$$\lim_{\varepsilon \to +0,\; w \to v} H_\varepsilon(w) = H(v) \quad \text{for all } v \in L \,,$$

and since

$$\text{mes}_{\,2n}\,((Y \times [0,1)^n) \setminus L) = 0 \,,$$

the proof is complete. ■

6 Asymptotically uniform distribution of toral projections

For any $\omega \in \mathbb{Z}^n$, define the elementary trigonometric polynomial $T_\omega : \mathbb{R}^n \to \mathbb{C}$ by

$$T_\omega(u) = \exp\left(2\pi i \omega^T u\right) \,. \tag{24}$$

Lemma 2 *Let $F : Y \to \mathbb{R}^n$ be a continuously differentiable mapping on a bounded open set $Y \subset \mathbb{R}^n$ such that the rows of the matrix*

$$\begin{bmatrix} I_n \\ F'(y) \end{bmatrix} \in \mathbb{R}^{2n \times n} \tag{25}$$

are rationally independent for almost all $y \in Y$. Then for any Jordan measurable set $B \subset Y$,

$$\lim_{\varepsilon \to +0} \left(\varepsilon^n \sum_{y \in B \bigcap Y_\varepsilon} T_\omega\left(\varepsilon^{-1} F(y)\right)\right) = \begin{cases} \text{mes}_{\,n} B & \textit{for} \quad \omega = 0 \\ 0 & \textit{for} \quad \omega \in \mathbb{Z}^n \setminus \{0\} \end{cases} . \tag{26}$$

Proof. The validity of (26) for $\omega = 0$ immediately follows from the identity $T_0 = 1$. Now prove the limit relation for an arbitrary $\omega \in \mathbb{Z}^n \setminus \{0\}$. For any $\alpha \in \mathbb{N}$, consider the discrete cube

$$Q_\alpha = \left\{ z = (z_k)_{1 \le k \le n} \in \mathbb{Z}^n : \ |z_k| \le \alpha \text{ for all } 1 \le k \le n \right\}$$

consisting of $(2\alpha+1)^n$ points of $\mathbb{Z}^n$. By appropriate translations of this cube, obtain a partition of $\mathbb{Z}^n$

$$\{ y + Q_\alpha : \ y \in (2\alpha+1)\, \mathbb{Z}^n \} .$$

It is straightforward to verify that for any fixed $\alpha \in \mathbb{N}$ and any Jordan measurable set $B \subset Y$,

$$\# \left(\left(B \bigcap Y_\varepsilon \right) \triangle \left(\left(B \bigcap Y_{(2\alpha+1)\varepsilon} \right) + \varepsilon\, Q_\alpha \right) \right) = o\left(\varepsilon^{-n} \right) \quad \text{as } \varepsilon \to +0 . \tag{27}$$

Let $F : Y \to \mathbb{R}^n$ be a continuously differentiable mapping satisfying the assumption of the lemma. For any given $\omega \in \mathbb{Z}^n \setminus \{0\}$ and any $\varepsilon > 0$, define $\varphi_\varepsilon : Y \to \mathbb{C}$ by

$$\varphi_\varepsilon = T_\omega \circ \varepsilon^{-1} \circ F$$

Since $|\varphi_\varepsilon| = 1$, from (27) it follows that for any $\alpha \in \mathbb{N}$,

$$\limsup_{\varepsilon \to +0} \left| \varepsilon^n \sum_{y \in B \bigcap Y_\varepsilon} \varphi_\varepsilon(y) \right| \le \limsup_{\varepsilon \to +0} \left(((2\alpha+1)\varepsilon)^n \sum_{y \in B \bigcap Y_{(2\alpha+1)\varepsilon}} \psi_{\alpha,\varepsilon}(y) \right) , \tag{28}$$

where $\psi_{\alpha,\varepsilon} : Y \to \mathbb{R}_+$ is the nonnegative-valued function given by

$$\psi_{\alpha,\varepsilon}(y) = (2\alpha+1)^{-n} \left| \sum_{z \in Q_\alpha} \varphi_\varepsilon(y + \varepsilon z) \right| .$$

Using the continuous differentiability of F, it is easy to show that

$$\lim_{\varepsilon \to +0} \psi_{\alpha,\varepsilon}(y) = (2\alpha+1)^{-n} \left| \sum_{z \in Q_\alpha} T_\omega\left(F'(y) z \right) \right| = \xi_\alpha \left(\left(F'(y) \right)^T \omega \right) \tag{29}$$

for any $\alpha \in \mathbb{N}$ and uniformly in y over any given compact subset of Y. Here, the function $\xi_\alpha : \mathbb{R}^n \to [0,1]$ is defined by

$$\xi_\alpha \left((u_k)_{1 \le k \le n} \right) = \prod_{k=1}^n \eta_\alpha(u_k) \tag{30}$$

where, in turn,

$$\eta_\alpha(v) = \begin{cases} 1 & \text{for } \ v \in \mathbb{Z} \\ \left| \frac{\sin(\pi(2\alpha+1)v)}{(2\alpha+1)\sin(\pi v)} \right| & \text{for } \ v \in \mathbb{R} \setminus \mathbb{Z} \end{cases} . \tag{31}$$

Since the convergence in (29) is uniform, the upper limit of the expression on the right hand side of (28) coincides with its limit and the inequality can thus be rewritten as

$$\limsup_{\varepsilon \to +0} \left| \varepsilon^n \sum_{y \in B \bigcap Y_\varepsilon} \varphi_\varepsilon(y) \right| \le \int_B \xi_\alpha \left(\left(F'(y) \right)^T \omega \right) \, dy . \tag{32}$$

From (30) and (31) it follows that

$$\lim_{\alpha\to+\infty} \xi_\alpha(u) = \left\{ \begin{array}{ll} 1 & \text{for} \quad u \in \mathbb{Z}^n \\ 0 & \text{for} \quad u \in \mathbb{R}^n \setminus \mathbb{Z}^n \end{array} \right. ,$$

so, applying the Lebesgue Dominated Convergence Theorem to the integral on the right of (32), obtain

$$\limsup_{\varepsilon\to+0} \left| \varepsilon^n \sum_{y\in B\bigcap Y_\varepsilon} \varphi_\varepsilon(y) \right| \le \operatorname{mes}_n \left\{ y \in B : \; (F'(y))^T \omega \in \mathbb{Z}^n \right\} .$$

Since $\omega \in \mathbb{Z}^n \setminus \{0\}$, then, by the assumption of the lemma, the right hand side of the last inequality is zero. Hence,

$$\lim_{\varepsilon\to+0} \left(\varepsilon^n \sum_{y\in B\bigcap Y_\varepsilon} \varphi_\varepsilon(y) \right) = 0 . \qquad \blacksquare$$

Lemma 3 *Let F be a mapping satisfying the assumptions of Lemma 2, and let $\varphi : \mathbb{R}^n \to \mathbb{C}$ be a continuous function, unit periodic in its n variables. Then for any Jordan measurable set $B \subset Y$,*

$$\lim_{\varepsilon\to+0} \left(\varepsilon^n \sum_{y\in B\bigcap Y_\varepsilon} \varphi\left(\varepsilon^{-1} F(y)\right) \right) = \operatorname{mes}_n B \int_{[0,1]^n} \varphi(u)\, du . \tag{33}$$

Proof. Let $\varphi : \mathbb{R}^n \to \mathbb{C}$ be a trigonometric polynomial,

$$\varphi = \sum_{\omega\in\Omega} c_\omega T_\omega \tag{34}$$

where T_ω are given by (24), c_ω are complex coefficients, and Ω is a finite subset of $\mathbb{Z}^n$ containing the zero vector. Clearly,

$$c_0 = \int_{[0,1]^n} \varphi(u)\, du . \tag{35}$$

On the other hand, application of Lemma 2 easily gives

$$\lim_{\varepsilon\to+0} \left(\varepsilon^n \sum_{y\in B\bigcap Y_\varepsilon} \varphi\left(\varepsilon^{-1} F(y)\right) \right) = c_0 \operatorname{mes}_n B . \tag{36}$$

Comparing (35) and (36) shows the validity of (33) for any trigonometric polynomial φ described by (34). The proof of the lemma now follows because the set of all such polynomials φ is dense in the Banach space of continuous functions $\psi : \mathbb{R}^n \to \mathbb{C}$ which are unit periodic in all n variables, endowed with the uniform metric. $\blacksquare$

Lemma 4 *Let F be a mapping satisfying the assumptions of Lemma 2. Then, for any bounded continuous function $\varphi : Y \times I\!\!R^n \to I\!\!R$ such that $\varphi(y,u)$ is unit periodic in the n coordinates of $u \in I\!\!R^n$ for any fixed $y \in Y$,*

$$\lim_{\varepsilon\to+0}\left(\varepsilon^n \sum_{y\in Y_\varepsilon} \varphi\left(y,\, \varepsilon^{-1} F(y)\right)\right) = \int\limits_{Y\times[0,1]^n} \varphi(z)\, dz\,. \tag{37}$$

Proof. Let B_k, $1 \le k \le N$, be Jordan measurable sets forming a partition of Y, and let $\psi_k : I\!\!R^n \to I\!\!R$ be continuous functions, unit periodic in their n variables. Define the function $\psi : Y \times I\!\!R^n \to I\!\!R$ by

$$\psi(y,u) = \sum_{k=1}^{N} \mathcal{I}_{B_k}(y)\, \psi_k(u)\,, \tag{38}$$

where $\mathcal{I}_M$ stands for the indicator function of a set M. From Lemma 3, it is easily seen that

$$\lim_{\varepsilon\to+0}\left(\varepsilon^n \sum_{y\in Y_\varepsilon} \psi\left(y,\, \varepsilon^{-1} F(y)\right)\right) = \sum_{k=1}^{N} \operatorname{mes}_n B_k \int\limits_{[0,1]^n} \psi_k(u)\, du\,.$$

Now, the limit relation (37) follows from the observation that for any compact set $K \subset Y$ and any $\delta > 0$ there exists a function ψ of form (38) such that

$$\sup_{z\in K\times[0,1)^n} |\varphi(z) - \psi(z)| < \delta\,, \qquad \sup_{z\in Y\times[0,1)^n} |\psi(z)| \le \sup_{z\in Y\times[0,1)^n} |\varphi(z)|\,. \qquad \blacksquare$$

Given a mapping $F : Y \to I\!\!R^n$, for each $\varepsilon > 0$, introduce a countably additive measure μ_ε defined on the Borel sets $B \subset Y \times [0,1)^n$ by

$$\mu_\varepsilon(B) = \varepsilon^n \,\#\left\{y \in Y_\varepsilon :\ \left(y,\, W\left(\varepsilon^{-1} F(y)\right)\right) \in B\right\} \tag{39}$$

where the mapping $W : I\!\!R^n \to [0,1)^n$ is given by

$$W\left((u_k)_{1\le k\le n}\right) = (\{\!\{u_k\}\!\})_{1\le k\le n}\,,$$

and can be regarded as a projection onto the n-dimensional torus (here, $\{\!\{v\}\!\} = v - \lfloor v \rfloor$ stands for the fractional part of a number v).

Theorem 3 *Let F be a mapping satisfying the assumptions of Lemma 2. Then the measure (39) converges weakly to the measure* mes_{2n} *as $\varepsilon \to +0$.*

Proof. From the definition of the measure μ_ε, for any function $\varphi : Y \times I\!\!R^n \to I\!\!R$ such that $\varphi(y,u)$ is unit periodic in the n coordinates of $u \in I\!\!R^n$ for any fixed $y \in Y$,

$$\varepsilon^n \sum_{y\in Y_\varepsilon} \varphi\left(y,\, \varepsilon^{-1} F(y)\right) = \int\limits_{Y\times[0,1)^n} \varphi(z)\, \mu_\varepsilon(dz)\,.$$

Hence, by Lemma 4, if the function φ is bounded and continuous,

$$\lim_{\varepsilon\to+0} \int\limits_{Y\times[0,1)^n} \varphi(z)\, \mu_\varepsilon(dz) = \int\limits_{Y\times[0,1)^n} \varphi(z)\, dz\,. \tag{40}$$

Since φ is otherwise arbitrary, (40) immediately implies the theorem. $\blacksquare$

7 Proof of Theorem 1

Let the measure μ_ε be defined on Borel subsets of $Y \times [0,1)^n$ by (39) with $F = f^{-1}$. By the conditionally periodic representation (20), for any function $\varphi : Y \times \mathbb{Z}_+ \to \mathbb{R}$,

$$\varepsilon^n \sum_{y \in Y_\varepsilon} \varphi\left(y,\, \# f_\varepsilon^{-1}(y)\right) = \int\limits_{Y \times [0,1)^n} \varphi\left(y,\, H_\varepsilon(y,u)\right) \mu_\varepsilon(dy \times du)\,. \tag{41}$$

Since

$$\begin{bmatrix} I_n \\ f'(x) \end{bmatrix} = \begin{bmatrix} F'(f(x)) \\ I_n \end{bmatrix} f'(x)$$

and f is a diffeomorphism, the assumption of Theorem 1 implies the rational independence of the rows of the matrix (25) for almost all $y \in Y$. Hence, Theorem 3 applies, and the measure μ_ε converges weakly to the restriction of $\mathrm{mes}_{\,2n}$ to the set $Y \times [0,1)^n$. Therefore, by Lemma 1 and [1, Theorem 5.5 on p. 34], if the function φ in (41) is bounded and such that $\varphi(y,k)$ is continuous almost everywhere in $y \in Y$ for any fixed $k \in \mathbb{Z}_+$, then

$$\begin{aligned} \varepsilon^n \sum_{y \in Y_\varepsilon} \varphi\left(y,\, \# f_\varepsilon^{-1}(y)\right) \;&\to \int\limits_{Y \times [0,1)^n} \varphi\left(y,\, H(y,u)\right) dy \times du \\ &= \int\limits_Y \sum_{k \geq 0} \varphi(y,k)\, q_k(y)\, dy \quad \text{as } \varepsilon \to +0. \end{aligned}$$

In particular, if B is a Jordan measurable subset of Y and $k \in \mathbb{Z}_+$, then for the indicator function $\varphi = \mathcal{I}_{B \times \{k\}}$,

$$\varepsilon^n \sum_{y \in Y_\varepsilon} \varphi\left(y,\, \# f_\varepsilon^{-1}(y)\right) = \varepsilon^n \# \left(B \bigcap Y_{\varepsilon,k}\right) \to \int\limits_B q_k(y)\, dy \quad \text{as } \varepsilon \to +0$$

and the theorem is proved. ∎

8 Proof of Theorem 2

From the definition (2), the set $Y_\varepsilon^{\mathrm{sur}}$ can be expressed in terms of the sets (6) as

$$Y_\varepsilon^{\mathrm{sur}} = \bigcup_{k \geq 1} Y_{\varepsilon,k} = Y_\varepsilon \setminus Y_{\varepsilon,0}\,.$$

Hence, by Theorem 1, for any Jordan measurable set $B \subset Y$,

$$\begin{aligned} \varepsilon^n \# \left(B \bigcap Y_\varepsilon^{\mathrm{sur}}\right) \;&=\; \varepsilon^n \# \left(B \bigcap Y_\varepsilon\right) - \varepsilon^n \# \left(B \bigcap Y_{\varepsilon,0}\right) \\ &\to\; \mathrm{mes}_{\,n} B - \int\limits_B q_0(y)\, dy = \int\limits_B (1 - q_0(y))\, dy \quad \text{as } \varepsilon \to +0\,, \end{aligned}$$

which, by (10), yields the relation (8). Turning now to (7), given a Jordan measurable set $A \subset X$, its image $B = f(A)$ is a Jordan measurable subset of Y and consequently, application of Theorem 1 gives

$$\lim_{\varepsilon\to+0}\left(\varepsilon^n\,\#\left(B\bigcap Y_{\varepsilon,1}\right)\right)=\int\limits_B q_1(y)\,dy\,. \tag{42}$$

By the definition (1),

$$\#\left(A\bigcap X_\varepsilon^{\mathrm{inj}}\right)=\#f_\varepsilon\left(A\bigcap X_\varepsilon^{\mathrm{inj}}\right) \tag{43}$$

and

$$\#\left(\left(B\bigcap Y_{\varepsilon,1}\right)\triangle f_\varepsilon\left(A\bigcap X_\varepsilon^{\mathrm{inj}}\right)\right)=o\left(\varepsilon^{-n}\right)\quad\text{as } \varepsilon\to+0\,. \tag{44}$$

From (42)–(44), it is easy to obtain the convergence

$$\lim_{\varepsilon\to+0}\left(\varepsilon^n\,\#\left(A\bigcap X_\varepsilon^{\mathrm{inj}}\right)\right)=\int\limits_B q_1(y)\,dy=\int\limits_A q_1(f(x))\,|\det f'(x)|\,dx$$

which, by (9), gives the limit (7), and the theorem is proved. ∎

9 Acknowledgements

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Figure captions

Figure 1. If $\alpha \le \max(|\cos\theta|, |\sin\theta|)$, then each point of the square $[0,1)^2$ is covered by at most one of the four rotated squares. The covered and uncovered areas equal α^2 and $1-\alpha^2$, respectively.

Figure 2. If $\max(|\cos\theta|, |\sin\theta|) < \alpha \le |\cos\theta| + |\sin\theta|$, then each point of the square $[0,1)^2$ is covered by at most two of the four rotated squares. The once covered and uncovered areas equal $-3\alpha^2 + 4\alpha(|\cos\theta| + |\sin\theta|) - 4|\cos\theta \sin\theta|$ and $(|\cos\theta| + |\sin\theta| - \alpha)^2$, respectively.

Figure 3. If $|\cos\theta| + |\sin\theta| < \alpha \le 2\max(|\cos\theta|, |\sin\theta|)$, then each point of the square $[0,1)^2$ is covered by at least one of the four rotated squares. The once covered area equals $(2\max(|\cos\theta|, |\sin\theta|) - \alpha)^2$.

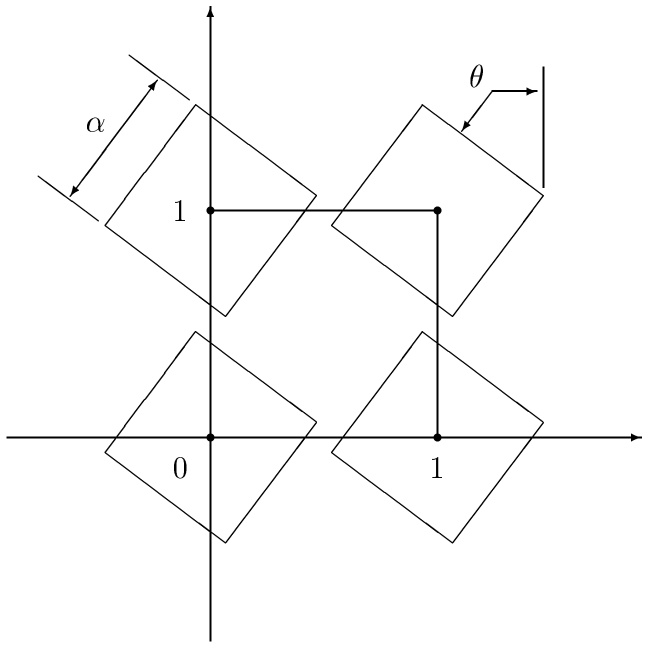
Figure 4. If $\alpha > 2\max(|\cos\theta|, |\sin\theta|)$, then each point of the square $[0,1)^2$ is covered by more than one of the four rotated squares.

Figure 5. The graph of the asymptotic density $p_{\rm inj}$ of injectivity points for the discretized mapping w^2.

Figure 6. The graph of the asymptotic density $p_{\rm sur}$ of surjectivity points for the discretized mapping w^2.

Figure 7. Injectivity points of the 0.01-discretized mapping w^2 on the quarter-disc $|w| < 1$, $\operatorname{Re} w > 0$, $\operatorname{Im} w > 0$. For $\|w\|_\infty < 1/4$ and $\|w\|_\infty > 1/2$ (beyond the solid curve), the asymptotic density $p_{\rm inj}(w)$ of these points equals 0 and 1, respectively.

Figure 8. Surjectivity points of the 0.01-discretized mapping w^2 on the target half-disc $|z| < 1$, $\operatorname{Im} z > 0$. For $\|\sqrt{z}\|_1 < 1/2$ (under the solid curve), the asymptotic density $p_{\rm sur}(z)$ of these points equals 1.



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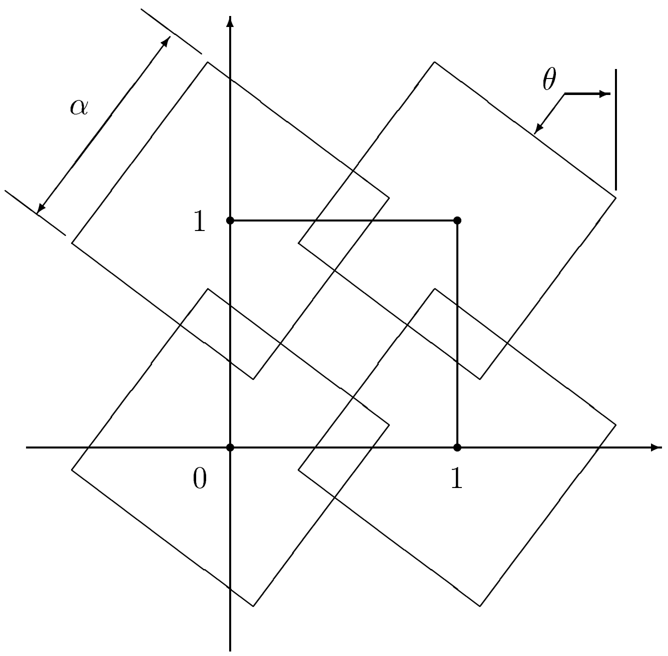


Figure 2:

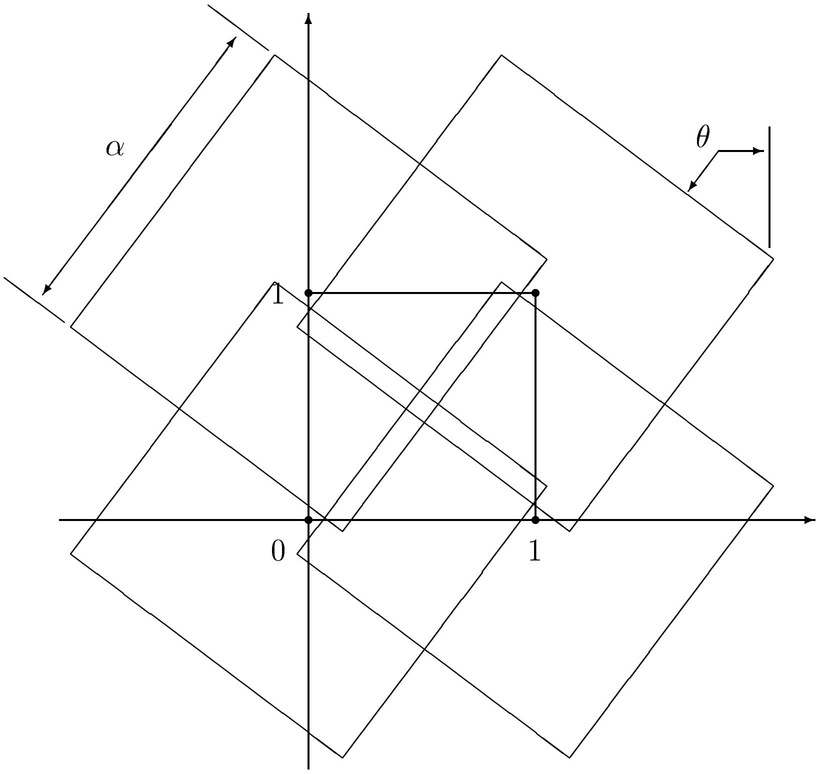


Figure 3:

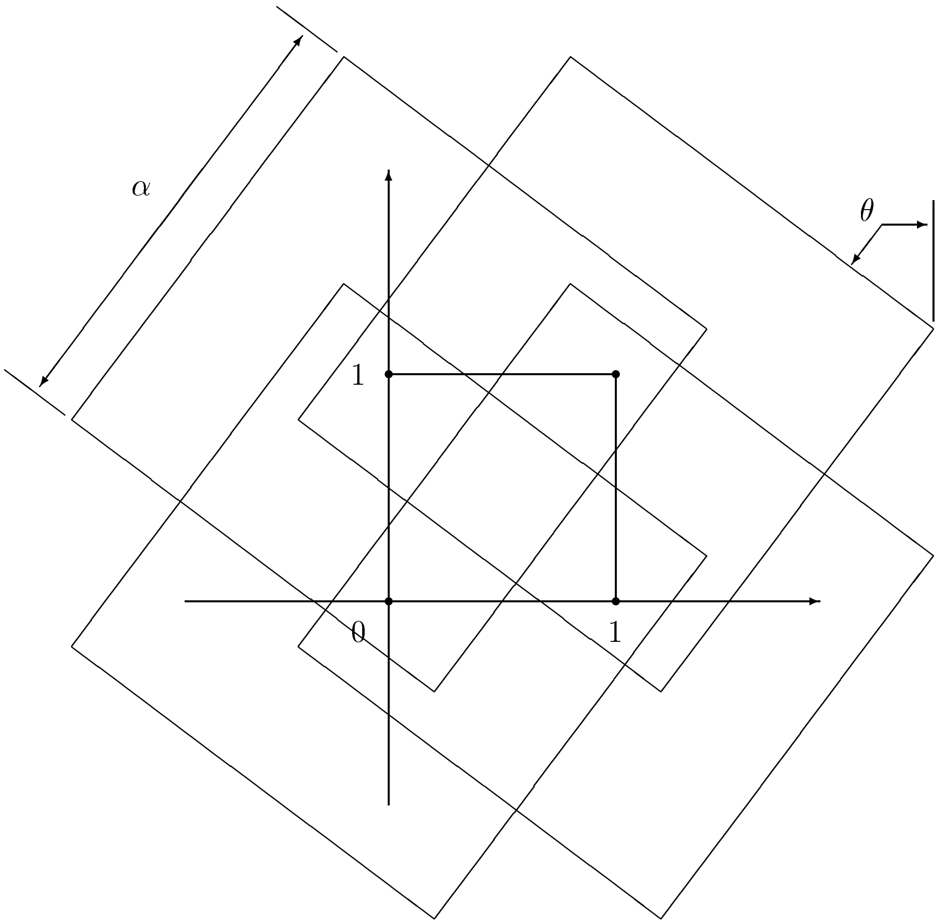


Figure 4:

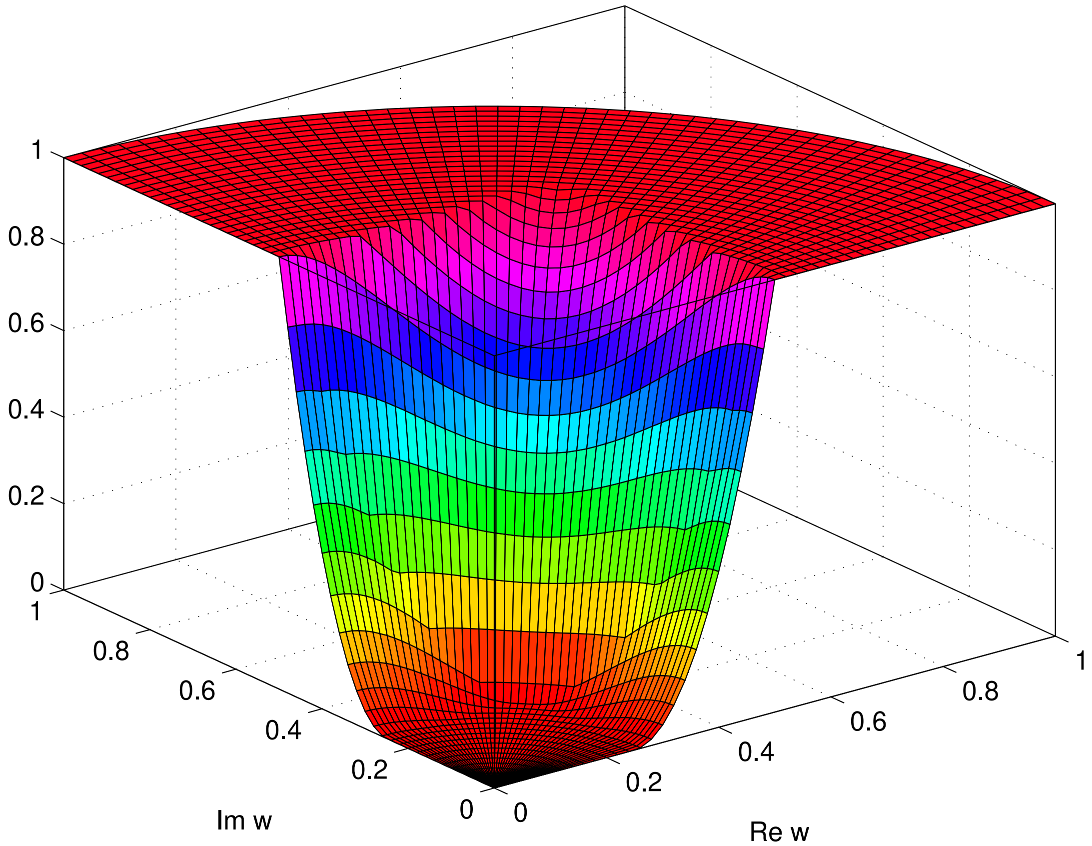
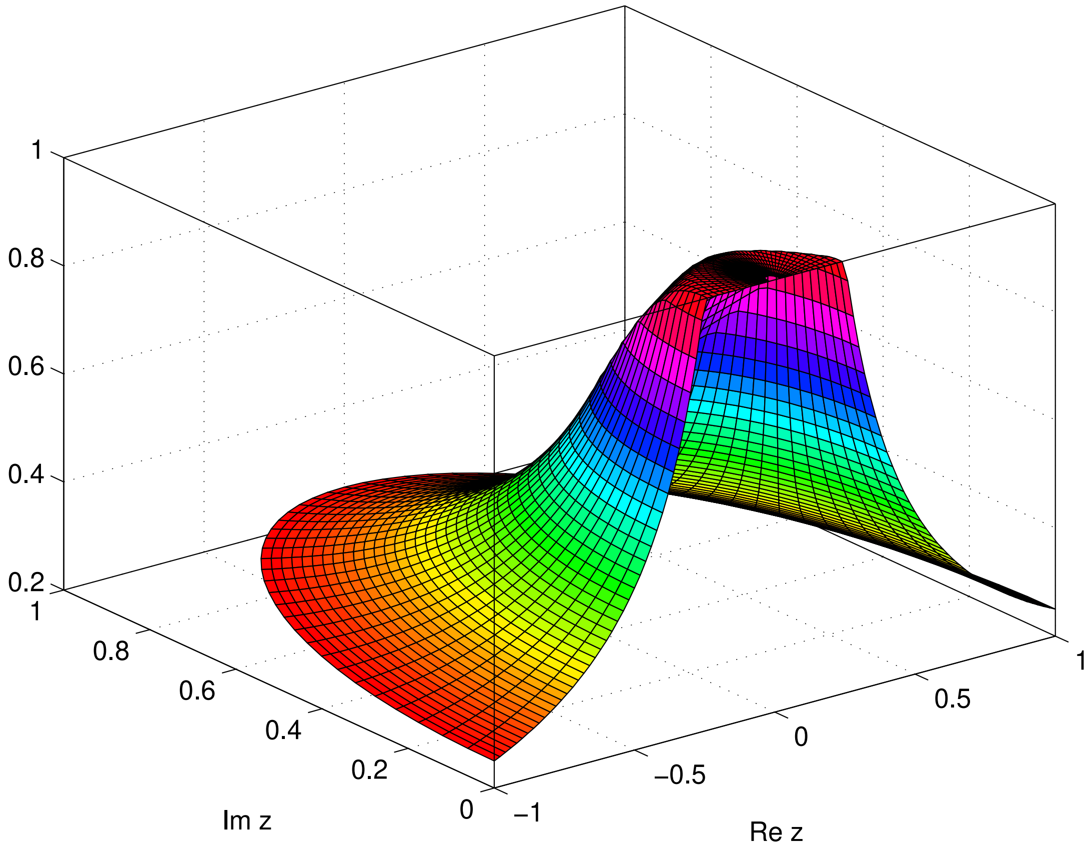


Figure 5:



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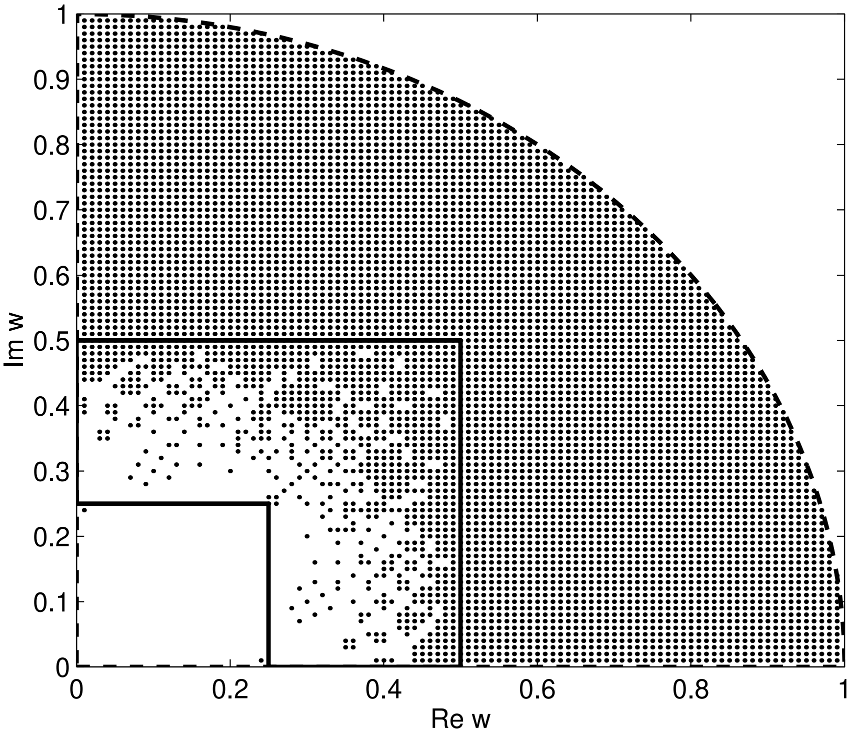

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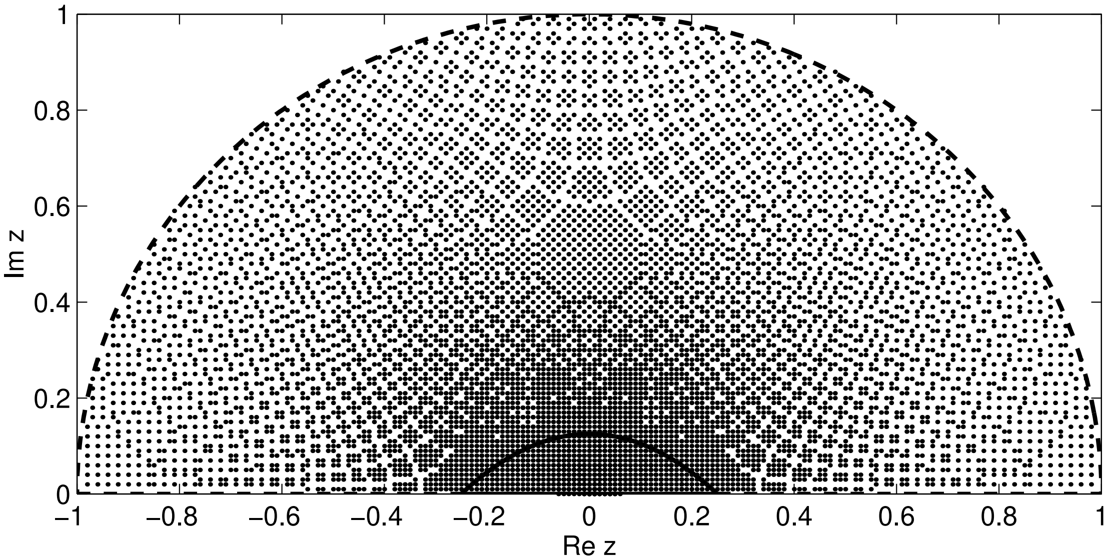

Figure 8: