THE UNIVERSITY OF QUEENSLAND

Second Semester Examination, November, 2020

MATH3404

OPTIMIZATION THEORY

Time: TWO hours for working.
Ten minutes for perusal before examination begins.

Candidates may attempt all Questions.
All questions are of equal value.

1.

(a) Find the local minima and maxima of

$$(x_1^2 - 4)^2 + x_2^2 \ .$$

Is there a global minimum or maximum?

(b) Find and test for nondegeneracy the critical points of

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2,$$

subject to the constraint $2x + 3y - z - 1 = 0$. Determine the local minimum of f. Is there a global maximum or minimum?

2.

(i) Find the extremal of

$$\int_0^{\pi/2} (x^2 - \dot{x}^2 - 2x \sin t)\, dt,$$

with end conditions $x(0) = 1, \ x(\pi/2) = 2$.

(ii) Find the extremal of

$$\int_0^T \frac{\dot{x}^2}{t^3}\, dt,$$

when $x(0) = 1$ and $x(T)$ lies on the curve $x = 2 + (t-1)^2$.

3.

(i) Explain what is meant by a corner condition.

(ii) Find two distinct nonsmooth extremals for

$$J[x] = \int_0^2 \dot{x}^2(\dot{x}+1)^2\,dt,\ x(0)=0,\ x(2)=-1.$$

4.

(a) Use the Lagrange multiplier technique to solve the isoperimetric problem of finding a minimizing curve for

$$J[x] = \int_0^1 (t^2 + x^2 - \dot{x}^2)\,dt,$$

subject to $x(0)=0,\ x(1)=0$ and the constraint $\int_0^1 x^2\,dt = 1$.

(b) Find the extremal of

$$J[x] = \int_0^2 \left(\frac{1}{2}\dot{x}^2 + x\dot{x} + x + \dot{x}\right)\,dt,$$

with endpoint conditions $x(0)=0,\ x(2)=2$. Use the Weierstrass excess function to show that the extremal is a minimizing curve.

5. Solve the problem of time-optimal control to the origin for

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2\\ \dot{x}_2 &= 4x_1 + x_2 + u,\ |u| \le 1.\end{aligned}$$

6.

$$\int_0^\infty (x_2^2 + 0.1u^2)dt,$$

subject to

$$\dot{x}_1 = -x_1 + u, \qquad \dot{x}_2 = x_1.$$

7. **(a)** Define the matrix exponential e^{tA}, where A is a constant square matrix. Find e^{tA} if

$$\begin{bmatrix} 1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$

(b) Show that $\frac{d}{dt}(e^{tA}X) = e^{tA}(\frac{dX}{dt} + AX)$ and $\frac{d}{dt}(Xe^{tA}) = (\frac{dX}{dt} + XA)e^{tA}$. Hence, solve the autonomous matrix differential equation

$$\frac{dW}{dt} = AW(t) + W(t)B, \quad W(0) = C.$$