6. $u_{tt} = u_{xx}$, $L=\pi$, $u(0,t) = u(0,\pi) = 0 \quad \forall t$

$$u(x,0) = f(x) = \begin{cases} kx/a, & 0 \le x \le a, \\ +k(x-\pi)/(a-\pi), & a < x \le \pi. \end{cases} \quad \& \; u_t(x,0)=0$$

Going through the same steps as in Q2,

$F_n(x) = \sin nx$, $G_n(t) = B_n \cos nt$, $B_n^* = 0$ from $u_t(x,0)=0$.

$u_n(t) = B_n \cos nt \sin nx$

$n = 1, 2, \ldots,$

where the B_n are the coeff. of Fourier sine series of f. That is, $B_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx \, dx$.

$$\frac{\pi B_n}{2k} = \int_0^a \frac{x}{a} \sin nx \, dx + \int_a^\pi \frac{x-\pi}{a-\pi} \sin nx \, dx$$

$$= \left[-\frac{x}{a}\frac{\cos nx}{n}\right]_0^a + \frac{1}{na}\int_0^a \cos nx \, dx - \left[\frac{x-\pi}{a-\pi}\frac{\cos nx}{n}\right]_a^\pi + \frac{1}{(a-\pi)n}\int_a^\pi \cos nx \, dx$$

$$= -\frac{\cos na}{n} + \frac{1}{n^2 a}\sin na + \frac{\cos na}{n} - \frac{1}{(a-\pi)n^2} \sin na$$

$$= \frac{\sin na}{n^2}\left(\frac{1}{a} - \frac{1}{a-\pi}\right) = \frac{\pi \sin na}{n^2(\pi a - a^2)}$$

$$B_n = 2k \sin na / n^2(\pi a - a^2).$$

$$u(x,t) = \sum_1^\infty u_n(x,t) = \sum_1^\infty B_n \sin nx \cos nt$$

$$= \frac{2k}{\pi a - a^2}\left(\frac{\sin a \sin x \cos t}{1^2} + \frac{\sin 2a \sin 2x \cos 2t}{2^2} + \cdots\right)$$

$$= \frac{2k}{\pi a - a^2} \sum_{n=1}^\infty \frac{\sin na \sin nx \cos nt}{n^2}$$