PDEs

LAPLACE'S EQN

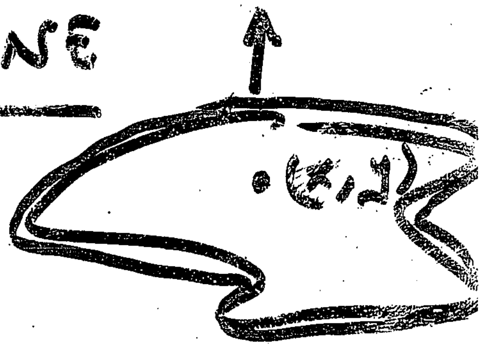
$$\nabla^2 \phi = 0$$

R^3 : $\nabla^2 = \underbrace{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}} + \frac{\partial^2}{\partial z^2}$

R^2 : $=$

R^1 : $\nabla^2 = \frac{\partial^2}{\partial x^2}$

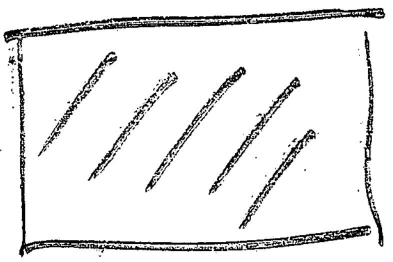
VIBRATING MEMBRANE



$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

HEAT EQUATION (Diffusion)

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u$$



Example

$L = \pi \; , \; c = 1$

$$u(x,0) = f(x) = x$$

BAR length L.

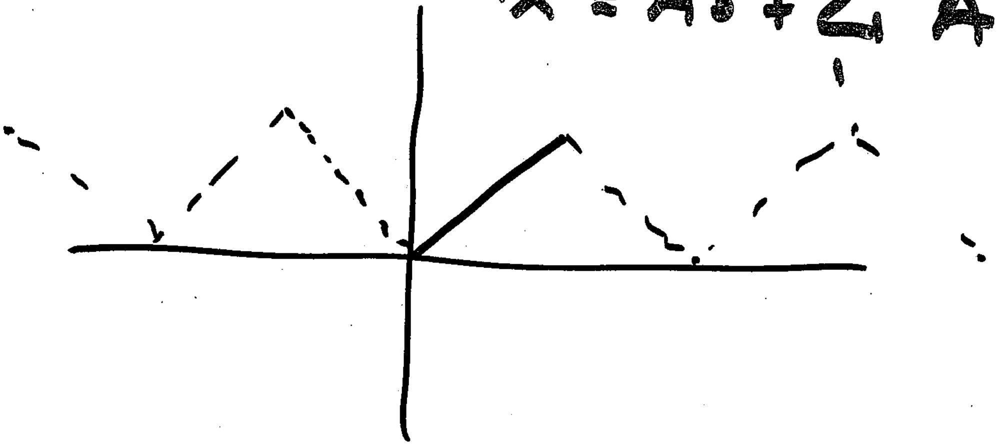
$$u(x,t) = A_0 + \sum_1^{\infty} A_n \cos\frac{n\pi x}{L} \cdot \exp\left(-\left(\frac{cn\pi}{L}\right)^2 t\right)$$

$L = \pi \Rightarrow$

$$u(x,t) = A_0 + \sum_1^{\infty} A_n \cos nx \, e^{-c^2 n^2 t}$$

Cosine series of $f(x) = x, \quad 0 < x < \pi$

$$x = A_0 + \sum_1^{\infty} A_n \cos nx$$



$$A_0 = \frac{1}{\pi}\int_0^{\pi} x \, dx = \pi/2$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\frac{x \sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} \frac{\sin nx}{n} \, dx$$

$$= \frac{2}{\pi n^2}\left[\cos nx\right]_0^{\pi} = \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

$$A_n = \begin{cases} 0, & n \text{ even} \\ -4/\pi n^2, & n \text{ odd} \end{cases}$$

$n = 2k-1$

$$u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} e^{-c^2(2k-1)^2}$$

(will accept an answer of form

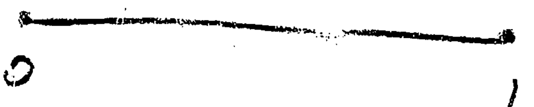
$$u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1} e^{-c^2t} + \frac{\cos 3x}{3^2} e^{-3^2c^2t} + \right)$$

Steady state as $t \to \infty$

$u(x,t) \longrightarrow \pi/2$

That is, uniform temperature in the bar

Bdy conditions in 1-D situation



$u(0,t) = \ldots$ ||

$u(L,t) = \ldots$

$u_x(0,t) = \ldots$ |||

$u_x(L,t) = \ldots$

$$u_{tt} = c^2 u_{xx} \qquad (*)$$

Make a change of variables.

$$\left.\begin{aligned} v &= x + ct \\ z &= x - ct \end{aligned}\right\} \ u(x,t) \text{ a function of } v, z.$$

chain rule $$u_x = u_v \frac{\partial v}{\partial x} + u_z \frac{\partial z}{\partial x} = u_v + u_z$$

$$u_{xx} = \frac{\partial}{\partial v}(u_v + u_z) + \frac{\partial}{\partial z}(u_v + u_z)$$

$$= u_{vv} + 2u_{vz} + u_{zz}$$

$$u_t = u_v \frac{\partial v}{\partial t} + u_z \frac{\partial z}{\partial t} = c(u_v - u_z)$$

$$u_{tt} = c^2 \frac{\partial}{\partial v}(u_v - u_z) - c^2 \frac{\partial}{\partial z}(u_v - u_z)$$

$$= c^2(u_{vv} - 2u_{vz} + u_{zz})$$

Equate these in (*)

$$c^2(\cancel{u_{vv}} - 2u_{vz} + \cancel{u_{zz}}) = \cancel{c^2}(\cancel{u_{vv}} + 2u_{vz} + \cancel{u_{zz}})$$

$$u_{vz} = 0$$

$$\frac{\partial}{\partial z}(u_v) = 0 \implies u_v = h(v)$$

$$u = \int h(v)\,dv + \psi(z)$$

$$u = \phi(v) + \psi(z)$$

$$= \phi(x+ct) + \psi(x-ct)$$

NOTE special technique which works for the wave equation, but not in general

Initial conditions:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

$$u_t(x,t) = c\,\phi'(x+ct) - c\,\psi'(x-c$$

(1) $u_t(x,0) = c\,\phi'(x) - c\,\psi'(x) = g(x)$

(2) $u(x,0) = \phi(x) + \psi(x) = f(x)$

$$c\phi(x) - c\psi(x) = c\phi(x_0) - c\psi(x_0) + \int_{x_0}^{x} g(x)\,dx$$

$$= k(x_0) + \int_{x_0} g(x)\,dx$$

Invoking (2)

(3) $2\phi(x) = f(x) + \frac{k(x_0)}{c} + \frac{1}{c}\int_{x_0}^{x} g$

(4) $2\psi(x) = f(x) - \frac{k(x_0)}{c} - \frac{1}{c}\int_{x_0}^{x} g$

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

$$= \frac{1}{2}\left\{f(x+ct) + f(x-ct)\right\}$$

$$+ \frac{1}{2c}\int_{x_0}^{x+ct} g(x)\,dx - \frac{1}{2c}\int_{x_0}^{x-ct} g(x)\,dx$$

$$= \frac{1}{2}\left\{f(x+ct) + f(x-ct)\right\}$$

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi$$

In particular, if the initial velocity $g(x) \equiv 0$, then

$$u(x,t) = \frac{1}{2}\left\{f(x+ct) + f(x-ct)\right\}$$

Classification of 2nd order PDEs

$$A\,u_{xx} + 2B\,u_{xy} + C\,u_{yy}$$
$$= F(x, y, \overset{u_t}{_\wedge} u, u_x, u_y)$$

A, B, C may be functions of x, y.

$AC - B^2 > 0$	ELLIPTIC
$AC - B^2 = 0$	PARABOLIC
$AC - B^2 < 0$	HYPERBOLIC

LAPLACE EQN $u_{xx} + u_{yy} = 0$
ELLIPTIC

HEAT EQN $u_t = c^2 u_{xx}$ PARABOLIC

WAVE EQN $u_{tt} - c^2 u_{xx} = 0$
2 2ND derivs in t, x HYPERBOLIC
$A = 1$, $C = -c^2$, $B = 0$

EXAMPLE TRICOMI EQN

$$y\, u_{xx} + u_{yy} = 0$$

Seek a solution by separation of variables

$$u(x, y) = F(x)\, G(y)$$

$$y\, F''(x)\, G(y) + F(x)\, G''(y) = 0$$

$$\frac{G''(y)}{y\, G(y)} = -\frac{F''(x)}{\underset{\uparrow}{F}(x)} = k$$

$$F''(x) = -k\overset{\downarrow}{F}(x)$$

$$G''(y) - k\, y\, G(y) = 0$$ AIRY EQN

Solutions of this (use series solns method for ODEs) called Airy functions.

HEAT EQN

$$u_t = c^2 \nabla^2 u \qquad c^2 = \frac{K}{\sigma\rho}$$

K = thermal conductivity

σ = specific heat

ρ = density

(In 1-D, $u_t = c^2 u_{xx}$)

CASE I

Bar of length L

$x = 0$, $x = L$, kept at const. temperature, 0 °C ; $u(x,t)$ temp. distn. in bar

BDY CONDNS

$$u(0,t) = u(L,t) = 0$$

INITIAL TEMP. DISTN

$$u(x,0) = f(x)$$

1. Separation of variables

$$u(x,t) = F(x)\,G(t)$$

$u_t = F\dot{G}$, $u_{xx} = F''G$

$$F\dot{G} = c^2 F''G$$

$$F''/F = c^{-2}\dot{G}/G = k \text{ const}$$

$$F'' - kF = 0$$

$$\dot{G} - kc^2 G = 0$$

2. Use the bdy conds to obtain information about k.

$k = 0$, $F(x) = Ax + B$

bdy conds $\Rightarrow$ A & $B = 0$, impos.

$k > 0$, $k = m^2$, $F(x) = Ce^{mx} + De^{-mx}$

bdy condns $\Rightarrow$ C & $D = 0$ no trivial solution, impossible.

$k < 0$, $k = -p^2$, $F'' + p^2 F = 0$

$$F(x) = A\cos px + B\sin px$$

$F(0) = 0 \Rightarrow A = 0$

$F(L) = 0 \Rightarrow B\sin pL = 0$

can't have $B = 0 \Rightarrow \sin pL = 0$

$pL = n\pi$, $n = 1, 2, \ldots$

$p_n = n\pi/L$, $n = 1, 2, \ldots$

That is, the B_n are the Fourier coefficients in the Fourier sine expansion of $f(x)$ in $[-L, L]$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

<u>Example</u> 4, p. 608

If $\rho = 10.6\ g/cm^3$, $L = 5$

$K = 1.04$ cal/cm sec

$\sigma = 0.056$, $c^2 = K/\sigma\rho$

$c = 1.752$

$$f(x) = k\sin(0.2\pi x)$$

Know that

$$u(x,t) = \sum_1 B_n \sin\frac{n\pi x}{5}\, e^{-\lambda_n^2 t}$$

$\lambda_n = cn\pi/5$

$n = 1, 2, \ldots$

$u(x,0) =$

$$\sum_1 B_n \sin\frac{n\pi x}{5} = k\sin\left(\frac{\pi x}{5}\right)$$

$B_1 = k$, $B_n = 0$, $n \geq 2$

$$u(x,t) = k\sin\left(\frac{\pi x}{5}\right) e^{-\frac{1.752\pi^2 t}{25}}$$

(better $\exp(-1.752\pi^2 t/25)$)

CASE II Insulated ends

$$u_x(0,t) = u_x(L,t) = 0.$$

As before

$$F'' - kF = 0$$

$$\dot{G} - kc^2 G = 0.$$

$k=0$ impossible.

$k = m^2$, $F(x) = Ae^{mx} + Be^{-mx}$

$$F'(0) = F'(L) = 0$$

$$mA - mB = 0$$

$$mAe^{mL} - mBe^{-mL} = 0$$

$\Rightarrow A = B = 0$ impossible

$k = -p^2$ $F(x) = A\cos px + B\sin p$

$$F'(x) = -pA\sin px + pB\cos px$$

$F'(0) = 0 \Rightarrow B = 0$

$F'(L) = 0 \Rightarrow \sin pL = 0$

$$pL = n\pi$$

$$p_n = n\pi/L \ , \ n = 0, 1, \ldots$$

$$F_n(x) = A_n \cos \frac{n\pi}{L} x$$

$n = 0, 1, 2, \ldots$

$$\dot{G} + (p_n c)^2 G = 0$$

$$G(t) = \exp(-\lambda_n^2 t)$$

$$\lambda_n = p_n c = c n\pi / L, \quad n = 0, 1, \dots$$

$$u_n(x,t) = A_n \cos\frac{n\pi x}{L} \exp(-\lambda_n^2 t)$$

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} \exp(-\lambda_n^2 t)$$

$u(x,0) = f(x)$ gives

$$\underbrace{A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}}_{\text{Fourier cosine series.}} = f(x)$$

$$A_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\, dx$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\, dx.$$

MATH 2100 MATH 2011

~~MT253~~ Assignment 9, ~~MT25~~6 Assignment 4

~~To be handed in Friday, 13 October, 2000~~

1. Find the temperature $u(x,t)$ in a bar of silver (length 10cm, constant cross section of area 1 cm^2, density 10.6 gm/cm^3, thermal conductivity 1.04 cal/cm sec °C, specific heat 0.056 cal/gm °C) that is perfectly insulated laterally and whose ends are kept at temperature 0 °C, whose initial temperature distribution is $f(x) = 5 - |x-5|$ °C.

2. Find the temperature in a bar insulated at both ends with

$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad u(x,0) = f(x),$$

where

$$f(x) = \begin{cases} 1 \text{ if } 0 < x < \frac{\pi}{2}, \\ 0 \text{ if } \frac{\pi}{2} < x < \pi. \end{cases}$$

3. Find the temperature $u(x,t)$ in a bar of length L that is kept at zero temperature at $x = 0$, assuming that the end $x = L$ is perfectly insulated, the initial temperature is a constant U_0 and $u_x(L,t) = 0$ (because of perfect insulation there).