Frequency Measurability, Algebras of Quasiperiodic Sets and Spatial Discretizations of Smooth Dynamical Systems

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Abstract. The roundoff errors in computer simulations of continuous dynamical systems caused by finiteness of machine arithmetic lead to spatially discrete systems with distinct functional properties. As models for the discretized systems in fixed-point arithmetic, autonomous dynamical systems on multi-dimensional uniform grids can be considered which are generated by composing the transition operator of the original system with a roundoff mapping. To study asymptotic properties of such model systems with increasing refinement of the grid, a probability theoretical approach is developed which is based on equipping the grid with an algebra of frequency measurable quasiperiodic subsets characterized by frequency functions. The approach is applied to spatial discretizations of smooth and invertible dynamical systems. It is shown that, under some nonresonance condition, events relating to mutual deviations of finite segments of trajectories of the discretized and original systems are representable by frequency measurable quasiperiodic sets with frequency functions amenable to explicit computation.

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1 Introduction

The roundoff errors in computer simulations of continuous dynamical systems caused by finiteness of machine arithmetic can lead to qualitative distortion of functional properties of the original systems [2, 7, 8].

As models for the resulting spatially discretized systems in fixed-point arithmetic with accuracy ε, autonomous dynamical systems $T_\varepsilon = R_\varepsilon \circ T$ on multi-dimensional cubic grids $X_\varepsilon = (\varepsilon\mathbb{Z}^n) \bigcap X$ can be considered which are generated by composing the transition operator T of the original system in phase space $X \subset \mathbb{R}^n$ with a roundoff operator R_ε modelling the computer discretization procedure.

Practical implementation of the original transition operator is, in general, a multi-stage procedure with not only the final but also intermediate quantities being rounded off. Nevertheless, the aforementioned composition T_ε is a qualitatively adequate model for the computer discretization of simple mappings T involving a few operations like multiplication or division performed with double precision.

Irregularity of increasingly oscillating, though vanishing, disturbances in such model systems, with $\varepsilon \to 0$, lead to persistent quantization effects. In particular, even if the original transition operator T is invertible, its discretization T_ε is, in general, neither surjective nor injective on the grid X_ε for arbitrarily small ε [5]. Because of this loss of invertibility, positive semi-trajectories of T_ε starting at distinct points can eventually couple. On the other hand, there can be points to which no points of the grid are mapped by T_ε. Informally, points of X_ε under the action of T_ε behave like particles of a compressible fluid evolving in discrete time. In k steps of its evolution, this hypothetical fluid fills in the set $X_{\varepsilon,k} = T_\varepsilon^k(X_\varepsilon) \bigcap X_\varepsilon$ of k-reachable points (which is normally a proper subset of X_ε) and never attends the set $X_\varepsilon \setminus X_{\varepsilon,k}$ again afterwards. Furthermore, the fluid is eventually absorbed by the set $C_\varepsilon = \bigcap_{k\geq 1} X_{\varepsilon,k}$ consisting of limit cycles of the mapping T_ε. A surprising feature of this behaviour for small ε is that the set C_ε is usually "sparse" in comparison with X_ε even if the original transition operator T has an absolutely continuous invariant measure with strictly positive density. Such *degeneration* or *computational collapse* of T_ε in its phase space X_ε is an intrinsically discrete phenomenon which, as discussed in [9], should have been described by probabilistic laws resembling those in the statistical mechanics of quantum physical systems of large number of particles.

Therefore, it seems reasonable to investigate the spatially discretized systems T_ε asymptotically from a probability theoretical viewpoint by equipping the grid X_ε with a probability structure which would suffice to describe the distortion of functional properties for a wide class of continuous dynamical systems T under unboundedly refining discretization $\varepsilon \to 0$. The paper is an attempt to provide a rigorous background towards this direction.

More precisely, we endow the grid X_ε with a totality of *quasiperiodic* subsets. Any such set is determined by a smooth vector-valued mapping F and a set-valued mapping S_ε parameterized by ε. It turns out that if F is *nonresonant* and S_ε is *graphically-continuously convergent* as $\varepsilon \to 0$, then the corresponding F-quasiperiodic set is *frequency measurable* in the sense of existence of an asymptotic absolutely continuous spatial distribution for its points. The density of the limiting distribution is called

a *frequency function*. It is important that the totality of frequency measurable F-quasiperiodic sets is an algebra with respect to set-theoretical operations. Moreover, considered at a given point, the frequency function for the elements of the algebra is a finitely additive probability measure.

In this paper, the original transition operator T is considered to be a smooth diffeomorphism which corresponds to a smooth and invertible dynamical system. Assuming that T is *iteratively nonresonant*, we consider the algebras of frequency measurable F-quasiperiodic subsets of X_ε where the mappings F are obtained by "stacking" finitely many iterates of T and of its inverse $U = T^{-1}$. The minimal algebra containing such algebras is called the *double-sided algebra*. The *positive algebra* is defined in a similar way with the aid of the mappings F formed by iterates of T. Similarly, the mappings F formed by iterates of U are used to define the *negative algebra*. Associated with the transition operator T, these three algebras are shown to be induced by appropriately defined *positive* and *negative quantization errors* which are responsible for the deviations of finite, though arbitrarily long, initial segments of positive and negative semi-trajectories of the discretized mapping T_ε from those of T. Therefore, events relating to such deviations are represented by frequency measurable quasiperiodic sets whose frequency functions afford effective computation, as demonstrated in [4, 6]. In particular, so are the sets of k-reachable points $X_{\varepsilon,k}$, which belong to the negative algebra and whose frequency functions q_k allow for lower bounds, with q_1 amenable to explicit computation.

It turns out that the double-sided algebra is invariant under T_ε and the mapping is measurable with respect to the algebra. A similar invariance under T_ε holds for the negative algebra, and measurability of T_ε is also valid with respect to the positive algebra. Furthermore, on the positive algebra, the mapping T_ε satisfies a mixing property.

The probability theoretical approach used in this paper is a development of the frequency-based approach which was proposed in [14] for quantized linear systems and then extended in [4] to investigate positive semi-trajectories of spatial discretizations of nonlinear systems. Note that similar, though different, quasiperiodic structures were thoroughly studied in [10] for rounded linear Hamiltonian systems in the plane (see also [13]).

The paper is organized as follows. In Section 2, the roundoff operator and related roundoff error in fixed-point arithmetic with accuracy ε are defined. Section 3 provides the definition of the frequency measurability of subsets of the ε-grid and basic properties of frequency functions. Quasiperiodic sets are defined in Section 4. Section 5 formulates sufficient conditions for frequency measurability of quasiperiodic sets based on the notions of nonresonance and graphical-continuous convergence of mappings. In Section 6, frequency-preserving transformations of quasiperiodic sets are described. In Section 7, spatial discretizations of smooth and invertible dynamical systems considered in the paper and related quantization errors are defined. Section 8 defines the algebras of frequency measurable quasiperiodic sets associated with the discretizations of iteratively nonresonant systems. In Section 9, the links between the associated algebras and the quantization errors are considered. In Sections 10–12, the properties of the associated algebras with respect to the discretized transition operator and its

set-valued inverse are formulated. Sections 14–22 provide auxiliary lemmas and proofs of the theorems.

2 Rounding off in fixed-point arithmetic

In fixed-point arithmetic with γ digits after the binary point, a real number u is represented by the nearest node $R_\varepsilon(u)$ of the uniform grid $\varepsilon\mathbb{Z}$ with stepsize $\varepsilon = 2^{-\gamma}$ (we do not take the overflow into account). With a convention to round up midpoints,

$$R_\varepsilon(u) = \varepsilon\lfloor u/\varepsilon + 1/2\rfloor = \varepsilon R_1(u/\varepsilon), \tag{1}$$

where $\lfloor\,\cdot\,\rfloor$ denotes the floor function. Associated with the *roundoff operator* R_ε is the *magnified roundoff error*

$$E_\varepsilon(u) = (u - R_\varepsilon(u))/\varepsilon = E_1(u/\varepsilon) \tag{2}$$

which takes values in the interval $[-1/2, 1/2)$. The roundoff operator and magnified roundoff error extend in an obvious way to spaces of vectors and matrices, where they act component-wise. In particular, when extended to $\mathbb{R}^n$, the mapping R_1 commutes with the additive group of translations of the n-dimensional integer lattice $\mathbb{Z}^n$,

$$R_1(u + z) = R_1(u) + z \quad \text{for all } u \in \mathbb{R}^n,\ z \in \mathbb{Z}^n,$$

and the corresponding mapping E_1 takes values in the full preimage of the zero vector under R_1 which is the half-open cube

$$V_n = R_1^{-1}(0) = \{u \in \mathbb{R}^n :\ R_1(u) = 0\} = [-1/2, 1/2)^n. \tag{3}$$

Clearly, the mapping E_1 is unit periodic in each of its n variables and identically maps the cube onto itself, that is

$$E_1(u + z) = u \quad \text{for all } u \in V_n,\ z \in \mathbb{Z}^n. \tag{4}$$

Although the roundoff operator R_ε will be interpreted in the sequel as a model for the computer discretization procedure in fixed-point arithmetic with accuracy ε, the latter will be considered a small positive parameter taking not only binary rational values $2^{-\gamma}$.

3 Frequency measurability of sets

In what follows, $X \subset \mathbb{R}^n$ will be a fixed open set. For arbitrary $\varepsilon > 0$, define an *ε-grid* of the set as

$$X_\varepsilon = X \bigcap (\varepsilon\mathbb{Z}^n). \tag{5}$$

Remind that a set $G \subset \mathbb{R}^n$ is called Jordan measurable if it is bounded and its boundary ∂G satisfies $\mathrm{mes}_n \partial G = 0$, where $\mathrm{mes}_n(\,\cdot\,)$ denotes the n-dimensional Lebesgue measure of a set. As is well-known, for any such set G, the following asymptotics holds:

$$\#(G \bigcap (\varepsilon\mathbb{Z}^n)) \sim \varepsilon^{-n}\,\mathrm{mes}_n G \quad \text{as } \varepsilon \to +0, \tag{6}$$

where $\#(\,\cdot\,)$ denotes the cardinality of a finite set.

Say that a subset A_ε of the grid X_ε, parameterized by $\varepsilon > 0$, is *frequency measurable* with *frequency function* $p : X \to [0,1]$, if for any Jordan measurable set $G \subset X$,

$$\lim_{\varepsilon \to +0} (\varepsilon^n \#(A_\varepsilon \bigcap G)) = \int\limits_G p(x)\, dx.$$

As can be seen, the frequency measurability is an asymptotic property of the set-valued mapping $0 < \varepsilon \mapsto A_\varepsilon \subset X_\varepsilon$ as $\varepsilon \to +0$, but not of its individual value A_ε for a particular ε. Informally, the property consists in existence of an absolutely continuous asymptotic spatial distribution for points of A_ε, and the frequency function is the density of this distribution.

Clearly, by (6), the grid X_ε itself is frequency measurable with the identically unit frequency function. A less trivial example of a frequency measurable set is $X \bigcap (\varepsilon M \mathbb{Z}^n)$, where $M \in \mathbb{Z}^{n \times n}$ is a given nonsingular matrix. This last set has the frequency function $|\det M|^{-1}$.

Note that the totality of frequency measurable subsets of X_ε is closed under union of disjoint sets and complementation of a set to the grid. More precisely, if $A_\varepsilon, B_\varepsilon$ are frequency measurable sets with frequency functions p, q and satisfying $A_\varepsilon \bigcap B_\varepsilon = \emptyset$ for all sufficiently small $\varepsilon > 0$, then $A_\varepsilon \bigcup B_\varepsilon$ and $X_\varepsilon \setminus A_\varepsilon$ are frequency measurable with frequency functions $p + q$ and $1 - p$, respectively.

4 Definition of quasiperiodic sets

Let $F : X \to \mathbb{R}^m$ be a continuously differentiable mapping and $S_\varepsilon : X \to 2^{\mathbb{R}^m}$ a set-valued mapping. Define a *F-quasiperiodic set* as

$$A_\varepsilon = \{x \in X_\varepsilon :\ \varepsilon^{-1} F(x) \in S_\varepsilon(x) + \mathbb{Z}^m\}, \tag{7}$$

where $A + B = \{a + b :\ a \in A,\ b \in B\}$ denotes the Minkowski sum of subsets A and B of a vector space (with the convention $A + \emptyset = \emptyset$). Note that by (2) and (4), for any set $G \subset \mathbb{R}^m$,

$$G + \mathbb{Z}^m = E_1(G) + \mathbb{Z}^m.$$

Therefore, the definition (7) is equivalent to

$$\begin{aligned} A_\varepsilon &= \{x \in X_\varepsilon :\ \varepsilon^{-1} F(x) \in E_1(S_\varepsilon(x)) + \mathbb{Z}^m\} \\ &= \{x \in X_\varepsilon :\ E_\varepsilon(F(x)) \in E_1(S_\varepsilon(x))\}, \end{aligned}$$

and hence, the mapping S_ε may be assumed, without loss of generality, to take its values in 2^{V_m}. It is easy to see that if B_ε is another F-quasiperiodic set,

$$B_\varepsilon = \{x \in X_\varepsilon :\ E_\varepsilon(F(x)) \in W_\varepsilon(x)\}$$

with a mapping $W_\varepsilon : X \to 2^{V_m}$, then

$$A_\varepsilon \bigcap B_\varepsilon = \{x \in X_\varepsilon :\ E_\varepsilon(F(x)) \in S_\varepsilon(x) \bigcap W_\varepsilon(x)\}, \tag{8}$$

$$A_\varepsilon \bigcup B_\varepsilon = \{x \in X_\varepsilon :\ E_\varepsilon(F(x)) \in S_\varepsilon(x) \bigcup W_\varepsilon(x)\}, \tag{9}$$

$$X_\varepsilon \setminus A_\varepsilon = \{x \in X_\varepsilon :\ E_\varepsilon(F(x)) \in V_m \setminus S_\varepsilon(x)\}. \tag{10}$$

Therefore, for any given F, the totality of F-quasiperiodic subsets of the grid X_ε is an algebra with respect to set-theoretical operations. It turns out that, under some conditions on the mappings F and S_ε, the corresponding F-quasiperiodic sets are frequency measurable in the sense of Section 3.

5 Frequency measurable quasiperiodic sets

Say that a matrix *resonates* if its rows are linearly dependent over the field of real rationals. A continuously differentiable mapping $F : X \to I\!R^m$ will be called *nonresonant* if its *resonance set*

$$\mathcal{R}(F) = \{x \in X : \text{ the matrix } \begin{bmatrix} I_n \\ F'(x) \end{bmatrix} \in I\!R^{(n+m)\times n} \text{ resonates}\} \tag{11}$$

satisfies $\mathrm{mes}_n \mathcal{R}(F) = 0$ (here, I_n denotes the identity matrix of order n, and $F'(x) \in I\!R^{m\times n}$ the Jacobi matrix of the mapping F).

Say that a mapping $f_\varepsilon : \Omega \to \Upsilon$ of a set $\Omega \subset I\!R^r$ to a separable metric space Υ, parameterized by $\varepsilon > 0$, *converges continuously* to a mapping f as $\varepsilon \to +0$ (with the notation $f_\varepsilon \xrightarrow{\mathrm{c}} f$), if the set $\{x \in \Omega : \ f_\varepsilon(y) \to f(x) \text{ as } (\varepsilon, y) \to (+0, x)\}$ has full r-dimensional Lebesgue measure.

Similarly, a set $G_\varepsilon \subset I\!R^r$ is said to converge continuously to a set G (with a similar notation $G_\varepsilon \xrightarrow{\mathrm{c}} G$), if its indicator function $\mathcal{I}_{G_\varepsilon} : I\!R^r \to \{0, 1\}$ is continuously convergent to $\mathcal{I}_G$ as $\varepsilon \to +0$. Note that such convergence is equivalent to

$$\mathrm{mes}_r(\mathrm{clos}(G \bigcup \limsup_{\varepsilon\to+0} G_\varepsilon) \setminus \mathrm{int}(G \bigcap \liminf_{\varepsilon\to+0} G_\varepsilon)) = 0, \tag{12}$$

where $\mathrm{int}(\,\cdot\,)$ and $\mathrm{clos}(\,\cdot\,)$ denote the interior and closure of a set, respectively, and $\liminf_{\varepsilon\to+0} G_\varepsilon = \bigcup_{\delta>0} \bigcap_{\varepsilon<\delta} G_\varepsilon$, $\limsup_{\varepsilon\to+0} G_\varepsilon = \bigcap_{\delta>0} \bigcup_{\varepsilon<\delta} G_\varepsilon$ are defined in a standard way. Clearly, (12) implies the equality $\mathrm{mes}_r \partial G = 0$.

A set-valued mapping $S_\varepsilon : X \to 2^{I\!R^m}$ on $X \subset I\!R^n$, parameterized by $\varepsilon > 0$, is said to *converge graphically-continuously* (g.c.) to a mapping S if the graph

$$\mathrm{Gr}\,(S_\varepsilon) = \{(x, y) : \ x \in X,\ y \in S_\varepsilon(x)\} \subset I\!R^{n+m}$$

converges continuously to $\mathrm{Gr}\,(S)$ as $\varepsilon \to +0$, that will be denoted as $S_\varepsilon \xrightarrow{\mathrm{g.c.}} S$ or $\lim^{\mathrm{g.c.}}_{\varepsilon\to+0} S_\varepsilon = S$. Clearly, the limiting mapping satisfies $\mathrm{mes}_m \partial S(x) = 0$ for mes_n-almost all $x \in X$. Note that the above-defined graphical-continuous convergence differs from the standard graphical convergence of set-valued mappings [1].

Theorem 1 *Let $F : X \to I\!R^m$ be nonresonant, and $S_\varepsilon : X \to 2^{V_m}$ g.c. convergent, with $\lim^{\mathrm{g.c.}}_{\varepsilon\to+0} S_\varepsilon = S$. Then the F-quasiperiodic set in (7) is frequency measurable with the frequency function $\mathrm{mes}_m S(x)$, $x \in X$.*

The proof of the theorem is given in Section 14. Note that the g.c. convergence commutes with set-theoretical operations. Consequently, by Theorem 1 and by (8)–(10), for any given nonresonant mapping F, the F-quasiperiodic subsets of X_ε with g.c.

convergent mappings S_ε form an algebra. This will be called the *algebra of frequency measurable F-quasiperiodic sets* and denoted by

$$\mathcal{A}(F) = \{A_\varepsilon \subset X_\varepsilon :\ A_\varepsilon \text{ is given by (7), with } S_\varepsilon \text{ g.c. convergent}\}. \tag{13}$$

Taken at a fixed but otherwise arbitrary point $x \in X$, the frequency function of such sets is a finitely additive probability measure on this algebra.

Let $F_1, \ldots, F_r$ be smooth mappings on X with values in $\mathbb{R}^{m_1}, \ldots, \mathbb{R}^{m_r}$, respectively. Suppose that the mapping $F : X \to \mathbb{R}^{m_1+\ldots+m_r}$ obtained by stacking the mappings F_k,

$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_r \end{bmatrix},$$

is nonresonant, and hence, so are all the mappings F_k. Then the algebras $\mathcal{A}(F_1)$, ..., $\mathcal{A}(F_r)$ are *independent* in the sense that, for any sets $A_{\varepsilon,k} \in \mathcal{A}(F_k)$ with frequency functions p_k, their intersection $\bigcap_{k=1}^r A_{\varepsilon,k}$ is a frequency measurable F-quasiperiodic set with frequency function $\prod_{k=1}^r p_k$.

6 Frequency-preserving transformations of quasiperiodic sets

Denote by $\mathcal{Q}_m$ the class of measurable mappings $Q : X \times \mathbb{R}^m \to V_m$ such that for any given $x \in X$, the mapping $Q(x, \cdot\,)$ is unit periodic in each of its m variables and the restriction $Q(x, \cdot\,)|_{V_m}$ is a mes_m-preserving bijection of the cube V_m.

Associate with any $Q \in \mathcal{Q}_m$ the mapping $Q^\ominus : X \times V_m \to V_m$ so that $Q^\ominus(x, \cdot\,) = Q(x, \cdot\,)|_{V_m}^{-1}$ for every fixed $x \in X$. Clearly, $Q^\ominus(x, Q(x, y)) = E_1(y)$.

For any $Q \in \mathcal{Q}_m$ and any mapping $S : X \to 2^{V_m}$, define the mapping $Q \bullet S : X \to 2^{V_m}$ by $(Q \bullet S)(x) = Q(x, S(x))$.

It is straightforward to verify that if $Q_\varepsilon \in \mathcal{Q}_m$ and $S_\varepsilon : X \to 2^{V_m}$ are two mappings, parameterized by $\varepsilon > 0$, and such that $Q_\varepsilon \xrightarrow{\text{c}} Q \in \mathcal{Q}_m$ and $S_\varepsilon \xrightarrow{\text{g.c.}} S$ as $\varepsilon \to +0$, then $Q_\varepsilon^\ominus \xrightarrow{\text{c}} Q^\ominus$ and $Q_\varepsilon \bullet S_\varepsilon \xrightarrow{\text{g.c.}} Q \bullet S$.

For a mapping $Q_\varepsilon \in \mathcal{Q}_m$, continuously convergent to $Q \in \mathcal{Q}_m$ as $\varepsilon \to +0$, and for a frequency measurable F-quasiperiodic set A_ε in (7), consider the set

$$\begin{aligned} \widetilde{A}_\varepsilon &= \{x \in X_\varepsilon :\ Q_\varepsilon(x, \varepsilon^{-1}F(x)) \in S_\varepsilon(x)\} \\ &= \{x \in X_\varepsilon :\ E_\varepsilon(F(x)) \in Q_\varepsilon^\ominus(x, S_\varepsilon(x))\}. \end{aligned}$$

The g.c. convergence of $S_\varepsilon : X \to 2^{V_m}$ to S implies that the F-quasiperiodic set $\widetilde{A}_\varepsilon$ is frequency measurable with the frequency function $\mathrm{mes}_m Q^\ominus(x, S(x)) = \mathrm{mes}_m S(x)$, $x \in X$. Therefore, $\widetilde{A}_\varepsilon$ can be regarded as the result of a *frequency-preserving* transformation of the set $A_\varepsilon \in \mathcal{A}(F)$, and these sets are equivalent in the sense. When applied to all the elements of the algebra $\mathcal{A}(F)$, this transformation, determined by a continuously convergent mapping $Q_\varepsilon \in \mathcal{Q}_m$, generates exactly the same algebra.

7 Spatially discretized dynamical systems

Let T be a continuously differentiable diffeomorphism of the open set $X \subset \mathbb{R}^n$. This will be interpreted as the transition operator of an autonomous smooth and invertible dynamical system with phase space X. Recalling (1), define an *ε-discretization* of T as the mapping

$$T_\varepsilon = R_\varepsilon \circ T|_{X_\varepsilon}. \tag{14}$$

Clearly, if $R_\varepsilon(X) \subset X$ then $T_\varepsilon(X_\varepsilon) \subset X_\varepsilon$. However, because of boundary effects, the grid X_ε is, in general, not invariant under the mapping T_ε. That is, iterating the T_ε can bring an initial point $x \in X_\varepsilon$ to outside the set X in which case the corresponding positive semi-trajectory of T_ε is not defined thereafter. Nevertheless, we shall systematically ignore this technical circumstance for the following reason. Since T is a smooth diffeomorphism of an open set X, for any $r \in \mathbb{N}$ and any compact set $K \subset X$, the inclusions

$$T_\varepsilon^k(K \bigcap X_\varepsilon) \subset X_\varepsilon, \quad 1 \le k \le r,$$

hold for all sufficiently small $\varepsilon > 0$. Informally, for any given r and for any arbitrarily "thin layer" $L = X \setminus K$, at least r first iterates of T_ε are well defined for all ε small enough and for all initial points $x \in X_\varepsilon \setminus L$ separated from the boundary ∂X. Since we shall only be concerned with asymptotic properties of a fixed, though arbitrarily large number of iterates of T_ε as $\varepsilon \to +0$, the mapping (14) may be regarded, for this purpose, as the transition operator of a spatially discrete autonomous dynamical system with phase space X_ε. Such a system models the computer implementation of the original dynamical system in fixed-point arithmetic with accuracy ε.

As was mentioned in Introduction, the invertibility of the original diffeomorphism T is normally not inherited by its discretization T_ε. Consequently, the set of *k-reachable* points,

$$X_{\varepsilon,k} = T_\varepsilon^k(X_\varepsilon) \bigcap X_\varepsilon = \{x \in X_\varepsilon : \ T_\varepsilon^{-k}(x) \neq \emptyset\} \tag{15}$$

is nonincreasing in $k \in \mathbb{Z}_+$, where $T_\varepsilon^{-k}(x) = \{y \in X_\varepsilon : \ T_\varepsilon^k(y) = x\}$ denotes the k-th full preimage of a point $x \in X_\varepsilon$ under T_ε, and the convention $X_{\varepsilon,0} = X_\varepsilon$ is used.

The distortion of functional properties of the original dynamical system under discretization can be described by the magnified deviations of positive and negative semi-trajectories of T_ε from those of T,

$$\delta^+_{\varepsilon,k}(x) = (T^k(x) - T_\varepsilon^k(x))/\varepsilon, \tag{16}$$

$$\delta^-_{\varepsilon,k}(x) = (U^k(x) - T_\varepsilon^{-k}(x))/\varepsilon, \tag{17}$$

where $U = T^{-1}$ denotes the inverse diffeomorphism. The mappings (16) and (17) can be expressed via appropriately defined quantization errors. Define the k-th *positive* and *negative quantization errors* as the mappings

$$E^+_{\varepsilon,k} = E_\varepsilon \circ T \circ T_\varepsilon^{k-1}, \quad E^-_{\varepsilon,k} = E_\varepsilon \circ U \circ U_\varepsilon^{k-1}, \tag{18}$$

respectively, where, similarly to (14),

$$U_\varepsilon = R_\varepsilon \circ U|_{X_\varepsilon} \tag{19}$$

is the ε-discretization of the inverse diffeomorphism U. As is shown in [4], the deviation $\delta^+_{\varepsilon,k}$ defined by (16) is related with the k first positive quantization errors by

$$\delta^+_{\varepsilon,k}(x) = \sum_{j=1}^{k} (T^{k-j})'(T^j(x)) E^+_{\varepsilon,j}(x) + o(1) \quad \text{as } \varepsilon \to +0,$$

where the estimate $o(1)$ holds uniformly in $x \in K \bigcap X_\varepsilon$ for any given compact set $K \subset X$. Similarly, the set-valued deviation $\delta^-_{\varepsilon,k}$ in (17) affords a representation via the negative quantization errors $E^-_{\varepsilon,1}, \ldots, E^-_{\varepsilon,k}$, although in a substantially more complicated way [6].

Therefore, the quantization errors $E^+_{\varepsilon,1}, \ldots, E^+_{\varepsilon,k}$ ($E^-_{\varepsilon,1}, \ldots, E^-_{\varepsilon,k}$) are responsible for the deviations $\delta^+_{\varepsilon,1}, \ldots, \delta^+_{\varepsilon,k}$ (respectively, $\delta^-_{\varepsilon,1}, \ldots, \delta^-_{\varepsilon,k}$) of the k-th initial segments of positive (respectively, negative) semi-trajectories of T_ε from those of T.

For example, the set $X_{\varepsilon,k}$ in (15) is an event relating to the k-th initial segments of the negative semi-trajectories, since $X_{\varepsilon,k} = \{x \in X_\varepsilon : \ \delta^-_{\varepsilon,k}(x) \neq \emptyset\}$. It turns out that, under a nonresonance condition on the diffeomorphism T, the sets $X_{\varepsilon,k}$ are all frequency measurable with some frequency functions q_k which form a nonincreasing sequence, $q_1(x) \geq q_2(x) \geq \ldots$ for any $x \in X$, with the function q_1 amenable to explicit computation.

8 Positive, negative and double-sided algebras of quasiperiodic sets

For any $a, b \in \mathbb{N}$, define the mappings $\mathcal{T}^-_a : X \to X^a$, $\mathcal{T}^+_b : X \to X^b$ and $\mathcal{T}_{a,b} : X \to X^{a+b}$ by stacking the appropriate iterates of the transition operator T and its inverse $U = T^{-1}$,

$$\mathcal{T}^-_a = \begin{bmatrix} U^a \\ \vdots \\ U \end{bmatrix}, \quad \mathcal{T}^+_b = \begin{bmatrix} T \\ \vdots \\ T^b \end{bmatrix}, \quad \mathcal{T}_{a,b} = \begin{bmatrix} \mathcal{T}^-_a \\ \\ \mathcal{T}^+_b \end{bmatrix}. \tag{20}$$

Say that the transition operator T is *iteratively nonresonant* if the mappings $\mathcal{T}^+_b$ are all nonresonant. Note that if T is such a diffeomorphism, then $\mathcal{T}_{a,b}$ is nonresonant for every $a, b \in \mathbb{N}$. Indeed, using the differential identities $(T^j)'(T^k(x))(T^k)'(x) = (T^{j+k})'(x)$, it is straightforward to verify that

$$\begin{bmatrix} I_n \\ \mathcal{T}'_{a,b}(T^a(x)) \end{bmatrix} (T^a)'(x) = \begin{bmatrix} (T^a)'(x) \\ I_n \\ \mathcal{T}^+_{a-1}{}'(x) \\ (T^{a+1})'(x) \\ \vdots \\ (T^{a+b})'(x) \end{bmatrix} = P_{a,b} \begin{bmatrix} I_n \\ \mathcal{T}^+_{a+b}{}'(x) \end{bmatrix}$$

where $P_{a,b}$ is a permutation matrix of order $(a+b+1)n$. Consequently, if T is an iteratively nonresonant diffeomorphism, the resonance sets $\mathcal{R}(\mathcal{T}_{a,b}) = T^a(\mathcal{R}(\mathcal{T}^+_{a+b}))$ satisfy $\mathrm{mes}_n \mathcal{R}(\mathcal{T}_{a,b}) = 0$ for all $a, b \in \mathbb{N}$.

Assuming the transition operator T to be iteratively nonresonant, denote by

$$\mathcal{A}_a^- = \mathcal{A}(\mathcal{T}_a^-), \quad \mathcal{A}_b^+ = \mathcal{A}(\mathcal{T}_b^+), \quad \mathcal{A}_{a,b} = \mathcal{A}(\mathcal{T}_{a,b}) \tag{21}$$

the algebras of frequency measurable quasiperiodic subsets of the grid X_ε corresponding to the mappings (20). Clearly, the algebras are all increasing in a, b. Furthermore, $\mathcal{A}_a^-$ and $\mathcal{A}_b^+$ are independent subalgebras of $\mathcal{A}_{a,b}$.

Associate with the transition operator T the *positive*, *negative* and *double-sided algebras* $\mathcal{A}^+$, $\mathcal{A}^-$ and $\mathcal{A}$ defined as the minimal algebras containing the appropriate algebras (21),

$$\mathcal{A}^+ = \bigcup_{k\in\mathbb{N}} \mathcal{A}_k^+, \quad \mathcal{A}^- = \bigcup_{k\in\mathbb{N}} \mathcal{A}_k^-, \quad \mathcal{A} = \bigcup_{a,b\in\mathbb{N}} \mathcal{A}_{a,b} = \bigcup_{k\in\mathbb{N}} \mathcal{A}_{k,k}. \tag{22}$$

Clearly, $\mathcal{A}^-$ and $\mathcal{A}^+$ are independent subalgebras of $\mathcal{A}$. Hence, application of the zero-one law of Kolmogorov [11] yields that any element of $\mathcal{A}^- \bigcap \mathcal{A}^+$ is a self-independent set with frequency function p satisfying $p(x) = (p(x))^2 \in \{0, 1\}$ for mes_n-almost all $x \in X$.

9 Associated algebras and quantization errors

For any $a, b \in \mathbb{N}$, define the mappings $\mathcal{E}_{\varepsilon,a}^- : X_\varepsilon \to V_{an}$, $\mathcal{E}_{\varepsilon,b}^+ : X_\varepsilon \to V_{bn}$ and $\mathcal{E}_{\varepsilon,a,b} : X_\varepsilon \to V_{(a+b)n}$ by stacking the appropriate quantization errors (18),

$$\mathcal{E}_{\varepsilon,a}^- = \begin{bmatrix} E_{\varepsilon,a}^- \\ \vdots \\ E_{\varepsilon,1}^- \end{bmatrix}, \quad \mathcal{E}_{\varepsilon,b}^+ = \begin{bmatrix} E_{\varepsilon,1}^+ \\ \vdots \\ E_{\varepsilon,b}^+ \end{bmatrix}, \quad \mathcal{E}_{\varepsilon,a,b} = \begin{bmatrix} \mathcal{E}_{\varepsilon,a}^- \\ \\ \mathcal{E}_{\varepsilon,b}^+ \end{bmatrix}. \tag{23}$$

Rephrasing [4, Lemmas 2–4] in terms of the classes $\mathcal{Q}_m$ defined in Section 6, there exist continuously convergent mappings $Q_{\varepsilon,a}^- \in \mathcal{Q}_{an}$ and $Q_{\varepsilon,b}^+ \in \mathcal{Q}_{bn}$ such that the mappings (23) are expressed via (20) as

$$\begin{aligned} \mathcal{E}_{\varepsilon,a}^-(x) &= Q_{\varepsilon,a}^-(x, \varepsilon^{-1}\mathcal{T}_a^-(x)), \\ \mathcal{E}_{\varepsilon,b}^+(x) &= Q_{\varepsilon,b}^+(x, \varepsilon^{-1}\mathcal{T}_b^+(x)), \\ \mathcal{E}_{\varepsilon,a,b}(x) &= Q_{\varepsilon,a,b}(x, \varepsilon^{-1}\mathcal{T}_{a,b}(x)). \end{aligned}$$

Here, the mapping $Q_{\varepsilon,a,b} : X \times \mathbb{R}^{(a+b)n} \to V_{(a+b)n}$ is defined by

$$Q_{\varepsilon,a,b}(x, \begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} Q_{\varepsilon,a}^-(x,u) \\ Q_{\varepsilon,b}^+(x,v) \end{bmatrix}, \quad x \in X,\ u \in \mathbb{R}^{an},\ v \in \mathbb{R}^{bn},$$

belongs to the class $\mathcal{Q}_{(a+b)n}$ and is also continuously convergent as $\varepsilon \to +0$. Hence, if the transition operator T is iteratively nonresonant and $S_\varepsilon : X \to 2^{V_{(a+b)n}}$ is a g.c. convergent mapping, the set

$$\{x \in X_\varepsilon :\ \mathcal{E}_{\varepsilon,a,b}(x) \in S_\varepsilon(x)\} = \{x \in X_\varepsilon :\ E_\varepsilon(\mathcal{T}_{a,b}(x)) \in Q_{\varepsilon,a,b}^{\ominus}(x, S_\varepsilon(x))\}$$

is an element of the algebra $\mathcal{A}_{a,b}$ in (21) obtained by a frequency-preserving transformation from the set

$$\{x \in X_\varepsilon : \ \varepsilon^{-1}\mathcal{T}_{a,b}(x) \in S_\varepsilon(x) + \mathbb{Z}^{(a+b)n}\}.$$

Therefore, the algebra $\mathcal{A}_{a,b}$ is induced by the quantization errors $E^-_{\varepsilon,1}, \ldots, E^-_{\varepsilon,a}$ and $E^+_{\varepsilon,1}, \ldots, E^+_{\varepsilon,b}$ which, as was discussed in Section 7, are responsible for the deviations $\delta^-_{\varepsilon,1}, \ldots, \delta^-_{\varepsilon,a}$ and $\delta^+_{\varepsilon,1}, \ldots, \delta^+_{\varepsilon,b}$ defined by (16)–(17). Similarly, the positive algebra $\mathcal{A}^+$ (negative algebra $\mathcal{A}^-$) in (22) consists of events relating to the deviations of initial segments of positive (respectively, negative) semi-trajectories of T_ε and T.

10 Measurability and mixing property with respect to positive algebra

Theorem 2 *Let the transition operator T be iteratively nonresonant. Then for any $k \in \mathbb{N}$ and any element A_ε of the algebra $\mathcal{A}^+_k$ in (21) with frequency function p, the set $T_\varepsilon^{-1}(A_\varepsilon)$ belongs to $\mathcal{A}^+_{k+1}$ and has the frequency function $p \circ T$.*

The proof of the theorem is based on lemmas of Section 15 and will be given in Section 16. Note that, by (22), the inclusions

$$T_\varepsilon^{-1}(\mathcal{A}^+_k) \subset \mathcal{A}^+_{k+1} \quad \text{for all } k \in \mathbb{N}, \tag{24}$$

stated by Theorem 2, immediately imply measurability of the mapping T_ε with respect to the positive algebra $\mathcal{A}^+$ in the sense that $T_\varepsilon^{-1}(\mathcal{A}^+) \subset \mathcal{A}^+$.

Theorem 3 *Let the transition operator T be iteratively nonresonant. Then for any $k \in \mathbb{N}$ and any $r \geq k$, the algebras $\mathcal{A}^+_k$ and $T_\varepsilon^{-r}(\mathcal{A}^+_k)$ are independent.*

The proof of the theorem is given in Section 17. Theorem 3 implies that for any $A_\varepsilon \in \mathcal{A}^+$ and for all sufficiently large $r \in \mathbb{N}$, the sets A_ε and $T_\varepsilon^{-r}(A_\varepsilon)$ are independent. This can be interpreted as a mixing property of T_ε on the positive algebra $\mathcal{A}^+$.

11 Invariance of negative algebra

In the sequel, the set of 1-reachable points $X_{\varepsilon,1}$ defined in (15) will be called the *surjectivity set* of the mapping T_ε.

Theorem 4 *Let the diffeomorphism T be nonresonant. Then the surjectivity set $X_{\varepsilon,1}$ is a frequency measurable U-quasiperiodic set with the frequency function*

$$q_1(x) = \operatorname{mes}_n E_1(U'(x)V_n), \quad x \in X. \tag{25}$$

The proof of the theorem is given in Section 18. It is straightforward to verify that

$$\min(1,\ \|T'(U(x))\|^{-n}) \le q_1(x) \le |\det U'(x)|,$$

where $\|A\| = \max_j \sum_k |a_{jk}|$ denotes the maximal row norm of a matrix $A = (a_{jk})$ induced by the Chebyshev vector norm. In particular, q_1 takes strictly positive values.

Theorem 5 *Let the transition operator T be iteratively nonresonant. Then for any $k \in \mathbb{N}$ and any element A_ε of the algebra $\mathcal{A}_k^-$ in (21) with frequency function p, the set $T_\varepsilon(A_\varepsilon) \bigcap X_\varepsilon$ belongs to $\mathcal{A}_{k+1}^-$ and its frequency function t satisfies*

$$t(x) \ge q_1(x)\, p(U(x)), \quad x \in X,$$

where q_1 is given by (25).

The proof of the theorem is based on lemmas of Section 19 and will be given in Section 20. Note that, by (22), the inclusions

$$T_\varepsilon(\mathcal{A}_k^-) \subset \mathcal{A}_{k+1}^- \quad \text{for all } k \in \mathbb{N}, \tag{26}$$

stated by Theorem 5, immediately imply invariance of the negative algebra $\mathcal{A}^-$ under the mapping T_ε in the sense that $T_\varepsilon(\mathcal{A}^-) \subset \mathcal{A}^-$.

12 Measurability with respect to and invariance of double-sided algebra

Theorem 6 *Let the transition operator T be iteratively nonresonant. Then for any $a, b \in \mathbb{N}$ the algebra $\mathcal{A}_{a,b}$ defined in (21) is transformed by the mapping T_ε and its set-valued inverse as*

$$T_\varepsilon^{-1}(\mathcal{A}_{a,b}) \subset \mathcal{A}_{a-1,b+1}, \quad T_\varepsilon(\mathcal{A}_{a,b}) \subset \mathcal{A}_{a+1,b-1}.$$

The proof of the theorem is similar to those of Theorems 2 and 5 and outlined in Section 21. The theorem implies that the double-sided algebra $\mathcal{A}$ in (22) is invariant under T_ε in the sense that $T_\varepsilon(\mathcal{A}) \subset \mathcal{A}$, and the mapping T_ε is measurable with respect to the algebra, that is, $T_\varepsilon^{-1}(\mathcal{A}) \subset \mathcal{A}$ (see Fig. 1). Moreover, when combined with Theorems 2 and 5, Theorem 6 yields that $T_\varepsilon^{-k}(A_\varepsilon) \in \mathcal{A}^+$ and $T_\varepsilon^{k}(A_\varepsilon) \in \mathcal{A}^-$ for any $A_\varepsilon \in \mathcal{A}$ and for all sufficiently large $k \in \mathbb{N}$.

13 Lower bounds for frequency functions of reachable points

Theorem 7 *Let the transition operator T be iteratively nonresonant. Then for any given $k \in \mathbb{N}$, the set $X_{\varepsilon,k}$ in (15) belongs to the algebra $\mathcal{A}_k^-$ defined by (21) and its frequency function q_k satisfies*

$$q_k(x) \ge \prod_{j=0}^{k-1} q_1(U^k(x)), \quad x \in X,$$

where q_1 is given by (25).

The proof of the theorem is given in Section 22. Suppose, in addition to the assumption of Theorem 7, that T has an invariant probability measure P. Then

$$\liminf_{k\to+\infty}(k^{-1}\int_X \ln q_k(x)P(dx)) \geq \int_X \ln q_1(x)\,P(dx).$$

Informally, this last inequality shows that if $q_k(x) \to 0$ as $k \to +\infty$, then, on average in the sense of P-measure, the convergence can not be faster than that of a geometric progression with parameter $\exp(\int_X \ln q_1(x)\,P(dx))$.

14 Proof of Theorem 1

Lemma 1 [4, Lemma 7] *Let $F : X \to I\!R^m$ be a nonresonant mapping on $X \subset I\!R^n$. Then for any compact set $K \subset X$ and any bounded continuous function $f : K \times V_m \to I\!R$,*

$$\lim_{\varepsilon\to+0}(\varepsilon^n \sum_{x\in K\bigcap X_\varepsilon} f(x, E_\varepsilon(F(x)))) = \int_{K\times V_m} f(z)\,dz.$$

The lemma can be restated in terms of weak convergence of measures [3] as follows. For any $\varepsilon > 0$, associate with the nonresonant mapping F a countably additive measure μ_ε on Borel sets $B \subset X \times V_m$ defined as

$$\mu_\varepsilon(B) = \varepsilon^n \#\{x \in X_\varepsilon : \begin{bmatrix} x \\ E_\varepsilon(F(x)) \end{bmatrix} \in B\}. \tag{27}$$

By Lemma 1, for any compact set $K \subset X$, the restriction of μ_ε to $K \times V_m$ converges weakly to the corresponding restriction of mes_{n+m},

$$\mu_\varepsilon|_{K\times V_m} \xrightarrow{\mathrm{w}} \mathrm{mes}_{n+m}|_{K\times V_m} \quad \text{as } \varepsilon \to +0. \tag{28}$$

Let G be an arbitrary Jordan measurable subset of X. Then, by the definitions (7) and (27),

$$\varepsilon^n \#(A_\varepsilon \bigcap G) = \mu_\varepsilon(B_\varepsilon), \tag{29}$$

where

$$B_\varepsilon = \{(x, y) :\ x \in G,\ y \in S_\varepsilon(x)\} = (G \times V_m) \bigcap \mathrm{Gr}\,(S_\varepsilon).$$

The assumption of the theorem that the mapping $S_\varepsilon : X \to 2^{V_m}$ is g.c. convergent to S implies the continuous convergence

$$B_\varepsilon \xrightarrow{\mathrm{c}} B = (G \times V_m) \bigcap \mathrm{Gr}\,(S) \quad \text{as } \varepsilon \to +0.$$

Hence, recalling the arbitrariness of a compact set $K \subset X$ in (28), and applying the results of [12] (see also [3, Theorem 5.5 on p. 34]), obtain that (29) has the limit

$$\lim_{\varepsilon\to+0}(\varepsilon^n \#(A_\varepsilon \bigcap G)) = \mathrm{mes}_{n+m}(B) = \int_G \mathrm{mes}_m S(x)dx.$$

Since this last convergence holds for any Jordan measurable $G \subset X$, the set A_ε is frequency measurable with frequency function $\mathrm{mes}_m S(x)$, and the proof of Theorem 1 is complete.

15 Preimages of quasiperiodic sets

Lemma 2 *The full preimage of an F-quasiperiodic set A_ε in (7) under the mapping T_ε is a Φ-quasiperiodic set*

$$\begin{aligned} T_\varepsilon^{-1}(A_\varepsilon) &= \{x \in X_\varepsilon :\ T_\varepsilon(x) \in A_\varepsilon\} \\ &= \{x \in X_\varepsilon :\ \varepsilon^{-1}\Phi(x) \in \Sigma_\varepsilon(x) + \mathbb{Z}^{n+m}\}, \end{aligned} \tag{30}$$

where the mappings $\Phi : X \to X \times \mathbb{R}^m$ and $\Sigma_\varepsilon : X \to 2^{V_n \times \mathbb{R}^m}$ are given by

$$\Phi(x) = \begin{bmatrix} T(x) \\ F(T(x)) \end{bmatrix}, \tag{31}$$

$$\Sigma_\varepsilon(x) = \{0\} \times (S_\varepsilon(R_\varepsilon(T(x))) + \Delta_\varepsilon(x)) + \begin{bmatrix} I_n \\ F'(T(x)) \end{bmatrix} V_n, \tag{32}$$

and

$$\Delta_\varepsilon(x) = \frac{F(T(x)) - F(R_\varepsilon(T(x)))}{\varepsilon} - F'(T(x))E_\varepsilon(T(x)). \tag{33}$$

Proof. ¿From the definitions (33) and (2), it follows that

$$\varepsilon^{-1}F(R_\varepsilon(T(x))) = \varepsilon^{-1}F(T(x)) - F'(T(x))\, E_1(\varepsilon^{-1}T(x)) - \Delta_\varepsilon(x),$$

and hence, by (7),

$$\begin{aligned} T_\varepsilon^{-1}(A_\varepsilon) &= \{x \in X_\varepsilon :\ \varepsilon^{-1}F(T_\varepsilon(x)) \in S_\varepsilon(T_\varepsilon(x)) + \mathbb{Z}^m\} \\ &= \{x \in X_\varepsilon :\ \varepsilon^{-1}F(T(x)) \in S_\varepsilon(T_\varepsilon(x)) \\ &\quad + F'(T(x))E_1(\varepsilon^{-1}T(x)) + \Delta_\varepsilon(x) + \mathbb{Z}^m\}. \end{aligned} \tag{34}$$

By (4), (34) implies that the inclusion $x \in T_\varepsilon^{-1}(A_\varepsilon)$ holds if and only if there exist $u \in V_n$ and $z \in \mathbb{Z}^n$ satisfying the relations

$$\varepsilon^{-1}T(x) = u + z, \quad \varepsilon^{-1}F(T(x)) \in S_\varepsilon(T_\varepsilon(x)) + F'(T(x))u + \Delta_\varepsilon(x) + \mathbb{Z}^m,$$

which can be written as

$$\varepsilon^{-1}\begin{bmatrix} T(x) \\ F(T(x)) \end{bmatrix} \in \{z\} \times (S_\varepsilon(T_\varepsilon(x)) + \Delta_\varepsilon(x) + \mathbb{Z}^m) + \begin{bmatrix} I_n \\ F'(T(x)) \end{bmatrix} u.$$

Taking the union of the sets on the right of the last inclusion over all $u \in V_n$ and $z \in \mathbb{Z}^n$ and recalling (32) yield the set $\Sigma_\varepsilon(x) + \mathbb{Z}^{n+m}$. Hence, by (31), the inclusion $x \in T_\varepsilon^{-1}(A_\varepsilon)$ is equivalent to $\varepsilon^{-1}\Phi(x) \in \Sigma_\varepsilon(x) + \mathbb{Z}^{n+m}$, which coincides with (30), thereby finishing the proof of the lemma. ■

Lemma 3 *Let the mapping $S_\varepsilon : X \to 2^{V_m}$ in (7) converge g.c. to S as $\varepsilon \to +0$, and let the mapping Φ defined by (31) be nonresonant. Then the set $T_\varepsilon^{-1}(A_\varepsilon)$ in (30) is frequency measurable with frequency function $\operatorname{mes}_m S(T(x))$, $x \in X$.*

Proof. Note that, by the nonresonance assumption on Φ, the mappings F and T are both nonresonant too. Indeed, T is so as a submapping of Φ. Furthermore,

$$\Phi'(x) = \left[\begin{array}{c} I_n \\ F'(T(x)) \end{array} \right] T'(x),$$

and consequently, since T is a smooth diffeomorphism of the set X, the resonance set $\mathcal{R}(F)$ in (11) satisfies the inclusion $\mathcal{R}(F) \subset T(\mathcal{R}(\Phi))$. Hence, $\mathrm{mes}_n \mathcal{R}(F) \leq \int_{\mathcal{R}(\Phi)} |\det T'(x)|\, dx$ which, by $\mathrm{mes}_n \mathcal{R}(\Phi) = 0$, implies $\mathrm{mes}_n \mathcal{R}(F) = 0$. Therefore, by Theorem 1, the F-quasiperiodic set A_ε is frequency measurable with frequency function $\mathrm{mes}_m S(x)$, $x \in X$. Show that Σ_ε in (32) is convergent g.c. to the mapping $\Sigma : X \to 2^{V_n \times I\!R^m}$ of kind

$$\Sigma(x) = \{0\} \times S(T(x)) + \left[\begin{array}{c} I_n \\ F'(T(x)) \end{array} \right] V_n. \tag{35}$$

It is easy to see that the graph of the set-valued mapping Σ_ε is given by

$$\begin{aligned} \mathrm{Gr}\,(\Sigma_\varepsilon) \;\; = \;\; & \{(x,u,v) :\; x \in X,\; u \in V_n, \\ & v \in S_\varepsilon(R_\varepsilon(T(x))) + \Delta_\varepsilon(x) + F'(T(x))u\}, \end{aligned} \tag{36}$$

and hence, its indicator function on the set $X \times V_n \times I\!R^m$ can be written as

$$\mathcal{I}_{\mathrm{Gr}\,(\Sigma_\varepsilon)}(x,u,v) = \mathcal{I}_{S_\varepsilon(R_\varepsilon(T(x)))}(v - \Delta_\varepsilon(x) - F'(T(x))u). \tag{37}$$

Similarly, for the mapping (35),

$$\mathrm{Gr}\,(\Sigma) = \{(x,u,v) \in X \times V_n \times I\!R^m :\; v \in S(T(x)) + F'(T(x))u\}, \tag{38}$$

and

$$\mathcal{I}_{\mathrm{Gr}\,(\Sigma)}(x,u,v) = \mathcal{I}_{S(T(x))}(v - F'(T(x))u). \tag{39}$$

The definitions (1) and (2) easily imply that

$$R_\varepsilon \circ T = T - \varepsilon E_\varepsilon \circ T. \tag{40}$$

Hence, by the uniform boundedness of the magnified roundoff error E_ε on $I\!R^n$, the mapping $R_\varepsilon \circ T$ is uniformly convergent to T,

$$\lim_{\varepsilon \to +0} \sup_{x \in X} |R_\varepsilon(T(x)) - T(x)| = 0. \tag{41}$$

Using (40) and the continuous differentiability of the mapping F on X, it is straightforward to verify that the mapping Δ_ε in (33) asymptotically vanishes in the sense that for any compact set $K \subset X$,

$$\lim_{\varepsilon \to +0} \sup_{x \in K} |\Delta_\varepsilon(x)| = 0. \tag{42}$$

Combining (36)–(39) with (41) and (42), obtain that

$$\begin{aligned}
&\{(x,u,v) \in X \times V_n \times I\!R^m : \\
&\mathcal{I}_{\mathrm{Gr}(\Sigma_\varepsilon)}(y,s,t) \not\to \mathcal{I}_{\mathrm{Gr}(\Sigma)}(x,u,v) \text{ as } (\varepsilon,y,s,t) \to (+0,x,u,v)\} \\
&\subset \{(x,u,v) \in X \times V_n \times I\!R^m : \\
&(T(x),\, v - F'(T(x))u) \in D\}, \qquad (43)
\end{aligned}$$

where $\not\to$ signifies "does not converge to", and

$$D = \{(x,y) \in X \times I\!R^m : \ \mathcal{I}_{S_\varepsilon(u)}(v) \not\to \mathcal{I}_{S(x)}(y) \quad \text{as } (\varepsilon,u,v) \to (+0,x,y)\}.$$

Since $\mathrm{mes}_{n+m} D = 0$ (by $S_\varepsilon \xrightarrow{\text{g.c.}} S$) and T is a smooth diffeomorphism of $X \subset I\!R^n$, the set on the left of the inclusion (43) has zero $(2n+m)$-dimensional Lebesgue measure and consequently, $\Sigma_\varepsilon \xrightarrow{\text{g.c.}} \Sigma$ and moreover, $E_1 \circ \Sigma_\varepsilon \xrightarrow{\text{g.c.}} E_1 \circ \Sigma$ as $\varepsilon \to +0$. Therefore, application of Theorem 1 to the Φ-quasiperiodic set $T_\varepsilon^{-1}(A_\varepsilon)$ yields its frequency measurability with frequency function $\mathrm{mes}_{n+m} E_1(\Sigma(x))$, $x \in X$. To calculate the function explicitly, note that, by (35),

$$\mathrm{mes}_{n+m} E_1(\Sigma(x)) = \int_{V_n} \mathrm{mes}_m E_1(S(T(x)) + F'(T(x))u)\, du = \mathrm{mes}_m S(T(x))$$

which completes the proof of the lemma. ■

16 Proof of Theorem 2

Let $A_\varepsilon \in \mathcal{A}_k^+$. That is, A_ε is a frequency measurable F-quasiperiodic set of kind (7), where the mapping $F = \mathcal{T}_k^+$ is given by (20) and $S_\varepsilon : X \to 2^{V_{kn}}$ is a g.c. convergent mapping. The frequency function p of the set A_ε is given by

$$p(x) = \mathrm{mes}_{kn} S(x), \quad x \in X,$$

where $S = \overset{\text{g.c.}}{\lim}_{\varepsilon \to +0} S_\varepsilon$. Applying Lemma 2, obtain that $B_\varepsilon = T_\varepsilon^{-1}(A_\varepsilon)$ is a Φ-quasiperiodic set, with the mapping $\Phi : X \to X^{k+1}$ given by

$$\Phi = \begin{bmatrix} T \\ \mathcal{T}_k^+ \circ T \end{bmatrix} = \mathcal{T}_{k+1}^+.$$

Hence, by (21), $B_\varepsilon \in \mathcal{A}_{k+1}^+$. Application of Lemma 3 yields that the frequency function of the set B_ε is

$$\mathrm{mes}_{kn} S(T(x)) = p(T(x)), \quad x \in X.$$

By the arbitrariness of $k \in I\!N$ and $A_\varepsilon \in \mathcal{A}_k^+$ in the considerations above, the proof of Theorem 2 is complete.

17 Proof of Theorem 3

Let $A_\varepsilon \in \mathcal{A}_k^+$. By the frequency-preserving transformations discussed in Section 9, any such set A_ε is representable with the aid of the k first positive quantization errors as

$$A_\varepsilon = \{x \in X_\varepsilon : \ \mathcal{E}_{\varepsilon,k}^+(x) \in S_\varepsilon(x)\}, \tag{44}$$

where the mapping $\mathcal{E}_{\varepsilon,k}^+$ is defined in (23) and $S_\varepsilon : X \to 2^{V_{kn}}$ is a g.c. convergent mapping, with $\operatorname{g.c.}\lim_{\varepsilon\to+0} S_\varepsilon = S$. Hence, for any $r \in \mathbb{N}$,

$$T_\varepsilon^{-r}(A_\varepsilon) = \{x \in X_\varepsilon : \ \mathcal{E}_{\varepsilon,k}^+(T_\varepsilon^r(x)) \in S_\varepsilon(T_\varepsilon^r(x))\}. \tag{45}$$

Note that, by (18),

$$\mathcal{E}_{\varepsilon,k}^+ \circ T_\varepsilon^r = \begin{bmatrix} E_{\varepsilon,r+1}^+ \\ \vdots \\ E_{\varepsilon,r+k}^+ \end{bmatrix}. \tag{46}$$

Now suppose that $r \geq k$. Then the mappings $\mathcal{E}_{\varepsilon,k}^+$ and (46) are formed by nonoverlapping collections of positive quantization errors, and the intersection of the sets (44) and (45) can be written as

$$A_\varepsilon \bigcap T_\varepsilon^{-r}(A_\varepsilon) = \{x \in X_\varepsilon : \ \mathcal{E}_{\varepsilon,r+k}^+(x) \in S_\varepsilon(x) \times V_{(r-k+1)n} \times S_\varepsilon(T_\varepsilon^r(x))\}. \tag{47}$$

The uniform convergence (41) implies a similar convergence of $(R_\varepsilon \circ T)^r$ to T^r as $\varepsilon \to +0$, which, in combination with $S_\varepsilon \xrightarrow{\text{g.c.}} S$, yields that $S_\varepsilon \circ (R_\varepsilon \circ T)^r \xrightarrow{\text{g.c.}} S \circ T^r$. Consequently, the set (47) has frequency function $p(x)\,p(T^r(x))$ (with $p(x) = \operatorname{mes}_{kn} S(x)$), that is, the product of the frequency functions of the sets (44) and (45). By the arbitrariness of $A_\varepsilon \in \mathcal{A}_k^+$ and $r \geq k$, this completes the proof of Theorem 3.

18 Proof of Theorem 4

First show that the surjectivity set $X_{\varepsilon,1}$ of the mapping T_ε defined in (15) is a U-quasiperiodic set.

Lemma 4 *The set $X_{\varepsilon,1}$ is representable as*

$$X_{\varepsilon,1} = \{x \in X_\varepsilon : \ E_\varepsilon(U(x)) \in N_\varepsilon(x)\} \tag{48}$$

where the mapping $N_\varepsilon : X \to 2^{V_n}$ is defined by

$$N_\varepsilon(x) = E_\varepsilon(U(x) - U(x + \varepsilon V_n)). \tag{49}$$

Proof. Using the property that $\varepsilon\mathbb{Z}^n$ is an additive group, from (1) and (3), obtain that the full preimage of a point $x \in X_\varepsilon$ under the mapping T_ε is representable as

$$\begin{aligned} T_\varepsilon^{-1}(x) &= \{y \in X_\varepsilon : \ T_\varepsilon(y) = x\} \\ &= (\varepsilon\mathbb{Z}^n) \bigcap U(x + \varepsilon V_n) \\ &= U_\varepsilon(x) + (\varepsilon\mathbb{Z}^n) \bigcap (U(x + \varepsilon V_n) - U_\varepsilon(x)) \\ &= U(x) + \varepsilon\,(M_\varepsilon(x,\, \varepsilon^{-1}U(x)) - E_1(\varepsilon^{-1}U(x))), \end{aligned} \tag{50}$$

where the notation (19) is used, and the mapping $M_\varepsilon : X \times I\!R^n \to 2^{Z\!\!Z^n}$ is given by

$$M_\varepsilon(x,u) = Z\!\!Z^n \bigcap ((U(x+\varepsilon V_n) - U(x))/\varepsilon + E_1(u)). \tag{51}$$

Clearly, $M(x,u)$ is unit periodic in each of the n coordinates of $u \in I\!R^n$ for any fixed $x \in X$. Now note that the set $Z\!\!Z^n \bigcap (G+u)$, where $G \subset I\!R^n$ and $u \in V_n$, is nonempty if and only if $u \in E_1(-G)$. Therefore, the representation (50) implies that

$$\begin{aligned} X_{\varepsilon,1} &= \{x \in X_\varepsilon : \ M_\varepsilon(x, \varepsilon^{-1}U(x)) \neq \emptyset\} \\ &= \{x \in X_\varepsilon : \ E_\varepsilon(U(x)) \in E_\varepsilon(U(x) - U(x+\varepsilon V_n))\} \end{aligned}$$

which, by the notation (49), coincides with (48), thereby completing the proof of the lemma. ■

It now remains to establish the frequency measurability of the set $X_{\varepsilon,1}$ claimed by Theorem 4. Since

$$\left[\begin{array}{c} I_n \\ U'(T(x)) \end{array} \right] T'(x) = \left[\begin{array}{c} T'(x) \\ I_n \end{array} \right],$$

and T is a nonresonant diffeomorphism (by assumption of the theorem), the resonance set $\mathcal{R}(U) = T(\mathcal{R}(T))$ satisfies $\mathrm{mes}_n \mathcal{R}(U) = 0$ which means that U is nonresonant too. Using (2) and the continuous differentiability of U, it is straightforward to verify that the mapping N_ε in (49) is g.c. convergent, as $\varepsilon \to +0$, to the mapping $N : X \to 2^{V_n}$ given by

$$N(x) = E_1(-U'(x)V_n). \tag{52}$$

Hence, application of Theorem 1 to the U-quasiperiodic set (48) yields its frequency measurability with frequency function

$$q_1(x) = \mathrm{mes}_n N(x) = \mathrm{mes}_n E_1(U'(x)V_n), \tag{53}$$

which coincides with (25), thereby completing the proof of the theorem. ■

19 Images of quasiperiodic sets

Lemma 5 *The image of an F-quasiperiodic set A_ε in (7) under the mapping T_ε is a Ψ-quasiperiodic set in the sense that*

$$T_\varepsilon(A_\varepsilon) \bigcap X_\varepsilon \ = \ \{x \in X_\varepsilon : \ \varepsilon^{-1}\Psi(x) \in \Xi_\varepsilon(x) + Z\!\!Z^{m+n}\} \tag{54}$$

where the mappings $\Psi : X \to I\!R^m \times X$ and $\Xi_\varepsilon : X \to 2^{I\!R^m \times V_n}$ are given by

$$\Psi(x) = \left[\begin{array}{c} F(U(x)) \\ U(x) \end{array} \right] \tag{55}$$

and

$$\begin{aligned} \Xi_\varepsilon(x) = & \bigcup_{u \in V_n} \bigcup_{y \in M_\varepsilon(x,u)} \\ & \left(\tfrac{F(U(x)) - F(U(x) + \varepsilon(y-u))}{\varepsilon} + S_\varepsilon(U(x) + \varepsilon(y-u)) \right) \times \{u\}. \end{aligned} \tag{56}$$

Proof. Note that $x \in T_\varepsilon(A_\varepsilon)$ if and only if $T_\varepsilon^{-1}(x) \bigcap A_\varepsilon \neq \emptyset$ which, by (7) and (50), is equivalent to existence of a point y satisfying

$$\begin{aligned} y &\in M_\varepsilon(x, \varepsilon^{-1}U(x)), \\ \varepsilon^{-1}F(U(x) + \varepsilon(y - E_1(\varepsilon^{-1}U(x)))) &\in S_\varepsilon(U(x) + \varepsilon(y - E_1(\varepsilon^{-1}U(x)))) + \mathbb{Z}^m. \end{aligned}$$

Hence, by (4), the inclusion $x \in T_\varepsilon(A_\varepsilon)$ holds if and only if there exists $u \in V_n$ such that

$$\begin{gathered} \varepsilon^{-1}F(U(x)) \in \bigcup_{y \in M_\varepsilon(x,u)} \left(\tfrac{F(U(x)) - F(U(x) + \varepsilon(y-u))}{\varepsilon} + S_\varepsilon(U(x) + \varepsilon(y-u)) \right) + \mathbb{Z}^m, \\ \varepsilon^{-1}U(x) \in u + \mathbb{Z}^n, \end{gathered}$$

which can be written as

$$\begin{gathered} \varepsilon^{-1} \begin{bmatrix} F(U(x)) \\ U(x) \end{bmatrix} \\ \in \bigcup_{y \in M_\varepsilon(x,u)} \left(\tfrac{F(U(x)) - F(U(x) + \varepsilon(y-u))}{\varepsilon} + S_\varepsilon(U(x) + \varepsilon(y-u)) \right) \times \{u\} + \mathbb{Z}^{m+n}. \end{gathered}$$

Taking the union of the sets on the right of the last inclusion over all $u \in V_n$ and recalling (56) yield the set $\Xi_\varepsilon(x) + \mathbb{Z}^{m+n}$. Hence, by (55), the inclusion $x \in T_\varepsilon(A_\varepsilon)$ is equivalent to $\varepsilon^{-1}\Psi(x) \in \Xi_\varepsilon + \mathbb{Z}^{m+n}$, which coincides with (54), thereby finishing the proof of the lemma. ■

Lemma 6 *Let the mapping $S_\varepsilon : X \to 2^{V_m}$ in (7) converge g.c. to S as $\varepsilon \to +0$, and let the mapping Ψ defined by (55) be nonresonant. Then the set (54) is frequency measurable and its frequency function t satisfies*

$$t(x) \geq q_1(x) \operatorname{mes}_m S(U(x)), \quad x \in X,$$

where q_1 is given by (25).

Proof. Note that for any $x \in X$ and $u \in \mathbb{R}^n$, the set $M_\varepsilon(x,u) \subset \mathbb{Z}^n$ in (51) is finite (possibly, empty) for all sufficiently small $\varepsilon > 0$. Denote by $\mathcal{Z}$ the totality of finite subsets of $\mathbb{Z}^n$, including the empty set. The set $\mathcal{Z}$ is denumerable and endowed with the discrete topology (with appropriate metrization, $\mathcal{Z}$ is a separable metric space).

It is easy to see that $M_\varepsilon \xrightarrow{c} M$ as $\varepsilon \to +0$, where the limiting mapping $M : X \times \mathbb{R}^n \to \mathcal{Z}$ is given by

$$M(x,u) = \mathbb{Z}^n \bigcap (U'(x)V_n + E_1(u)). \tag{57}$$

Indeed, taking the continuous differentiability of U into account,

$$\lim_{(\varepsilon,y,v) \to (+0,x,u)} M_\varepsilon(y,v) = M(x,u) \quad \text{for all } (x,u) \in (X \times V_n) \setminus D, \tag{58}$$

where $\lim(\,\cdot\,)$ corresponds to the discrete topology of $\mathcal{Z}$, and D is the discontinuity set of M on $X \times V_n$. By (4) and by the central symmetry $\partial V_n = -\partial V_n$,

$$D = \{(x,u) : \ x \in X, \ u \in E_1(U'(x)\partial V_n)\}.$$

For any given $x \in X$, the set $E_1(U'(x)\partial V_n)$ is contained by a union of countably many hyperplanes in $I\!R^n$ and consequently, has zero n-dimensional Lebesgue measure. Hence, $\mathrm{mes}_{2n} D = 0$ in (58), thereby implying that $M_\varepsilon \xrightarrow{\mathrm{c}} M$.

In combination with $S_\varepsilon \xrightarrow{\mathrm{g.c.}} S$ and with the smoothness of F, the continuous convergence $M_\varepsilon \xrightarrow{\mathrm{c}} M$ easily implies that Ξ_ε in (56) is g.c. convergent to the mapping Ξ given by

$$\Xi(x) = \bigcup_{u \in V_n} ((-F'(U(x))\,(M(x,u) - u) + S(U(x))) \times \{u\}),$$

and moreover, $E_1 \circ \Xi_\varepsilon \xrightarrow{\mathrm{g.c.}} E_1 \circ \Xi$ as $\varepsilon \to +0$. Therefore, by Theorem 1, the set (54) is frequency measurable with the frequency function t satisfying

$$\begin{aligned} t(x) &= \mathrm{mes}_{m+n} E_1(\Xi(x)) \\ &= \int_{V_n} \mathrm{mes}_m E_1(-F'(U(x))M(x,u) + S(U(x)))du \\ &\geq \mathrm{mes}_n\{u \in V_n :\ M(x,u) \neq \emptyset\}\, \mathrm{mes}_m S(U(x)) \\ &= q_1(x)\, \mathrm{mes}_m S(U(x)), \end{aligned}$$

where (52), (53) and (57) have been used. This last inequality completes the proof of the lemma. ■

20 Proof of Theorem 5

The proof is similar to that of Theorem 2. Let $A_\varepsilon \in \mathcal{A}_k^-$. That is, A_ε is a frequency measurable F-quasiperiodic set defined by (7), where the mapping $F = \mathcal{T}_k^-$ is given by (20) and $S_\varepsilon : X \to 2^{V_{kn}}$ is a g.c. convergent mapping. The frequency function of the set A_ε is

$$p(x) = \mathrm{mes}_{kn} S(x), \quad x \in X,$$

where $S = \overset{\mathrm{g.c.}}{\lim}_{\varepsilon \to +0} S_\varepsilon$. Applying Lemma 5, obtain that $B_\varepsilon = T_\varepsilon(A_\varepsilon) \bigcap X_\varepsilon$ is a Ψ-quasiperiodic set, with the mapping $\Psi : X \to X^{k+1}$ given by

$$\Psi = \begin{bmatrix} \mathcal{T}_k^- \circ U \\ U \end{bmatrix} = \mathcal{T}_{k+1}^-.$$

Hence, by (21), $B_\varepsilon \in \mathcal{A}_{k+1}^-$. Application of Lemma 6 yields that the frequency function t of the set B_ε satisfies

$$t(x) \geq q_1(x)\, \mathrm{mes}_{kn} S(U(x)) = q_1(x)\, p(U(x)), \quad x \in X.$$

By the arbitrariness of $k \in I\!N$ and $A_\varepsilon \in \mathcal{A}^-$ in the considerations above, the proof of Theorem 5 is complete.

21 Proof of Theorem 6

Let $A_\varepsilon \in \mathcal{A}_{a,b}$. That is, A_ε is a frequency measurable F-quasiperiodic set of kind (7), where the mapping $F = \mathcal{T}_{a,b}$ is given by (20) and $S_\varepsilon : X \to 2^{V_{(a+b)n}}$ is a g.c. convergent

mapping. Applying Lemmas 2 and 5, obtain that the sets $T_\varepsilon^{-1}(A_\varepsilon)$ and $T_\varepsilon(A_\varepsilon)\bigcap X_\varepsilon$ are Φ- and Ψ-quasiperiodic, respectively, with

$$\Phi = \begin{bmatrix} T \\ \mathcal{T}_{a,b} \circ T \end{bmatrix} = \begin{bmatrix} T \\ \mathcal{T}^-_{a-1} \\ E \\ T^2 \\ \vdots \\ T^{b+1} \end{bmatrix} = K_{a,b} \begin{bmatrix} \mathcal{T}^-_{a-1} \\ E \\ \mathcal{T}^+_{b+1} \end{bmatrix},$$

$$\Psi = \begin{bmatrix} \mathcal{T}_{a,b} \circ U \\ U \end{bmatrix} = \begin{bmatrix} U^{a+1} \\ \vdots \\ U^2 \\ E \\ \mathcal{T}^+_{b-1} \\ U \end{bmatrix} = L_{a,b} \begin{bmatrix} \mathcal{T}^-_{a+1} \\ E \\ \mathcal{T}^+_{b-1} \end{bmatrix},$$

where $K_{a,b}$ and $L_{a,b}$ are permutation matrices of order $(a+b+1)n$, and E the identity mapping on the set X. However, the presence of the identity mapping is redundant, since the representations (30) and (54) reduce to

$$T_\varepsilon^{-1}(A_\varepsilon) = \{x \in X_\varepsilon : \ E_\varepsilon(\mathcal{T}_{a-1,b+1}(x)) \in \widetilde{\Sigma}_\varepsilon(x)\}, \tag{59}$$

$$T_\varepsilon(A_\varepsilon)\bigcap X_\varepsilon = \{x \in X_\varepsilon : \ E_\varepsilon(\mathcal{T}_{a+1,b-1}(x)) \in \widetilde{\Xi}_\varepsilon(x)\}, \tag{60}$$

where

$$\widetilde{\Sigma}_\varepsilon(x) = \{\begin{bmatrix} u \\ v \end{bmatrix} : \ u \in V_{(a-1)n}, \ v \in V_{(b+1)n}, \ \begin{bmatrix} u \\ 0 \\ v \end{bmatrix} \in K_{a,b}^T E_1(\Sigma_\varepsilon(x))\}, \tag{61}$$

$$\widetilde{\Xi}_\varepsilon(x) = \{\begin{bmatrix} u \\ v \end{bmatrix} : \ u \in V_{(a+1)n}, \ v \in V_{(b-1)n}, \ \begin{bmatrix} u \\ 0 \\ v \end{bmatrix} \in L_{a,b}^T E_1(\Xi_\varepsilon(x))\}, \tag{62}$$

and the set-valued mappings Σ_ε and Ξ_ε are defined by (32) and (56). Consequently, the sets (59) and (60) are $\mathcal{T}_{a-1,b+1}$- and $\mathcal{T}_{a+1,b-1}$-quasiperiodic, respectively. The g.c. convergence of the mappings $\widetilde{\Sigma}_\varepsilon$ and $\widetilde{\Xi}_\varepsilon$ in (61) and (62) can be established in the same way as it was done for the mappings Σ_ε and Ξ_ε in the proofs of Lemmas 3 and 6 (the details are cumbersome and therefore omitted). Hence, application of Theorem 1 gives

$$T_\varepsilon^{-1}(A_\varepsilon) \in \mathcal{A}_{a-1,b+1}, \quad T_\varepsilon(A_\varepsilon)\bigcap X_\varepsilon \in \mathcal{A}_{a+1,b-1},$$

which, by the arbitrariness of $A_\varepsilon \in \mathcal{A}_{a,b}$ and $a, b \in I\!N$, yields the assertion of Theorem 6, thereby completing the proof. ■

22 Proof of Theorem 7

The proof proceeds by induction on $k \in I\!N$. The basis of induction is established by Theorem 4. Suppose that the assertion of Theorem 7 holds for some $k \in I\!N$. Then, by

(26),

$$X_{\varepsilon,k+1} = T_\varepsilon(X_{\varepsilon,k}) \bigcap X_\varepsilon \in \mathcal{A}^-_{k+1}.$$

Furthermore, application of Theorem 5 yields that the frequency function q_{k+1} of $X_{\varepsilon,k+1}$ satisfies

$$q_{k+1}(x) \geq q_1(x)\, q_k(U(x)) \geq \prod_{j=0}^{k} q_1(U^j(x)), \quad x \in X,$$

which completes the inductive step, thereby finishing the proof of Theorem 7.

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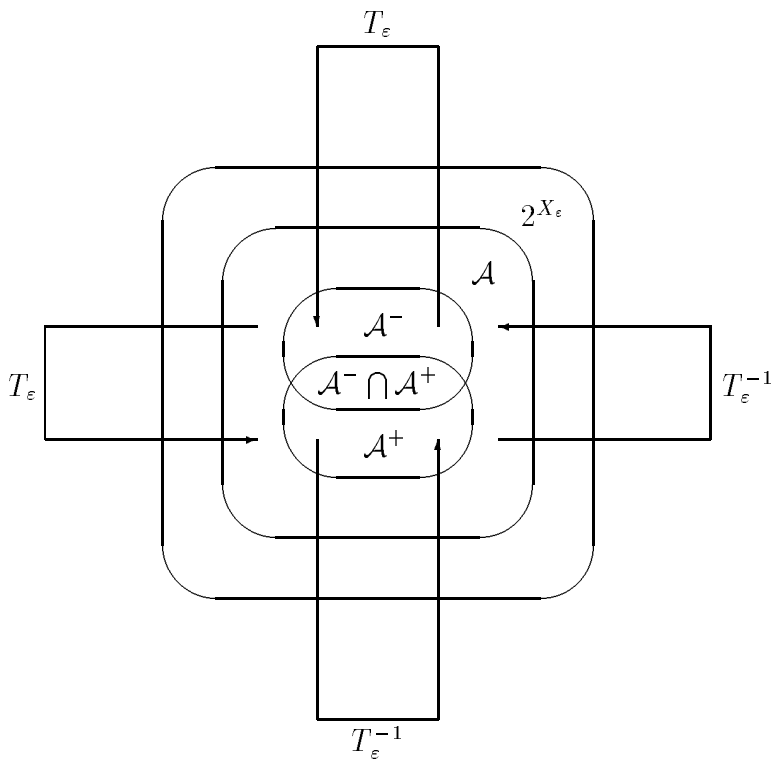

Figure 1: For an iteratively nonresonant transition operator T, the double-sided algebra $\mathcal{A}$ consists of frequency measurable subsets of the grid X_ε and is invariant under the discretized mapping T_ε and its set-valued inverse T_ε^{-1}. The positive and negative algebras $\mathcal{A}^+$ and $\mathcal{A}^-$ are invariant under T_ε^{-1} and T_ε, respectively. All the elements of the intersection $\mathcal{A}^+ \bigcap \mathcal{A}^-$ have frequency functions taking values in $\{0, 1\}$.