COLOR LAYER

red

FOURIER SERIES

- **Evaluation of FS;**
- **ODEs (Forced Oscillations);**
- **PDEs: Heat Equation & FS.**

POWER SERIES

- **Series Solution of DEs;**
- **Examples.**

Fourier Series and DEs.
Forced oscillations:

$$my'' + cy' + ky = r(t).$$

If $r(t)$ is a sine or cosine function and damping occurs ($c > 0$), then steady state solution is a harmonic oscillation with period same as $r(t)$. If $r(t)$ is not of this form, but has period p, the steady state solution is a superposition of harmonic oscillations of period np, $n = 1, 2, \ldots$. If one of these is close to the resonant frequency, then the corresponding oscillation can be the dominant response of the system to the input $r(t)$.

Example 1.

$$y'' + 0.04y' + 9y = r(t), \tag{1}$$

$$r(t) = \begin{cases} \frac{\pi}{4}t & \text{if } -\frac{\pi}{2} < t < \frac{\pi}{2}, \\ \frac{\pi}{4}(\pi - t) & \text{if } \frac{\pi}{2} < t < \frac{3\pi}{2}, \end{cases} \quad r(t+2\pi) = r(t).$$

Represent $r(t)$ by a FS

$$r(t) = \frac{1}{1^2}\sin t - \frac{1}{3^2}\sin 3t + \frac{1}{5^2}\sin 5t - \dots$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\sin(2k-1)t}{(2k-1)^2} .$$

For $k = 1, 2, \dots$ consider the DEs

$$y'' + 0.04y' + 9y = \frac{(-1)^{k+1}\sin(2k-1)t}{(2k-1)^2} . \quad (2)$$

The solution to (20) is the superposition of all the solutions to (21). From earlier work on forced oscillations, the steady state solution $y_{2k-1}(t) = y_p(t)$ and is of the form

$$y_{2k-1}(t) = A_{2k-1}\cos(2k-1)t + B_{2k-1}\sin(2k-1)t$$

where $A_{2k-1},\ B_{2k-1}$ are undetermined coefficients. Substituting in (21),

$$\begin{aligned}(9-(2k-1)^2)A_{2k-1} &+ \frac{2k-1}{25}B_{2k-1} \\ &= 0 \\ -\frac{2k-1}{25}A_{2k-1} &+ (9-(2k-1)^2)B_{2k-1} \\ &= \frac{(-1)^{k+1}}{(2k-1)^2}\ .\end{aligned}$$

Solving for $A_{2k-1},\ B_{2k-1}$ gives

$$\begin{aligned}A_{2k-1} &= \frac{(-1)^{k+1}}{25(2k-1)D_{2k-1}}\ , \\ B_{2k-1} &= \frac{(-1)^{k+1}(9-(2k-1)^2)}{(2k-1)^2 D_{2k-1}}\ ,\end{aligned}$$

where

$$D_{2k-1} = [9-(2k-1)^2]^2 + \left[\frac{2k-1}{25}\right]^2\ .$$

The amplitude of y_{2k-1} is

$$C_{2k-1} = \left[A_{2k-1}^2 + B_{2k-1}^2\right]^{1/2},$$

which is

$$\left[\frac{1}{[25(2k-1)D_{2k-1}]^2} + \frac{[9-(2k-1)^2]^2}{(2k-1)^4 D_{2k-1}^2}\right]^{1/2} .$$

That is,

$$\frac{1}{(2k-1)^2 D_{2k-1}} \left[\left[\frac{2k-1}{25}\right]^2 + [9-(2k-1)^2]^2\right]^{1/2}$$

$$\text{Hence,} \quad C_{2k-1} = \frac{1}{(2k-1)^2 \sqrt{D_{2k-1}}} .$$

Some numerical values:

$$\begin{aligned} C_1 &= 0.1250 \\ C_3 &= 0.9269 \\ C_5 &= 0.0025 \\ C_7 &= 0.0005 \\ C_9 &= 0.0002 \end{aligned}$$

The values of all C_{2k-1} are quite small, except for $2k-1=3$, when D_3 is very small and $C_3 = 0.9259$ is so large that it dominates the other harmonics.

Summary of steps.

$$y'' + by' + cy = r(t), \qquad (*)$$

where $r(t)$ is a periodic forcing term, $r(t + 2L) = r(t)$, and the plant is $y'' + by' + cy = 0$. What is the response of the system to the forcing term and what is the dominant frequency of the response?

- Expand $r(t) = \sum r_k(t)$ in a FS.
- $y'' + by' + cy = r_k(t)$ has particular soln $y_k(t)$.
- Particular soln of (*) is $y_p = \sum y_k$.
- Complete soln $y = y_h + y_p$.

If the plant is damped, this means that the particular solution $\sum y_k$ is the steady state soln.

There may be a harmonic r_p of the FS $\sum r_k$ of the forcing term $r(t)$ which is close to resonance with the plant – then y_p is the dominant response to the forcing term.

PDEs and Fourier Series. A vibrating string of length L and fixed at both ends can be described by its displacement $u(x,t)$ at time t and position x along the string. There are two independent variables x and t and instead of a DE we have a **Partial Differential Equation**. The displacement $u(x,t)$ can be shown to satisfy a PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho},$$

called the **one-dimensional wave equation.**

Boundary conditions:

Since the string is fixed at ends $x = 0$ and $x = L$,

$$u(0,t) = 0, \quad u(L,t) = 0, \qquad \forall\ t > 0.$$

Initial conditions: With initial displacement $f(x)$ and initial velocity $g(x)$,

$$u(x,0) = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \quad 0 \le x \le L.$$

PROBLEM: Solve the PDE satisfying the boundary and initial conditions.

- **Separation of Variables** gives two ODEs, one in t and the other in x;
- Solve these ODEs to satisfy the **boundary conditions**;
- Using **Fourier Series** the solutions are superposed to obtain a solution of the wave equation satisfying the **initial conditions.**

Separation of variables looks for a solution as a product with the variables "separated":

$$\begin{aligned} u(x,t) &= F(x)G(t). \\ \frac{\partial^2 u}{\partial t^2} &= F\ddot{G}, \qquad \frac{\partial^2 u}{\partial x^2} = F''G \\ F\ddot{G} &= c^2 F''G \\ \frac{\ddot{G}}{c^2 G} &= \frac{F''}{F} = k. \end{aligned}$$

Now both sides must be **constant** k because LHS is a function of t only and the RHS a function only of x.

$$\begin{aligned} F'' - kF &= 0 \\ \ddot{G} - c^2 kG &= 0. \end{aligned}$$

Boundary conditions: For all t

$$\begin{aligned} u(0,t) &= F(0)G(t) = 0, \\ u(L,t) &= F(L)G(t) = 0. \\ G &\not\equiv 0 \Rightarrow F(0) = 0, \quad F(L) = 0. \end{aligned}$$

Solving for F: now, $k = 0$ gives $F'' = 0 \Rightarrow F = ax + b$. Hence $a = b = 0$ and $F \equiv 0$, $u = 0$.

For $k = \mu^2 > 0$, $F = Ae^{\mu x} + Be^{-\mu x}$ and $F \equiv 0$ again. So the only interesting possibility is if $k = -p^2 < 0$.

$$\begin{aligned} F'' + p^2 F &= 0, \\ F(x) &= A \cos px + B \sin px \\ F(0) &= A = 0, \; F(L) = B \sin pL = 0, \\ \sin pL &= 0 \Rightarrow p = \frac{n\pi}{L} \\ F_n(x) &= \sin \frac{n\pi}{L} x, \; n = 1, 2, \ldots \end{aligned}$$

Now solve for G with $k = -p^2 = -(n\pi/L)^2$

$$\begin{aligned} \ddot{G} + \lambda_n^2 G &= 0, \quad \lambda_n = \frac{cn\pi}{L}, \\ G_n(t) &= B_n \cos \lambda_n t + B_n^* \sin \lambda_n t. \end{aligned}$$

So the functions $u_n(x,t) = F_n(x) G_n(t)$,

$$u_n(x,t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x,$$

$n = 1, 2, \ldots$, are solutions of the PDE satisfying the boundary conditions. These are the **eigenfunctions** and $\lambda_n = cn\pi/L$ are the **eigenvalues** of the problem.

A single $u_n(x,t)$ will not satisfy the initial conditions in general. However, the sum of the u_n is also a solution of the wave eqn:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\ &= \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x, \\ u(x,0) &= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x). \end{aligned}$$

Choose the B_n so that $u(x,0)$ is the Fourier sine series of $f(x)$:

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

Similarly, differentiating u,

$$\begin{aligned} \left.\frac{\partial u}{\partial t}\right|_{t=0} &= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L} = g(x) \\ B_n^* \lambda_n &= \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx \\ B_n^* &= \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx. \end{aligned}$$

In particular, if $u_t(x,0)=0$, that is, $g(x)=0$, then $B_n^*=0$ and

$$u(x,t)=\sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin\frac{n\pi x}{L} \ .$$

Example. Find the displacement $u(x,t)$ of the vibrating string of length $L=\pi$ with fixed ends and $c^2=1$, whose initial velocity is zero and initial displacement is given by $f(x)$ as shown.

Since initial velocity $g(x)=0$, $B_n^*=0$ and the solution is

$$u(x,t)=\sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin\frac{n\pi x}{L}$$

Because $c = 1$, $L = \pi$, $\lambda_n = cn\pi/L = n$ and

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos nt \sin nx.$$

The B_n are the Fourier sine coefficients for the half-range expansion of $f(x)$:

$$\begin{aligned}
B_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \\
&= \frac{2}{\pi}\left\{\int_0^1 x\sin nx\,dx \right. \\
&\quad \left. + \frac{1}{\pi-1}\int_1^{\pi}(\pi - x)\sin nx\,dx\right\} \\
&= \frac{2}{\pi}\left[\frac{-1}{n}\cos n + \frac{1}{n^2}\sin n\right] \\
&\quad + \frac{2}{\pi(\pi-1)}\left[\frac{\pi-1}{n}\cos n + \frac{1}{n^2}\sin n\right] \\
&= \frac{2\sin n}{(\pi-1)n^2} \\
u(x,t) &= \frac{2}{\pi-1}\sum_{n=1}^{\infty}\frac{\sin n}{n^2}\cos nt\sin nx.
\end{aligned}$$

Series Solution of DEs.

In the general case

$$y'' + p(t)y' + q(t)y = 0,$$

the solutions might be nonelementary functions which are **special functions**: Legendre polynomials, Bessel functions, etc. These are solved by using power series. Express $p(t)$, $q(t)$ as power series in t or $t - t_0$. Assume a solution, convergent in $|t| < R$,

$$\begin{aligned} y &= a_0 + a_1 t + a_2 t^2 + \ldots = \sum_{m=0}^{\infty} a_m t^m \\ y' &= a_1 + 2a_2 t + 3a_3 t^2 + \ldots = \sum_{m=1}^{\infty} m a_m t^{m-1} \\ y'' &= 2a_2 + 6a_3 t + \ldots = \sum_{m=2}^{\infty} m(m-1) a_m t^{m-2} \end{aligned}$$

and substitute these into the DE. Collect powers of t and equate sum of coefficients of like powers to zero. This gives recurrence relations which can be solved for the a_m.

Example 1: $y'' - y = 0$.
Substituting the series into the equation:

$$(2a_2 + 3 \cdot 2a_3 t + 4 \cdot 3a_4 t^2 + \cdots) \\ -(a_0 + a_1 t + a_2 t^2 + \ldots) = 0$$

$$2a_2 - a_0 + (3 \cdot 2a_3 - a_1)t \\ +(4 \cdot 3a_4 - a_2)t^2 + \ldots \\ +[(m+2)(m+1)a_{m+2} - a_m]t^m + \ldots = 0.$$

$$\begin{aligned} 2a_2 - a_0 &= 0 \\ 3 \cdot 2a_3 - a_1 &= 0 \\ 4 \cdot 3a_4 - a_2 &= 0, \ldots \\ (m+2)(m+1)a_{m+2} - a_m &= 0, \ \ldots \end{aligned}$$

So $a_2,\ a_4,\ \ldots$ can be expressed in terms of a_0, and $a_3,\ a_5,\ \ldots$ in terms of a_1.

$$\begin{aligned} a_2 &= \frac{a_0}{2!},\ a_4 = \frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \ldots \\ a_3 &= \frac{a_1}{3!},\ a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}, \ldots \end{aligned}$$

So the series is

$$y = a_0 + a_1 t + \frac{a_0}{2!}t^2 + \frac{a_1}{3!}t^3 + \frac{a_0}{4!}t^4 + \ldots$$

This can be written as $y = y_1 + y_2$, where

$$\begin{aligned} y_1 &= a_0 \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots\right) = a_0 \cosh t \\ y_2 &= a_1 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots\right) = a_1 \sinh t \\ y &= a_0 \cosh t + a_1 \sinh t. \end{aligned}$$

We saw earlier that e^t, e^{-t} is a basis of solutions of this DE, but so also are

$$\frac{e^t + e^{-t}}{2} = \cosh t, \quad \frac{e^t - e^{-t}}{2} = \sinh t.$$

Both these series converge for $|t| < \infty$, because, for $\cosh t$,

$$\begin{aligned} R &= \lim_{m \to \infty} \left| \frac{a_{2m}}{a_{2m+2}} \right| \\ &= \lim_{m \to \infty} (2m+2)(2m+1) = \infty, \end{aligned}$$

and similarly for $\sinh t$. (Ratio Test)

Example. $y'' + ty' + 2y = 0.$

$$\begin{aligned} y &= \sum_{m=0}^{\infty} a_m t^m, \\ ty' &= \sum_{m=1}^{\infty} m a_m t^m, \\ y'' &= \sum_{m=2}^{\infty} m(m-1) t^{m-2}. \end{aligned}$$

Substituting these into the DE,

$$\sum_{m=2}^{\infty} m(m-1) a_m t^{m-2} + \sum_{m=0}^{\infty} [m a_m + 2a_m] t^m.$$

$2a_2 + 2a_0 = 0,\ 12a_4 + 2a_2 + 2a_2 = 0, \ldots$

$12a_4 + 2a_2 + 2a_2 = 0,\ 20a_5 + 3a_3 + 2a_3 = 0, \ldots$

and for general m,

$(m+1)(m+2)a_{m+2} + m a_m + 2a_m = 0.$

So, solving for a_{m+2},

$$a_{m+2} = -\frac{1}{m+1} a_m,\ m = 0, 1, 2, \ldots.$$

Taking even $k = 0, 2, 4, \ldots,$

$$a_2 = -\frac{a_0}{1},\ a_4 = -\frac{a_2}{3} = \frac{a_0}{1 \cdot 3}, \ldots$$

$$a_{2k} = (-1)^k \frac{a_0}{1 \cdot 3 \cdot \ldots (2k-1)}, \ldots .$$

Taking odd $k = 1, 3, 5, \ldots,$

$$a_3 = -\frac{a_1}{2},\ a_5 = -\frac{a_3}{4} = \frac{a_1}{2 \cdot 4}, \ldots$$

$$a_{2k+1} = (-1)^k \frac{a_1}{2 \cdot 4 \cdot \ldots (2k)}, \ldots .$$

Hence,

$$\begin{aligned} y(t) \quad &= \quad a_0 \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{1 \cdot 3 \cdot \ldots (2k-1)} t^{2k} \right] \\ & \quad\quad + a_1 \sum_{k=0} \frac{(-1)^k}{2 \cdot 4 \cdot \ldots (2k)} t^{2k+1} \quad . \end{aligned}$$

Both series converge for all t. For example, in the second series,

$$R = \lim_{k \to \infty} \left| \frac{a_{2k+1}}{a_{2k+3}} \right| = \lim_{k \to \infty} 2(k+1) = \infty.$$