STEADY STATE 2-D HEAT FLOW

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c^2 \nabla^2 u$$

If the heat flow is in a steady state, that is $u_t = 0$. The heat eqn then becomes the

2-D LAPLACE'S EQN

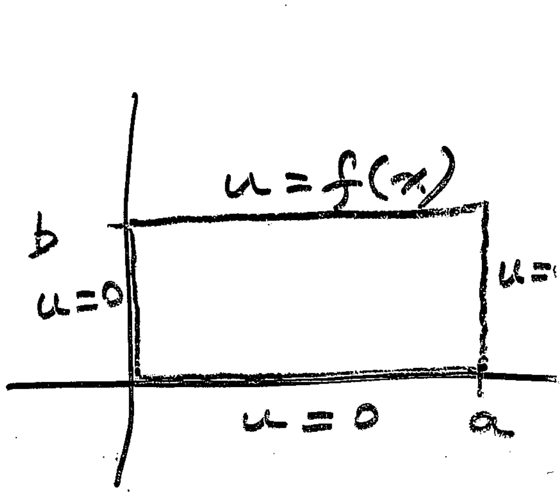
$$\nabla^2 u = 0.$$

Consider the eqn in a region R in xy-plane, with a bdy C

Boundary Value Problem:

- # u given on C (DIRICHLET PROBLEM)
- # $\frac{\partial u}{\partial n}$, normal derivative is prescribed on C (NEUMANN PROBLEM)
- # MIXED PROBLEM

RECTANGLE R



4. Look for separable solutions

$$u(x,y) = F(x)\,G(y)$$

$$u_{xx} = F''(x)\,G(y)$$

$$u_{yy} = F(x)\,G''(y)$$

$\nabla^2 u = 0$ becomes

$$F''G + FG'' = 0$$

$$F''/F = -G''/G = k$$

Boundary conditions $\begin{cases} u(0,y)=0 \\ u(a,y)=0 \end{cases}$

and $u(x,0)=0$, $u(x,b)=f(x)$

The first two give $F(0)=0$, $F(a)=0$

$$F'' - kF = 0 \qquad G'' + kG = 0$$

$k=0$, $F(x) = Ax+B$, bdy condns $\Longrightarrow F(x)\equiv 0$

$k>0$, $F(x) = Ae^{\mu x} + Be^{-\mu x}$
$= \mu^2$ bdy condns $\Longrightarrow A=B=0$, $F\equiv 0$

$k<0$, $\quad F'' + p^2 F = 0$
$k=-p^2 \quad F(x) = A\cos px + B\sin px$

$F(0)=0 \Longrightarrow A=0$

$F(a)=0 \Longrightarrow \sin pa = 0$, $\cancel{p} = n\pi/a$
$n=1,2,\ldots$

$G'' - p_n^2 G = 0$, $p_n = n\pi/a$
eigenvalues

$$G_n(y) = A_n e^{p_n y} + B_n e^{-p_n y}$$

But $u(x,0) = 0 \Rightarrow G_n(0) = 0$

$$A_n + B_n = 0$$

$$G_n(y) = A_n\left(e^{p_n y} - e^{-p_n y}\right)$$

$$= \underset{A_n^*}{2A_n} \sinh(p_n y)$$

Eigenfunctions $F_n(x)\, G_n(y)$

$$A_n^* \sin(p_n x) \sinh(p_n y)$$

$$u(x,y) = \sum_{n=1}^{\infty} A_n^* \sin(p_n x) \sinh(p_n y).$$

Remaining bdy condition is

$$u(x,b) = f(x)$$

$$f(x) = \sum_{n=1}^{\infty} A_n^* \sinh(p_n b) \sin p_n x$$

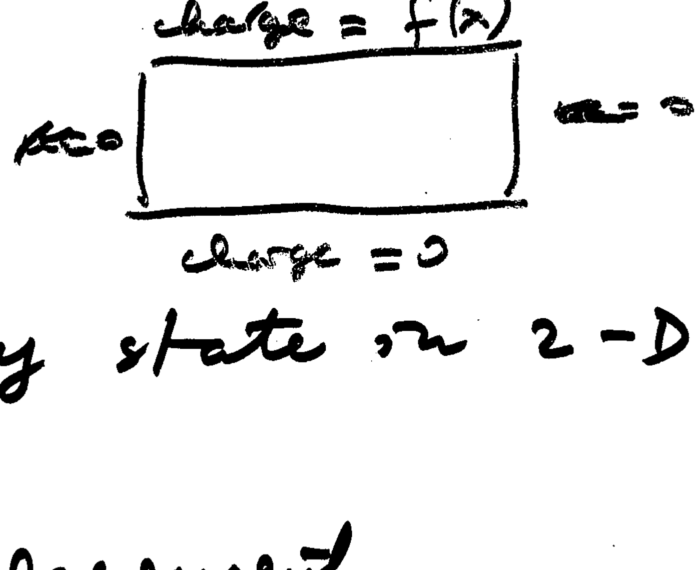
$$= \sum_{n=1}^{\infty} A_n^* \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi}{a} x\right)$$

Fourier sine series, half-range on $0 < x < a$

$$A_n^* \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

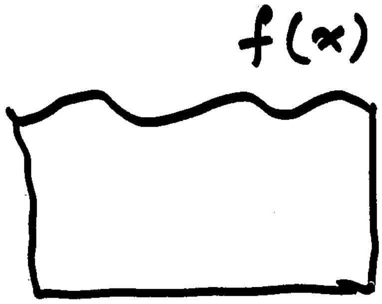
$$A_n^* = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

<u>Note</u> # Laplace's equation also governs the electrostatic potential. If $u(x,y)$ is potential at x, y, $\nabla^2 u = 0$ & in plate



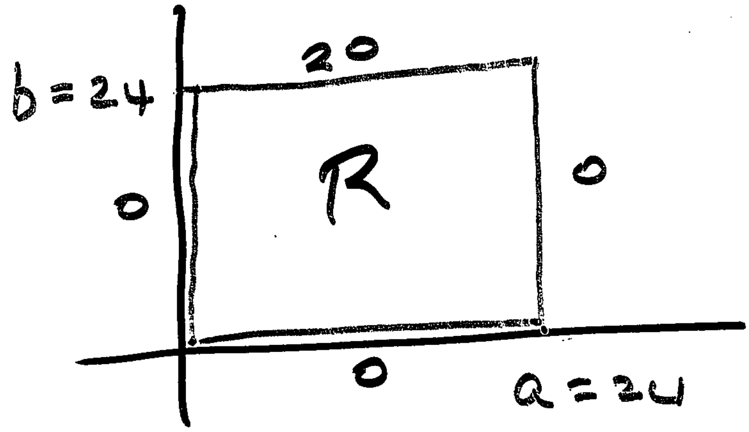
Also steady state in 2-D wave equation

$u(x,y)$ = displacement of a rectangular elastic membrane fixed along the boundary



1. Example 1-D Heat equation.

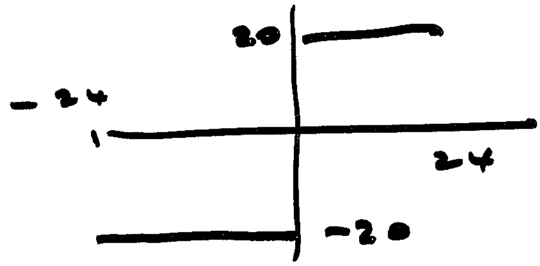
Know the form of the solution:



$$u(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{24} \sinh\frac{n\pi y}{24}$$

$$A_n^* \sinh n\pi = \frac{1}{12}\int_0^{24} 20 \sin\frac{n\pi x}{24}\,dx$$

What is the F.S.?



$$A_n^* \sinh n\pi = \frac{20}{12}\left[-\cos\frac{n\pi x}{24}\right]_0^{24} \frac{24}{n\pi}$$

$$= \frac{40}{n\pi}\left[1 - \cos n\pi\right].$$

$$= \begin{cases} 80/n\pi, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$A_{2k} = 0$

$$A_{2k-1} = \frac{80/\pi(2k-1)}{\sinh(2k-1)\pi}$$

Finally,

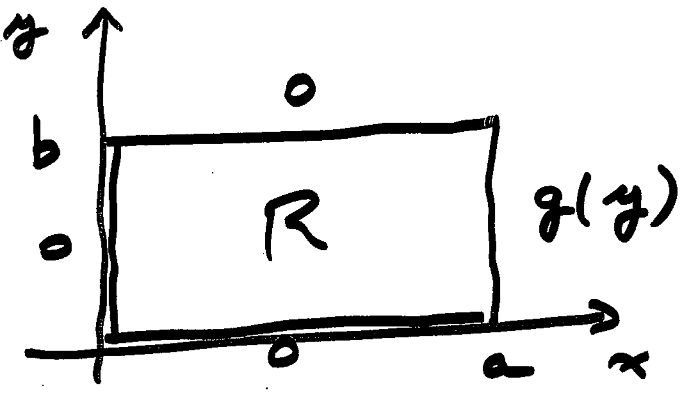
$u(x, y)$

$$= \frac{80}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)\sinh(2k-1)\pi} \sin\left\{\frac{(2k-1)\pi x}{24}\right\} \sinh\left\{\frac{(2k-1)\pi y}{24}\right\}$$

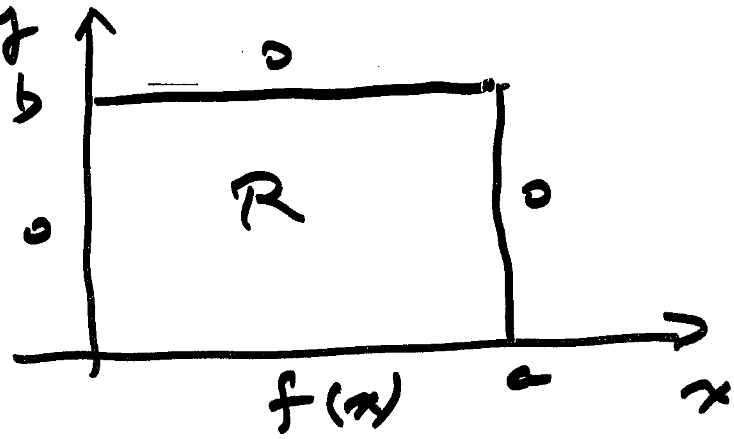
Solution with different bdy condns:

Interchange the role of x & y.

1)



(2)



$$u(x, b) = 0 \quad \& \quad u(x, 0) = f(x)$$

As before $\quad F_n(x) = \sin \frac{n\pi x}{a}, \quad n = 1, 2, \dots$

and $\quad G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$

$u(x,b) = 0 \implies G_n(b) = 0$

$$A_n e^{n\pi b/a} + B_n e^{-n\pi b/a} = 0$$

$$B_n = -e^{2n\pi b/a} A_n$$

and $\sinh(-x) = -\sinh x$

3. $$G_n(y) = A_n \left(e^{n\pi y/a} - e^{2n\pi b/a} e^{-n\pi y/a}\right)$$

$$= A_n e^{n\pi b/a} \times$$

$$\left\{ e^{\frac{n\pi}{a}(y-b)} - e^{-\frac{n\pi}{a}(y-b)} \right\}$$

$$= \underbrace{2A_n e^{n\pi b/a}}_{A_n^*} \sinh\left(\frac{n\pi(y-b)}{a}\right)$$

Hence $u(x,y)$

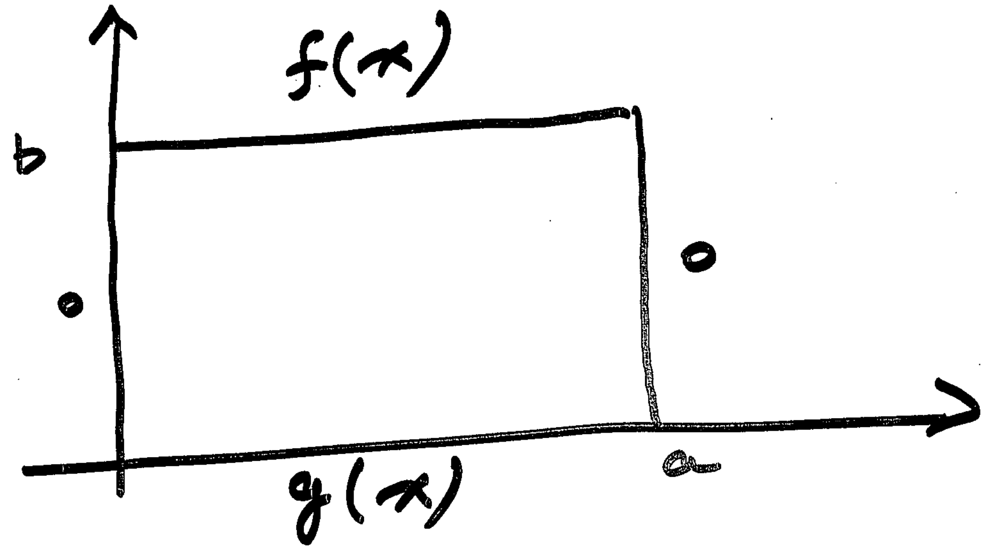
$$= \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a} \sinh \frac{n\pi(y-b)}{a}$$

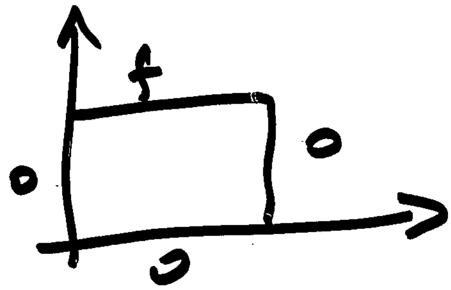
Now $u(x,0) = f(x)$

$$\sum_1 A_n^* \left(-\sinh \frac{n\pi b}{a}\right) \sin \frac{n\pi x}{a}$$
$$= f(x)$$

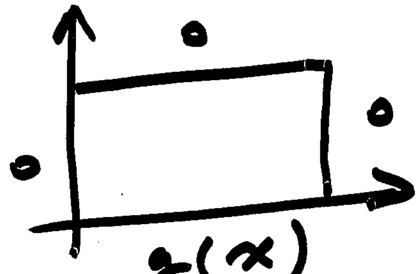
$\Rightarrow -A_n^* \sinh\frac{n\pi b}{a}$ are the F. sine coefficients of $f(x)$

$$\Rightarrow A_n^* = \frac{-2}{a \sinh\frac{n\pi b}{a}} \int_0^a f(x) \sin\frac{n\pi x}{a}\, dx$$





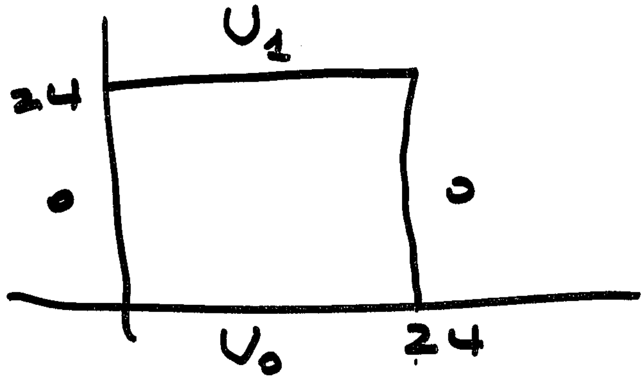
$u_I(x, y)$



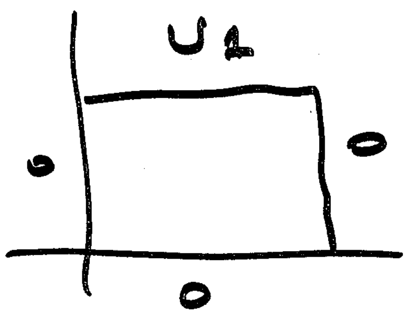
$u_{II}(x, y)$

Solution $u(x, y) = u_I(x, y) + u_{II}(x, y)$

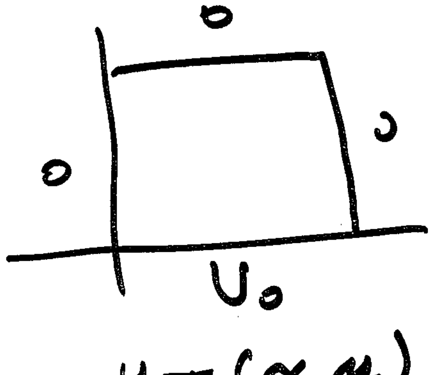
Ex



steady state solution



$u_I(x, y)$



$u_{II}(x, y)$

5. The solution for $u_I(x,y)$ will be quite similar to the previous example

$$u_I(x,y) = \sum_{n=1}^{\infty}{}' A_n^* \sin\frac{n\pi x}{24} \sinh\frac{n\pi y}{24}$$

$$A_n^* \sinh n\pi = \frac{1}{12}\int_0^{24} U_1 \sin\frac{n\pi x}{24}\,dx$$

$$= \frac{U_1}{12}\cdot\frac{24}{n\pi}\left[-\cos\frac{n\pi x}{24}\right]_0^{24}$$

$$= \frac{2U_1}{n\pi}\left[1-\cos n\pi\right]$$

$$A_{2k}^* = 0$$

$$A_{2k-1}^* = \frac{4U_1}{(2k-1)\sinh(2k-1)\pi}\ . \quad a=24$$

$$u_{II}(x,y) = \sum_n{}' A_n^* \sin\frac{n\pi x}{a} \sinh\frac{n\pi(y-a)}{a}$$

$$A_n^* = \frac{-2}{a\sinh n\pi}\int_0^a U_0 \sin\frac{n\pi x}{a}\,dx$$

$a=24$

$$A_n^* = \frac{-U_0}{12\sinh n\pi}\int_0^{24} \sin\frac{n\pi x}{24}\,dx$$

$$= \frac{2U_0}{n\pi\sinh n\pi}\left[\cos\frac{n\pi x}{24}\right]_0^{24}$$

$$= \frac{2U_0}{n\pi \sinh n\pi}\left[\cos n\pi - 1\right]$$

$$A^*_{2k} = 0$$

$$A^*_{2k-1} = \frac{-4U_0}{(2k-1)\pi \sinh (2k-1)\pi}$$

$$u_{II}(x, y)$$

$$= \frac{-4U_0}{\pi} \sum_{k=1}^{\infty} \frac{\sin \frac{(2k-1)\pi}{24} \sinh \frac{(2k-1)\pi (y-24)}{24}}{(2k-1) \sinh (2k-1)\pi}.$$