6. Recall, for even functions,

$$\int_0^\infty F(p)\,dp = \frac{1}{2}\int_{-\infty}^\infty F(p)\,dp$$

So, we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(v)\cos p(x-v)\,dv \right) dp.$$

On the other hand, since

$\int_{-\infty}^\infty f(v)\sin p(x-v)\,dv$ is an odd function of p, call it $G(p)$

& $\int_{-\infty}^\infty G(p)\,dp = 0$. Use

$$e^{ip(x-v)} = \cos p(x-v) + i\sin p(x-v)$$

to obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(v)\,e^{ip(x-v)}\,dv\,dp$$

(COMPLEX) Fourier Integral representation.

__Example__ Find F.T. of

$$f(x) = \begin{cases} e^{2ix}, & -1 < x < 1 \\ 0, & \text{elsewhere } (|x| \geq 1) \end{cases}$$

$$\hat{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-ipv} \, dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{2iv} e^{-ipv} \, dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i(2-p)v} \, dv$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(2-p)v}}{i(2-p)} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(2-p)} - e^{-i(2-p)}}{i(2-p)} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2 \sin(2-p)}{2-p}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin(2-p)}{2-p}$$

1.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\, e^{-ipv}\, dv\right) e^{ipx}\, dp$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\, e^{-ipv}\, dv\right) e^{ipx}\, d$$

FOURIER TRASFORM OF $f(x)$

$$\boxed{\hat{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\, e^{-ipv}\, dv}$$

Then the Fourier integral representation becomes

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(p)\, e^{ipx}\, dp}$$

FOURIER INVERSION FORMULA

$$\hat{f}(p) = \mathfrak{F}(f)$$

$$f(x) = \mathfrak{F}^{-1}(\hat{f})$$

EXISTS # $f(x)$ piecewise continuou on every finite interval.

$\int_{-\infty}^{\infty} |f(x)|\, dx$ exists.

3\. Some Properties of F.T

$$\hat{f}(p) = \mathcal{F}(f)$$

\# Linearity

$$\mathcal{F}(af + bg) = a\hat{f}(p) + b\hat{g}(p)$$

because

$$\int_{-\infty}^{\infty} (a f(x) + b g(x)) e^{-ipx} dx$$

$$= a \int_{-\infty}^{\infty} f(x) e^{-ipx} dx + b \int_{-\infty}^{\infty} g(x) e^{-ipx} dx$$

\# $\mathcal{F}(f') = ip\hat{f}(p)$

Note For $\int_{-\infty}^{\infty} |f(x)| dx$ to converge, necessary that $f(x) \to 0$ as $x \to \pm\infty$.

$$\mathcal{F}(f') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-ipx} dx$$

integrate by parts

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[f(x) e^{-ipx} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-ip) e^{-ipx} dx \right.$$

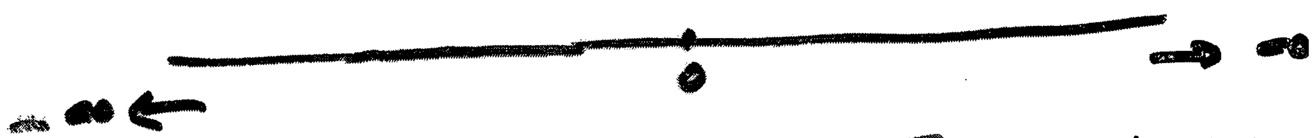
$$= 0 + \frac{ip}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ipx} dx$$

4. # $\mathcal{F}(f'') = ip\,\mathcal{F}(f')$

$$= (ip)^2\,\mathcal{F}(f)$$

$$= -p^2\,\mathcal{F}(f) = -p^2\hat{f}(p)$$

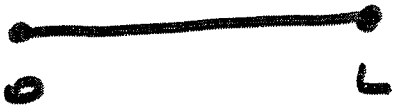
APPLICATION Use Fourier integrals in the heat equation for the infinite bar.



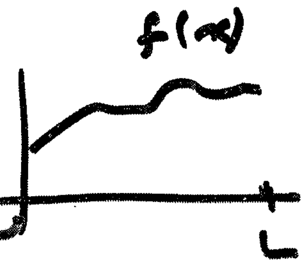
Laterally insulated. Boundary conditions $u(x,t) \to 0$ as $x \to \pm\infty$.

Initial condition $u(x,0) = f(x)$.

The principal difference from the situations we've looked at previously is



that $f(x)$ $0 < x < L$ was continued/extended as a PERIODIC FUNCTION



BUT HERE, IN THE INFINITE CASE, $f(x)$ is not periodic in general.

$u_t = c^2 u_{xx}$

Look for separable solutions

$$u(x,t) = F(x)\,G(t)$$

$$F\dot{G} = c^2 F'' G$$

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = k.$$

$$F'' - kF = 0$$

$$\dot{G} - c^2 k G = 0$$

1. $k = \mu^2 > 0$

$$F(x) = Ae^{\mu x} + Be^{-\mu x}$$

$u(x,t) \longrightarrow 0$ as $x \to \infty \Rightarrow A = 0$

$u(x,t) \longrightarrow 0$ as $x \to -\infty \Rightarrow B = 0$

Not possible

2. $k = 0$, similarly.

3. $k = -p^2$

$$F'' + p^2 F = 0$$

$$\dot{G} + c^2 p^2 G = 0$$

__Note__ Hasn't given the "form" that the eigenvalues take, as in the finite bar ($p_n = n\pi/L$).

6. $F(x) = A\cos px + B\sin px$

$G(t) = e^{-c^2p^2t}$

Can't use F.S. because

- Don't have eigenvalues $\lambda_n = c p_n$

- $f(x)$ is not periodic.

$$u(x,t;p) = (A\cos px + B\sin px)e^{-c^2p^2t}$$

$(A = A(p),\ B = B(p)$ depend on $p)$

(Compare the finite bar

$$u_n(x,t) = (A_n\cos p_n x + B_n \sin p_n x) \cdot e^{-\lambda_n^2 t}\Big)$$

Instead of a sum of all the eigenfunctions, use an integral

$$u(x,t) = \int_0^\infty u(x,t;p)\,dp$$

$$= \int_0^\infty (A(p)\cos px + B(p)\sin px)e^{-c^2p^2t}\,dp$$

Use the initial value

$$u(x,0) = f(x).$$

$$u(x,0) = \int_0^\infty \left(A(p)\cos px + B(p)\sin px\right) dp$$

$$= f(x)$$ FOURIER INTEGRAL REPRESENTATION

$$\Rightarrow \quad A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos pv \, dv$$

$$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin pv \, dv$$

$$u(x,0) = \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^{\infty} f(v)\cos p(x-v)\, dv\right) dp$$

$$\Rightarrow u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left(\int_{-\infty}^{\infty} e^{-c^2p^2t} \cdot \cos p(x-v)\, dp\right) dv$$

But $$\int_0^\infty e^{-s^2}\cos 2bs \, ds = \frac{\sqrt{\pi}}{2} e^{-b^2}$$

Put $p = s/c\sqrt{t}$, $b = \dfrac{x-v}{2c\sqrt{t}}$

& obtain

$$\int_0^\infty e^{-c^2p^2t}\cos p(x-v)\,dp = \frac{\sqrt{\pi}}{2c\sqrt{t}}\, e^{-(x-v)^2/4c^2t}$$

Hence

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\, e^{-\frac{(x-v)^2}{4c^2t}}\,dv$$

<u>Example</u> Solve the problem with $f(x) = \begin{cases} 1, & x>0 \\ 0, & x<0. \end{cases}$

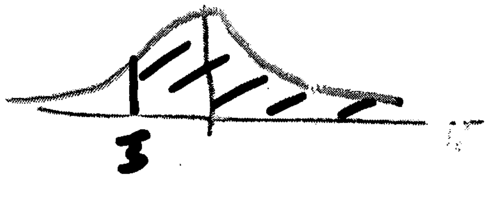
<u>SOLUTION</u> Have

$$u(x,t) = \frac{1}{2c\sqrt{t\pi}}\int_0^\infty e^{-\frac{(x-v)^2}{4c^2t}}\,dv$$

change of variable $z = \frac{v-x}{2c\sqrt{t}}$

$$\frac{dv}{2c\sqrt{t}} = dz\ ; \quad v=\infty,\ z=\infty; \quad v=0,\ z=\frac{-x}{2c\sqrt{t}}$$

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2c\sqrt{t}}}^{\infty} e^{-z^2}\,dz$$



FOURIER TRANSFORM & PDE

As done with the Laplace transformation when $[0, \infty)$ was present, it is natural to use the F.T. $\mathcal{F}$ when the domain of $(-\infty, \infty)$ is a DE.

EXAMPLE Heat in a uniform bar on $(-\infty, \infty)$,

$$u(x,0) = f(x), \quad -\infty < x < \infty$$

$$\left.\begin{aligned} u(x,t) &\to 0 \\ \frac{\partial u}{\partial x}(x,t) &\to 0 \end{aligned}\right\} \quad |x| \to \infty$$

In particular, find $u(x,t)$ when $f(x) = \begin{cases} U_0, & |x|<1 \\ 0, & |x|>1 \end{cases}$.

SOLUTION $u_t = c^2 u_{xx}$

Take F.T. with respect to x (contrast with Laplace when $t \in [0, \infty)$ & $\mathcal{L}$ with respect to t)

4. $\hat{u} = \tilde{\mathcal{F}}(u)$, $\hat{u}(p,t)$

$$\hat{u}_t = c^2 \tilde{\mathcal{F}}(u_{xx})$$
$$= c^2(-p^2\hat{u}).$$

$$\partial \hat{u}/\partial t = -c^2 p^2 \hat{u}.$$

First order ODE

$$\hat{u}(p,t) = C(p)e^{-c^2p^2t}$$

$$\hat{u}(p,0) = \hat{f}(p) \quad (u(x,0) = f(x$$
$$= C(p)$$

$$\hat{u}(p,t) = \hat{f}(p)\, e^{-c^2p^2t}$$

(**Note** $\hat{f}(p) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x)e^{-ipx}dx$

$$= \frac{U_0}{\sqrt{2\pi}}\int_{-1}^{1} e^{-ipx}dx$$

$$= \frac{U_0}{\sqrt{2\pi}}\,\frac{1}{-ip}\left[e^{-ipx}\right]_{-1}^{1}$$

$$= \frac{2U_0}{\sqrt{2\pi}}\,\frac{\sin p}{p}$$).

5. Using the inversion formula

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p)\, e^{-c^2p^2t} e^{ipx}\, dp$$

$$\hat{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\, e^{-ipv}\, dv$$

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(v)\, e^{-ipv}\, dv \right) e^{-c^2p^2t} e^{ipx}\, dp$$

$$e^{-c^2p^2t} e^{i(px - pv)}$$

$$= e^{-c^2p^2t} \Big(\underbrace{\cos p(x-v)}_{\text{even}} + i \underbrace{\sin p(x-v)}_{\text{ODD}} \Big)$$

$$\int_{-\infty}^{\infty} = 2\int_0^{\infty} \qquad\qquad \int_{-\infty}^{\infty} = 0$$

$$u(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \left(\int_0^{\infty} e^{-c^2p^2t} \cos p(x-v)\, dp \right) dv$$

This gives the same formula as using the Fourier cosine & sine integrals.

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v)\, e^{-\frac{(x-v)^2}{4c^2t}}\, dv$$

6.

$$= \frac{U_0}{2c\sqrt{\pi t}} \int_{-1}^{1} e^{-(x-v)^2/4c^2 t}\, dv$$

Take $z = \dfrac{v-x}{2c\sqrt{t}}$ etc.

$$u(x,t) = \int_{-(1+x)/2c\sqrt{t}}^{(1-x)/2c\sqrt{t}} e^{-z^2}\, dz$$

& can write in terms of erf & erfc.

FIN

ERROR FUNCTION

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz .$$

\# $\operatorname{erf}(\infty) = 1$

$z \to -s$

\# $\operatorname{erf}(-x) = -\operatorname{erf}(x)$

\# $$u(x,t) = \frac{1}{\sqrt{\pi}} \left(\int_0^\infty e^{-z^2}\, dz + \int_{-x/2c\sqrt{t}}^{0} e^{-z^2}\, dz \right)$$

$$= \underline{1} \overset{?\;-}{\cancel{+}} \operatorname{erf}\left(x/2c\sqrt{t} \right)$$

$$= \operatorname{erfc}\left(x/2c\sqrt{t} \right)$$

COMPLEMENTARY ERROR FN