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SYSTEMS OF DEs

- **Constant coefficients: eigenvalue problem;**

- **Classification of critical point;**

  # Node;

  # Saddle point;

  # Centre;

  # Focus.

- **Nonhomogeneous equations.**

Systems of DEs

Every nth order DE

$$y^{(n)} = F\left(t, y, y', \ldots, y^{(n-1)}\right)$$

is reduced to a system of n 1st order DEs by

$$y_1 = y,\; y_2 = y',\; y_3 = y'',\; \ldots,\; y_n = y^{(n-1)}.$$

The system is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\cdot \\ &\cdot \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, \ldots, y_n). \end{aligned}$$

Example. $y'' + \frac{c}{m}y' + \frac{k}{m}y = 0$ becomes

$$\begin{aligned} y_1' &= 0.y_1 + y_2 \\ y_2' &= -\frac{k}{m}y_1 - \frac{c}{m}y_2. \end{aligned}$$

Let $\mathbf{y}^T = (y_1\ y_2)$. In matrix form,

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \mathbf{y} = \mathbf{A}\mathbf{y}.$$

Characteristic equation is

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} \\ &= \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0. \end{aligned}$$

Same as for mass on a spring DE. For solution, try $\mathbf{y} = \mathbf{x}e^{\lambda t}$. Then

$$\mathbf{y}' = \lambda \mathbf{x} e^{\lambda t}, \text{ or } \mathbf{A}\mathbf{x} = \lambda \mathbf{x},$$

and λ is an eigenvalue of $\mathbf{A}$, with eigenvector $\mathbf{x}$. To illustrate, let $m = 1$, $c = 3$, $k = 2$. Then $\lambda^2 + 3\lambda + 2 = 0$ has roots $\lambda_1 = -1$, $\lambda_2 = -2$ with eigenvectors $\mathbf{x}^{(1)} = (1 \quad -1)^T$ and $\mathbf{x}^{(2)} = (1 \quad -2)^T$.

Solution is thus

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t},$$

or, in components,

$$\begin{aligned} y_1 &= c_1 e^{-t} + c_2 e^{-2t} \\ y_2 &= -c_1 e^{-t} - 2c_2 e^{-2t} = y_1' \end{aligned}$$

Homogeneous, Const Coefficients

$$\mathbf{y}' = \mathbf{A}\mathbf{y},$$

where the $n \times n$ matrix $\mathbf{A}$ is constant. Try

$$\mathbf{y} = \mathbf{x}e^{\lambda t} \Rightarrow \mathbf{y}' = \lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{x}e^{\lambda t}.$$

This becomes an eigenvalue problem:

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \ .$$

Solutions are $\mathbf{x}e^{\lambda t}$, where λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ the corresponding eigenvector.

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Assume $\mathbf{A}$ has

- basis of eigenvectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$
- corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.

Solutions of DE are

$$\mathbf{y}^{(1)} = \mathbf{x}^{(1)} e^{\lambda_1 t}, \ldots, \mathbf{y}^{(n)} = \mathbf{x}^{(n)} e^{\lambda_n t}$$

with Wronskian

$$W(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}) = \begin{vmatrix} x_1^{(1)} e^{\lambda_1 t} & \cdots & x_1^{(n)} e^{\lambda_n t} \\ x_2^{(1)} e^{\lambda_1 t} & \cdots & x_2^{(n)} e^{\lambda_n t} \\ \cdot & \cdots & \cdot \\ x_n^{(1)} e^{\lambda_1 t} & \cdots & x_n^{(n)} e^{\lambda_n t} \end{vmatrix}$$

$$= e^{(\lambda_1 t + \ldots + \lambda_n t)} \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ x_2^{(1)} & \cdots & x_2^{(n)} \\ \cdot & \cdots & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix}$$

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The exponential $\neq 0$, nor is the determinant, because the columns are the lin indept eigenvectors forming a basis. So, when the constant matrix $\mathbf{A}$ has a linearly indept set of eigenvectors, the corresponding solutions $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}$ are a basis of solutions for $\mathbf{y}' = \mathbf{A}\mathbf{y}$, and a general solution is

$$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + \ldots + c_n \mathbf{x}^{(n)} e^{\lambda_n t}.$$

Example 1: Node

$$\mathbf{y}' = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-3}{2} \end{bmatrix} \mathbf{y}; \qquad \begin{aligned} y_1' &= -\tfrac{3}{2} y_1 + \tfrac{1}{2} y_2 \\ y_2' &= \tfrac{1}{2} y_1 - \tfrac{3}{2} y_2 \end{aligned}$$

Characteristic equation

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} -3/2 - \lambda & 1/2 \\ 1/2 & -3/2 - \lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2$$

Eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$. Eigenvectors satisfy $(A - \lambda I)\mathbf{x} = \mathbf{0}$,

$$(-3/2 - \lambda) x_1 + x_2/2 = 0.$$

For $\lambda_1 = -1,\ -x_1 + x_2 = 0 \Rightarrow \mathbf{x}^{(1)} = (1 \quad 1)^T$;

For $\lambda_2 = -2,\ x_1 + x_2 = 0 \Rightarrow \mathbf{x}^{(2)} = (1 \quad -1)^T$;

General solution

$$\begin{aligned} \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} \\ &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} \end{aligned}$$

Each choice of arbitrary constants $c_1,\ c_2$ gives a path in the y_1, y_2-plane. For $c_2 = 0,\ c_1 > 0$ is a ray $y_1 = y_2$ in the first quadrant; $c_2 = 0,\ c_1 < 0$ is the ray $y_1 = y_2$ in the third quadrant. For $c_1 = 0$ and $c_2 < 0$ or $c_2 > 0$, obtain the rays $y_1 = -y_2$ in 4th and 2nd quadrants. If both $c_1 \neq 0,\ c_2 \neq 0$, there is a curve tangent to the $\mathbf{x}^{(1)}$ direction at $\mathbf{0}$.

Improper Node

There are only two directions at $\mathbf{0}$.

Proper Node

There are solution curves in any direction at the origin $\mathbf{0}$. For example,

$$\mathbf{y}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{y}$$

has a proper node at $\mathbf{0}$ because the general solution is $\mathbf{y} = c_1(1\ 0)^T e^{2t} + c_2(0\ 1)^T e^{2t}$, or $c_2 y_1 = c_1 y_2$.

Example 2: Saddle Point

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 7 & -8 \\ 4 & -5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

- Node has real eigenvalues of the **same sign**, and solution curves all travel in the same direction: either towards $\mathbf{0}$ or away from $\mathbf{0}$.

- Saddle point has two real eigenvalues of **opposite** sign: so there is an **attractive direction** ($\lambda_2 < 0$) and a **repelling direction** ($\lambda_1 > 0$).

Characteristic eqn

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 7-\lambda & -8 \\ 4 & -5-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0,$$

$\lambda_1 = 3$, $\lambda_2 = -1$. Eigenvectors given by
$(7-\lambda)x_1 - 8x_2 = 0$.

$$\lambda_1 = 3 \Rightarrow 4x_1 = 8x_2,\ \mathbf{x}^{(1)} = (2\ 1)^T$$
$$\lambda_2 = -1 \Rightarrow 8x_1 = 8x_2,\ \mathbf{x}^{(2)} = (1\ 1)^T$$

General solution

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}$$
$$= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Take $c_2 = 0$ to give two outward rays, then $c_1 = 0$ giving the inward rays. For other $c_1,\ c_2$, the path is first attracted (when c_1e^{3t} small) to $\mathbf{0}$, then repelled (when e^{3t} term grows and $e^{-t} \to 0$).

Initial conditions $y_1(0) = 3,\ y_2(0) = 0$ give

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \begin{array}{rcl} 2c_1 + c_2 & = & 3 \\ c_1 + c_2 & = & 0 \end{array}$$

and $c_1 = 3,\ c_2 = -3$. Solution:

$$\mathbf{y} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} - 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Another saddle at $\mathbf{0}$ given by

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{y},$$

with general solution

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$$

or $y_1 = c_1 e^t,\ y_2 = c_2 e^{-2t} \ \Rightarrow y_1^2 y_2 = const.$

Centre Eigenvalues pure imaginary:

$$\mathbf{y}' = \begin{bmatrix} 0 & -6 \\ 3/2 & 0 \end{bmatrix} \mathbf{y}.$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & -6 \\ 3/2 & -\lambda \end{vmatrix} = \lambda^2 + 9 = 0,$$

so, eigenvalues $\lambda = \pm 3i$. Eigenvector constraint $-\lambda x_1 - 6x_2 = 0$. For $\lambda = 3i$, $-3ix_1 - 6x_2 = 0$ and $\mathbf{x}^{(1)} = (2 \quad -i)^T$. Similarly, for $\lambda = -3i$, $\mathbf{x}^{(2)} = (2 \quad i)^T$. general solution

$$\mathbf{y} = c_1 \begin{bmatrix} 2 \\ -i \end{bmatrix} e^{3it} + c_2 \begin{bmatrix} 2 \\ i \end{bmatrix} e^{-3it}.$$

This solution is **complex** and we obtain a **real** solution as follows. Since $e^{i\theta} = \cos\theta + i\sin\theta$,

$$\begin{bmatrix} 2 \\ -i \end{bmatrix} e^{3it} = \begin{bmatrix} 2\cos 3t \\ \sin 3t \end{bmatrix} + i \begin{bmatrix} 2\sin 3t \\ -\cos 3t \end{bmatrix},$$

$$\begin{bmatrix} 2 \\ i \end{bmatrix} e^{-3it} = \begin{bmatrix} 2\cos 3t \\ \sin 3t \end{bmatrix} - i \begin{bmatrix} 2\sin 3t \\ -\cos 3t \end{bmatrix}.$$

The real and imaginary parts

$$\mathbf{u} = \begin{bmatrix} 2\cos 3t \\ \sin 3t \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2\sin 3t \\ -\cos 3t \end{bmatrix}$$

are thus a basis of solutions because

$$W(\mathbf{u}, \mathbf{v}) = \begin{vmatrix} 2\cos 3t & 2\sin 3t \\ \sin 3t & -\cos 3t \end{vmatrix} = -2 \neq 0.$$

General solution

$$\mathbf{y}(t) = A \begin{bmatrix} 2\cos 3t \\ \sin 3t \end{bmatrix} + B \begin{bmatrix} 2\sin 3t \\ -\cos 3t \end{bmatrix}$$

Focus (Spiral Point) Complex eigenvalues (nonzero real part) give a spiral of solutions around $\mathbf{0}$, either $\to \mathbf{0}$ as $t \to \infty$ or being repelled from $\mathbf{0}$ as $t \to \infty$.

$$\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ -4 & -1 \end{bmatrix} \mathbf{y} \ .$$

Characteristic equation

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & 1 \\ -4 & -1-\lambda \end{vmatrix} = \lambda^2+2\lambda+5 = 0$$

gives eigenvalues $\lambda = -1 \pm 2i$. Eigenvectors determined by $(-1-\lambda)x_1 + x_2 = 0$. If $\lambda = -1+2i$, $-2ix_1 + x_2 = 0$ and $\mathbf{x}^{(1)} = (1 \quad 2i)^T$. If $\lambda = -1-2i$, $2ix_1 + x_2 = 0$ and $\mathbf{x}^{(2)} = (1 \quad -2i)^T$. General solution

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(-1+2i)t} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{(-1-2i)t}.$$

A real solution is obtained as before:

$$\begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(-1+2i)t} = \begin{bmatrix} e^{-t}\cos 2t \\ -2e^{-t}\sin 2t \end{bmatrix} + i \begin{bmatrix} e^{-t}\sin 2t \\ 2e^{-t}\cos 2t \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{(-1-2i)t} = \begin{bmatrix} e^{-t}\cos 2t \\ -2e^{-t}\sin 2t \end{bmatrix} - i \begin{bmatrix} e^{-t}\sin 2t \\ 2e^{-t}\cos 2t \end{bmatrix}.$$

Real and imaginary parts are real solutions, and are a basis because the Wronskian

$$\begin{vmatrix} e^{-t}\cos 2t & e^{-t}\sin 2t \\ -2e^{-t}\sin 2t & 2e^{-t}\cos 2t \end{vmatrix} = 2e^{-2t} \neq 0.$$

General solution

$$\mathbf{y} = A \begin{bmatrix} e^{-t}\cos 2t \\ -2e^{-t}\sin 2t \end{bmatrix} + B \begin{bmatrix} e^{-t}\sin 2t \\ 2e^{-t}\cos 2t \end{bmatrix}$$

In components, $y_1 = e^{-t}(A\cos 2t + B\sin 2t)$, and $y_2 = 2e^{-t}(B\cos 2t - A\sin 2t)$.
Eliminate t, $y_1^2 + y_2^2/4 = (A^2 + B^2)e^{-2t}$, a **spiral.**

No Basis of Eigenvectors

If $\mathbf{A}$ does not have a basis of eigenvectors, with for example a double eigenvalue μ for which there is only one eigenvector $\mathbf{x}$, we only have one solution $\mathbf{y}^{(1)} = \mathbf{x}e^{\mu t}$. To obtain a second lin indept soln, try

$$\mathbf{y}^{(2)} = \mathbf{x}te^{\mu t} + \mathbf{u}e^{\mu t} \quad \mathbf{u} = ?$$

in the DE $\mathbf{y}^{(2)\,\prime} = \mathbf{A}\mathbf{y}^{(2)}$. That is,

$$\begin{aligned} \frac{d}{dt}(\mathbf{x}te^{\mu t} + \mathbf{u}e^{\mu t}) &= \mathbf{x}e^{\mu t} + \mu\mathbf{x}te^{\mu t} + \mu\mathbf{u}e^{\mu t} \\ &= \mathbf{A}(\mathbf{x}te^{\mu t} + \mathbf{u}e^{\mu t}). \end{aligned}$$

Since $\mathbf{A}\mathbf{x} = \mu\mathbf{x}$, dividing by $e^{\mu t}$,

$$\mathbf{x} + \mu\mathbf{u} = \mathbf{A}\mathbf{u} \;\Rightarrow\; (\mathbf{A} - \mu\mathbf{I})\mathbf{u} = \mathbf{x},$$

which can be solved for $\mathbf{u}$.

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Example. $\mathbf{y}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{y}.$

Characteristic eqn $\lambda^2 - 4\lambda + 4 = 0$ with double root $\lambda = 2$. Eigenvectors satisfy $(3-\lambda)x_1 + x_2 = 0$ and so $\mathbf{x}^{(1)} = (1 \quad -1)^T$. But

$$(\mathbf{A} - 2\mathbf{I})\mathbf{u} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

and $\mathbf{u} = (0 \quad 1)^T$ works, giving

$$\begin{aligned} \mathbf{y} &= c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} \\ &= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{2t}. \end{aligned}$$

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Nonhomogeneous Systems

We will also solve these later with Laplace Transforms. Here are a few examples of solution by Undetermined Coefficients:

Example 1

$$\mathbf{y}' = \begin{bmatrix} -1 & 4 \\ 1 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2t^2 + 6t \\ 4t^2 + 6t + 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t}.$$

General solution of homog eqn

$$\mathbf{y}_h = c_1 e^{-2t} \begin{bmatrix} -4 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Look for $\mathbf{y}_p = \mathbf{u} + \mathbf{v}t + \mathbf{w}t^2 + \mathbf{z}e^{-t}$ and determine the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$. Substituting,

$$\begin{aligned} \mathbf{y}_p' = \mathbf{v} + 2\mathbf{w}t \quad &= \quad \begin{bmatrix} -1 & 4 \\ 1 & 2 \end{bmatrix} (\mathbf{u} + \mathbf{v}t + \mathbf{w}t^2) \\ &\quad + \begin{bmatrix} 2t^2 + 6t \\ 4t^2 + 6t + 1 \end{bmatrix}. \end{aligned}$$

Equating terms in t^2,

$$0 = -w_1 + 4w_2 + 2,\ \ 0 = w_1 + 2w_2 + 4$$
$$\Rightarrow w_1 = -2,\ \ w_2 = -1.$$

Using the terms in t:

$$2w_1 = -v_1 + 4v_2 + 6,\ \ 2w_2 = v_1 + 2v_2 + 6$$
$$\Rightarrow v_1 = -2,\ \ v_2 = -3,$$

and from the constant terms

$$v_1 = -u_1 + 4u_2,\ \ v_2 = u_1 + 2u_2 + 1$$
$$\Rightarrow u_1 = -2,\ \ u_2 = -1.$$

So, the general solution is

$$\begin{aligned} \mathbf{y} &= \mathbf{y}_h + \mathbf{y}_p \\ &= c_1 e^{-2t} \begin{bmatrix} -4 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &\quad + \begin{bmatrix} -2 - 2t - 2t^2 \\ -1 - 3t - t^2 \end{bmatrix} \end{aligned}$$

Example 2: Modification Rule

$$\mathbf{y}' = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-3}{2} \end{bmatrix} \mathbf{y} \; + \; \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t} \; .$$

General solution of homog eqn is a node (see Example on node):

$$\mathbf{y}_h = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Since $\lambda = -1$ is an eigenvalue of $\mathbf{A}$, we must modify the function $\mathbf{y}_p$ to try as
$\mathbf{y}_p = \mathbf{u} t e^{-t} + \mathbf{v} e^{-t}$.

$$\begin{aligned} \mathbf{y}_p' &= \mathbf{u} e^{-t} - \mathbf{u} t e^{-t} - \mathbf{v} e^{-t} \\ &= \mathbf{A}(\mathbf{u} t e^{-t} + \mathbf{v} e^{-t}) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Cancelling e^{-t} and equating the coefficients of t gives $\mathbf{A}\mathbf{u} = -\mathbf{u}$, so $\mathbf{u}$ is an eigenvector of $\mathbf{A}$, and must be of the form $\mathbf{u} = \alpha(1 \;\; 1)^T$.

Equating the constant terms gives

$$\mathbf{A}\mathbf{v} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = -\mathbf{v} + \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} ,$$

$$\begin{aligned} -\frac{v_1}{2} + \frac{v_2}{2} &= \alpha - 3 \\ \frac{v_1}{2} - \frac{v_2}{2} &= \alpha - 1 \ . \end{aligned}$$

These eqns have a solution only if

$$\alpha - 3 = -(\alpha - 1),$$

and $\alpha = 2$, & then $v_1 - v_2 = 2$. Any solution of this will do, say $v_1 = 2,\ v_2 = 0$, so $\mathbf{v} = (2\ 0)^T$ and

$$\mathbf{y}_p = 2te^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{y} &= c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &\qquad + 2te^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{aligned}$$