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LAPLACE TRANSFORM

Differentiation & Integration of Transforms;

Convolution;

Partial Fraction Formulas;

Systems of DEs;

Periodic Functions.

Differentiation of Transforms.

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, F'(s) = -\int_0^\infty e^{-st}(tf(t))\,dt$$

$$\mathcal{L}(tf(t)) = -F'(s), \quad \mathcal{L}^{-1}(F'(s)) = -tf(t).$$

Example. Find inverse transforms of

$$\frac{s}{(s^2+\beta^2)^2}, \quad \frac{1}{(s^2+\beta^2)^2}, \quad \frac{s^2}{(s^2+\beta^2)^2}.$$

Differentiating $\mathcal{L}(\sin\beta t) = \frac{\beta}{s^2+\beta^2}$,

$$\mathcal{L}(t\sin\beta t) = -\frac{d}{ds}\frac{\beta}{s^2+\beta^2} = \frac{2\beta s}{(s^2+\beta^2)^2}\ .$$

Similarly,

$$\mathcal{L}(t\cos\beta t) = -\frac{(s^2+\beta^2)-2s^2}{(s^2+\beta^2)^2} = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}.$$

Hence,

$$\begin{aligned}\mathcal{L}(t\cos\beta t - \frac{1}{\beta}\sin\beta t) &= \frac{s^2-\beta^2}{(s^2+\beta^2)^2} - \frac{1}{s^2+\beta^2} \\ &= -\frac{2\beta^2}{(s^2+\beta^2)^2}.\end{aligned}$$

Similarly,

$$\mathcal{L}(t\cos\beta t + \frac{1}{\beta}\sin\beta t) = \frac{2s^2}{(s^2+\beta^2)^2}\ .$$

Integration of Transforms. If $\lim_{t\to 0+} f(t)/t$ exists, then

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma)\,d\sigma$$

$$\mathcal{L}^{-1}\left(\int_s^\infty F(\sigma)\,d\sigma\right) = \frac{f(t)}{t}\ ,$$

because

$$\begin{aligned}\int_s^\infty F(\sigma)\,d\sigma &= \int_s^\infty \left[\int_0^\infty e^{-\sigma t} f(t)\,dt\right]\,d\sigma \\ &= \int_0^\infty \left[\int_s^\infty e^{-\sigma t} f(t)\,d\sigma\right]\,dt \\ &= \int_0^\infty f(t)\frac{e^{-st}}{t}\,dt = \mathcal{L}\left(\frac{f(t)}{t}\right)\ .\end{aligned}$$

Note: If $G(s) = \int_s^\infty F(\sigma)\, d\sigma$, then

$$F(s) = -G'(s).$$

Example. $\mathcal{L}^{-1}\left(\ln(1 - a^2/s^2)\right)$. First, find $F(s)$ with $G(s) = \int_s^\infty F(\sigma)\, d\sigma$.

Put $G(s) = \ln(1 - a^2/s^2)$.

$$\begin{aligned} F(s) = -G'(s) &= -\frac{d}{ds}\ln\left(1 - \frac{a^2}{s^2}\right) = \frac{-2a^2}{s(s^2 - a^2)} \\ &= \frac{2}{s} - \frac{1}{s-a} - \frac{1}{s+a} \\ f(t) &= \mathcal{L}^{-1}(F) = 2 - e^{-at} - e^{at} \\ &= 2(\cosh at - 1). \end{aligned}$$

$$\mathcal{L}^{-1}\left(\ln\left(1 - \frac{a^2}{s^2}\right)\right) = -\frac{f(t)}{t} = \frac{2(1 - \cosh at)}{t}.$$

CONVOLUTION.

$\mathcal{L}(fg) \neq \mathcal{L}(f)\mathcal{L}(g), \quad \mathcal{L}^{-1}(F(s)G(s)) \neq f(t)g(t),$

but there is a relation between them, which is called convolution: if $H(s) = F(s)G(s)$,

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Example. $H(s) = \frac{2s}{(s^2+1)^2} = 2\,\frac{s}{s^2+1} \cdot \frac{1}{s^2+1}$

$$\begin{aligned} h(t) &= 2\cos t * \sin t \\ &= 2\int_0^t \cos\tau \sin(t-\tau)\,d\tau \\ &= \int_0^t (\sin t + \sin(2\tau - t))\,d\tau \\ &= \left[\tau \sin t - \frac{1}{2}\cos(2\tau - t)\right]_0^t = t\sin t. \end{aligned}$$

Example. $H(s) = \frac{1}{s(s^2+\omega^2)}$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1,\ \mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{1}{\omega}\sin\omega t.$$

By convolution,

$$\begin{aligned} h(t) = \frac{1}{\omega}\sin\omega t * 1 &= \int_0^t \frac{1}{\omega}\sin\omega\tau\,d\tau \\ &= \left[-\frac{1}{\omega^2}\cos\omega\tau\right]_0^t \\ &= \frac{1}{\omega^2} - \frac{1}{\omega^2}\cos\omega t. \end{aligned}$$

Proof of convolution: By 2nd shift,

$$\begin{aligned} e^{-st}G(s) &= \mathcal{L}(g(t-\tau)u(t-\tau)) \\ &= \int_0^\infty e^{-st}g(t-\tau)u(t-\tau)\,dt \\ &= \int_\tau^\infty e^{-st}g(t-\tau)\,dt \\ F(s)G(s) &= \int_0^\infty e^{-s\tau}f(\tau)G(s)\,d\tau \\ &= \int_0^\infty f(\tau)\int_\tau^\infty e^{-st}g(t-\tau)\,dt\,d\tau. \end{aligned}$$

Changing the order of integration,

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-st}\int_0^t f(\tau)g(t-\tau)\,d\tau\,dt \\ &= \int_0^\infty e^{-st}(f*g)(t)\,dt = \mathcal{L}(f*g). \end{aligned}$$

Differential Equation:

$$\begin{aligned} y'' + ay' + by &= r(t), \; y(0) = y'(0) = 0, \\ Y(s) &= R(s)Q(s), \\ y(t) &= r(t) * q(t), \quad q(t) = \mathcal{L}^{-1}(Q), \end{aligned}$$

where $Q(s) = 1/(s^2 + as + b)$ is the transfer function.

$$y(t) = \int_0^t q(t-\tau)r(\tau)\,d\tau.$$

Example.

$$y'' + y = \sin 3t, \quad y(0) = y'(0) = 0.$$

Now $Q(s) = 1/(s^2+1)$, $q(t) = \sin t$.

$$\begin{aligned} y(t) &= r(t) * q(t) \\ y(t) &= \int_0^t \sin 3\tau \sin(t-\tau)\,d\tau \\ &= \frac{1}{2}\int_0^t (\cos(t-4\tau) - \cos(t+2\tau))\,d\tau \\ &= \frac{3}{8}\sin t - \frac{1}{8}\sin 3t. \end{aligned}$$

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Integral Equations.

Integral equations which are in a convolution form can be solved very easily:

$$\begin{aligned} y(t) &= f(t) + (y * g)(t) \Rightarrow \\ Y(s) &= F(s) + Y(s)G(s) \Rightarrow Y(s) = \frac{F(s)}{1 - G(s)}. \end{aligned}$$

Example: $y(t) = e^{-t} - 2\int_0^t y(\tau)\cos(t-\tau)\,d\tau$. This is in convolution form

$$\begin{aligned} y &= e^{-t} - 2y * \cos t \\ Y(s) &= \frac{1}{s+1} - 2Y(s)\frac{s}{s^2+1} \\ Y(s) &= \frac{s^2+1}{(s+1)^3} \\ &= \frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3} \\ y(t) &= (1-t)^2 e^{-t}. \end{aligned}$$

Revision of Partial Fractions.

Solutions of subsidiary equation appear as

$$Y(s) = \frac{F(s)}{G(s)}\ , \ \text{deg}(F) < \text{deg}(G),$$

where F, G have real coefficients, and it is necessary to expand the rational function as partial fractions. There are four cases commonly occurring, where $G(s)$ has a factor:

- Simple factor $s - a$;
- Repeated factor $(s - a)^m$;
- Irreducible quadratic $(s - \alpha)^2 + \beta^2$;

Simple factor:

$$Y(s) = \frac{F(s)}{G(s)} \text{ has partial fraction } \frac{A}{s-a}.$$

$$y(t) \quad \text{has a term } Ae^{at}.$$

Example 1.

$$Y(s) = \frac{-7s-1}{s^3-7s+6} = \frac{A_1}{s-1} + \frac{A_2}{s-2} + \frac{A_3}{s+3}$$

$$-7s-1 = A_1(s-2)(+3) + A_2(s-1)(s+3) + A_3(s-1)(s-2).$$

$$\mathbf{s=1}: \quad -4A_1 = -8, \quad A_1 = 2.$$

$$\mathbf{s=2}: \quad 5A_2 = -15, \quad A_2 = -3.$$

$$\mathbf{s=-3}: \quad 20A_3 = 20, \quad A_3 = 1.$$

$$y(t) = \mathcal{L}^{-1}(Y) = 2e^t - 3e^{2t} + e^{-3t}$$

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Example 2. Repeated factor

$y'' + 4y' + 4y = 2e^{-2t},\ y(0) = 3,\ y'(0) = -10.$

$$s^2Y - 3s + 10 + 4(sY - 3) + 4Y = \frac{2}{s+2},$$

$$(s^2+4s+4)Y = 3s+2+\frac{2}{s+2} = \frac{3s^2+8s+6}{s+2},$$

$$\begin{aligned} Y(s) &= \frac{F(s)}{G(s)} = \frac{3s^2+8s+6}{(s+2)^3} \\ &= \frac{A_3}{(s+2)^3} + \frac{A_2}{(s+2)^2} + \frac{A_1}{s+2}\ . \end{aligned}$$

$$3s^2 + 8s + 6 = A_3 + A_2(s+2) + A_1(s+2)^2.$$

$$\begin{aligned} \textbf{cft of } s^2 &: \quad A_1 = 3 \\ \textbf{cft of } s &: \quad A_2 + 4A_1 = 8, \quad A_2 = -4 \\ \textbf{const} &: \quad A_3 + 2A_2 + 4A_1 = 6, \quad A_3 = 2 \\ y(t) &= 3e^{-2t} - 4te^{-2t} + t^2e^{-2t}. \end{aligned}$$

Irreducible quadratic factor $(s-\alpha)^2+\beta^2$

$Y(s)$ has a partial fraction

$$\frac{A_1 s + A_2}{(s-\alpha)^2+\beta^2} = \frac{A_1(s-\alpha)+\alpha A_1 + A_2}{(s-\alpha)^2+\beta^2}$$

$$= A_1\,\frac{s-\alpha}{(s-\alpha)^2+\beta^2} + \frac{\alpha A_1 + A_2}{(s-\alpha)^2+\beta^2}$$

and the inverse transform is

$$y(t) = e^{\alpha t}\left(A_1 \cos\beta t + \frac{\alpha A_1 + A_2}{\beta}\sin\beta t\right) .$$

Example: $y'' + 4y = \sinh t, \ y(0) = y'(0) = 0,$

$$s^2Y - sy(0) - y'(0) + 4Y = \frac{1}{s^2 - 1},$$

$$\begin{aligned} Y(s) &= \frac{1}{(s^2+4)(s-1)(s+1)} \\ &= \frac{A_1 s + A_2}{s^2+4} + \frac{B}{s-1} + \frac{C}{s+1} \end{aligned}$$

$$\begin{aligned} 1 &= (A_1 s + A_2)(s-1)(s+1) \\ &\qquad +B(s+1)(s^2+4) \\ &\qquad\quad +C(s-1)(s^2+4) \end{aligned}$$

$$\begin{aligned} \mathrm{s} = 1 &: \quad 10B = 1, \quad B = 1/10. \\ \mathrm{s} = -1 &: \quad -10C = 1, \quad C = -1/10. \\ \textbf{cft } s^3 &: \quad A_1 + B + C = 0, \quad A_1 = 0. \\ \textbf{const} &: \quad -A_2 - 2 + 4B - 4C = 1, \ A_2 = -2/10. \end{aligned}$$

$$y(t) = -\frac{1}{10}\sin 2t + \frac{1}{10}(e^t - e^{-t})$$

Systems of DEs.
Example 1.

$$\mathbf{y}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}, \ \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Taking the Laplace transform,

$$s\mathbf{Y} - \mathbf{y}(0) = \mathbf{AY} + \frac{1}{s+2} \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} s+3 & -1 \\ -1 & s+3 \end{bmatrix} \mathbf{Y} = \frac{1}{s+2} \begin{bmatrix} -6 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{Y} &= \frac{1}{(s+2)^2(s+4)} \begin{bmatrix} s+3 & 1 \\ 1 & s+3 \end{bmatrix} \begin{bmatrix} -6 \\ 2 \end{bmatrix} \\ &\quad + \frac{1}{(s+2)(s+4)} \begin{bmatrix} s+3 & 1 \\ 1 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{(s+2)^2(s+4)} \begin{bmatrix} s^2 - s - 10 \\ 3s+2 \end{bmatrix} \\ Y_1(s) &= \frac{s^2 - s - 10}{(s+2)^2(s+4)} \Rightarrow y_1(t), \\ Y_2(s) &= \frac{3s+2}{(s+2)^2(s+4)} \Rightarrow y_2(t). \end{aligned}$$

$$\begin{aligned}
Y_1(s) &= \frac{A_1}{(s+2)^2} + \frac{B_1}{s+2} + \frac{C_1}{s+4} \\
s^2 - s - 10 &= A_1(s+4) + B_1(s+2)(s+4) \\
&\qquad + C_1(s+2)^2 \\
\mathbf{s = -2} &: \quad -4 = 2A_1, \quad A_1 = -2. \\
\mathbf{s = -4} &: \quad 10 = 4C_1, \quad C_1 = 5/2. \\
\textbf{cft } s^2 &: \quad 1 = B_1 + C_1, \quad B_1 = -3/2. \\
Y_2(s) &= \frac{A_2}{(s+2)^2} + \frac{B_2}{s+2} + \frac{C_2}{s+4} \\
3s + 2 &= A_2(s+4) + B_2(s+2)(s+4) \\
&\qquad + C_2(s+2)^2 \\
\mathbf{s = -2} &: \quad -4 = 2A_2, \quad A_2 = -2. \\
\mathbf{s = -4} &: \quad -10 = 4C_2, \quad c_2 = -5/2. \\
\textbf{cft } s^2 &: \quad 0 = B_2 + C_2, \quad B_2 = 5/2.
\end{aligned}$$

$$\begin{aligned}
y_1(t) &= -2te^{-2t} - 3e^{-2t}/2 + 5e^{-4t}/2 \\
y_2(t) &= -2te^{-2t} + 5e^{-2t}/2 - 5e^{-4t}/2.
\end{aligned}$$

Example 2. Two masses on springs.

$$\begin{aligned} y_1'' &= -ky_1 + k(y_2 - y_1) \\ y_2'' &= -k(y_2 - y_1) - ky_2, \\ y_1(0) &= y_2(0) = 1, \\ y_1'(0) &= \sqrt{3k},\ y_2'(0) = -\sqrt{3k}. \end{aligned}$$

Let $Y_1 = \mathcal{L}(y_1),\ Y_2 = \mathcal{L}(y_2)$.

$$\begin{aligned} s^2Y_1 - s - \sqrt{3k} &= -kY_1 + k(Y_2 - Y_1) \\ s^2Y_2 - s + \sqrt{3k} &= -k(Y_2 - Y_1) - kY_2 \\ (s^2 + 2k)Y_1 - kY_2 &= s + \sqrt{3k} \\ -kY_1 + (s^2 + 2k)Y_2 &= s - \sqrt{3k}. \end{aligned}$$

Solving for $Y_1,\ Y_2$,

$$Y_1 = \frac{(s+\sqrt{3k})(s^2+2k)+k(s-\sqrt{3k})}{(s^2+2k)^2-k^2}$$

$$Y_2 = \frac{(s^2+2k)(s-\sqrt{3k})+k(s+\sqrt{3k})}{(s^2+2k)^2-k^2}$$

$$\begin{aligned}(s^2+2k)^2-k^2 &= [(s^2+2k)-k][(s^2+2k)+k] \\ &= (s^2+k)(s^2+3k)\end{aligned}$$

$$Y_1 = \frac{s}{s^2+k}+\frac{\sqrt{3k}}{s^2+3k}$$

$$Y_2 = \frac{s}{s^2+k}-\frac{\sqrt{3k}}{s^2+3k}$$

$$\begin{aligned}y_1(t) &= \cos\sqrt{k}t+\sin\sqrt{3k}t \\ y_2(t) &= \cos\sqrt{k}t-\sin\sqrt{3k}t\end{aligned}$$

Periodic Functions. $f(t)$ defined for all $t > 0$, and has period $p > 0$,

$$f(t+p) = f(t) \quad \forall t > 0.$$

$$\begin{aligned}
\mathcal{L}(f) &= \int_0^p e^{-st} f\,dt + \int_p^{2p} e^{-st} f\,dt \\
&\qquad + \int_{2p}^{3p} e^{-st} f\,dt + \ldots \\
&= \int_0^p e^{-s\tau} f(\tau)\,d\tau + \int_0^p e^{-s(\tau+p)} f(\tau)\,d\tau \\
&\qquad + \int_0^p e^{-s(\tau+2p)} f(\tau)\,d\tau + \ldots \\
&= [1 + e^{-sp} + e^{-2sp} + \ldots] \int_0^p e^{-s\tau} f(\tau)\,d\tau \\
&= \frac{1}{1-e^{-ps}} \int_0^p e^{-s\tau} f(\tau)\,d\tau.
\end{aligned}$$

This is because in the integral $\int_{kp}^{(k+1)p} e^{-st} f\,dt$, we can substitute $t = \tau + kp$ to obtain

$$\int_0^p e^{-s(\tau+kp)} f(\tau+kp)\,d\tau = e^{-kp} \int_0^p e^{-s\tau} f(\tau)\,d\tau,$$

because $f(\tau + kp) = f(\tau)$.

Example 1.

$$f(t) = \begin{cases} 1 & \text{if } 0 < t < \pi, \\ -1 & \text{if } \pi < t < 2\pi, \end{cases} \quad f(t+2\pi) = f(t).$$

$$\begin{aligned} \mathcal{L}(f) &= \frac{1}{1-e^{-2\pi s}} \left(\int_0^{\pi} e^{-st}\,dt + \int_{\pi}^{2\pi} -1.e^{-st}\,dt \right) \\ &= \frac{1}{1-e^{-2\pi s}}\,\frac{1}{s}\left(1 - 2e^{-\pi s} + e^{-2\pi s}\right) \\ &= \frac{(1-e^{-\pi s})^2}{s(1-e^{-2\pi s})} \\ &= \frac{1-e^{-\pi s}}{s(1+e^{-\pi s})} = \frac{1}{s}\coth\left(\frac{\pi s}{2}\right). \end{aligned}$$

Example 2. Halfwave Rectifier:

$$f(t) = \begin{cases} \sin \omega t & \text{if } 0 < t, \pi/\omega, \\ 0 & \text{if } \pi/\omega < t < 2\pi/\omega, \end{cases}$$

$$f(t + 2\pi/\omega) = f(t).$$

$$\mathcal{L}(f) = \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt.$$

The integral is the imaginary part of

$$\begin{aligned} \int_0^{\pi/\omega} e^{(-s+i\omega)t} \, dt &= \frac{1}{-s + i\omega} \left[e^{(-s+i\omega)t} \right]_0^{\pi/\omega} \\ &= \frac{-s - i\omega}{s^2 + \omega^2} (-e^{-s\pi/\omega} - 1), \end{aligned}$$

and so

$$\begin{aligned} \mathcal{L}(f) &= \frac{\omega(1 + e^{-s\pi/\omega})}{(s^2 + \omega^2)(1 - e^{-2s\pi/\omega})} \\ &= \frac{\omega}{(s^2 + \omega^2)(1 - e^{-s\pi/\omega})}. \end{aligned}$$

Example 3. Rectifier of $\sin \omega t$**:**

$$f(t) = \sin \omega t, \ 0 \le t \le \frac{\pi}{\omega}, \ f(t + \frac{\pi}{\omega}) = f(t).$$

$$\mathcal{L}(f) = \frac{1}{1 - e^{-\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt.$$

$$\begin{aligned} \int_0^{\pi/\omega} e^{-st} e^{i\omega t} \, dt &= \int_0^{\pi/\omega} e^{-(s+i\omega)t} \, dt \\ &= \left[\frac{-e^{-(s-i\omega)t}}{s - i\omega} \right]_0^{\pi/\omega} \\ &= \frac{1 - e^{-(s-i\omega)\pi/\omega}}{s - i\omega} \\ &= \frac{1 + e^{-\pi s/\omega}}{s - i\omega} \\ &= \frac{(1 + e^{-\pi s/\omega})(s + i\omega)}{s^2 + \omega^2}. \end{aligned}$$

Since $e^{-st}\sin\omega t$ is the imaginary part of $e^{-(s-i\omega t)}$, take the imaginary part of the integral and multiply by $1/(1-e^{-\pi s/\omega})$ to obtain

$$\begin{aligned}\mathcal{L}(f) &= \frac{1}{1-e^{-\pi s/\omega}}\;\frac{\omega(1+e^{-\pi s/\omega})}{s^2+\omega^2}\\ &= \frac{1}{e^{\pi s/2\omega}-e^{-\pi s/2\omega}}\;\frac{\omega(e^{\pi s/2\omega}+e^{-\pi s/2\omega})}{s^2+\omega^2}\\ &= \frac{\omega}{s^2+\omega^2}\;\frac{\cosh(\pi s/2\omega)}{\sinh(\pi s/2\omega)}\;.\end{aligned}$$