# MODELLING RANDOM FLUCTUATIONS IN A BISTABLE TELECOMMUNICATIONS NETWORK

by

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## REFERENCES


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# A SYMMETRIC FULLY-CONNECTED NETWORK

$N$ - nodes

$K = \binom{N}{2} = \frac{1}{2}N(N-1)$ - links

$C$ - circuits on each link

Poisson traffic offered at rate $\nu$

Exponentially distributed holding times (mean 1)

$r$ retries (initially take $r = 1$)

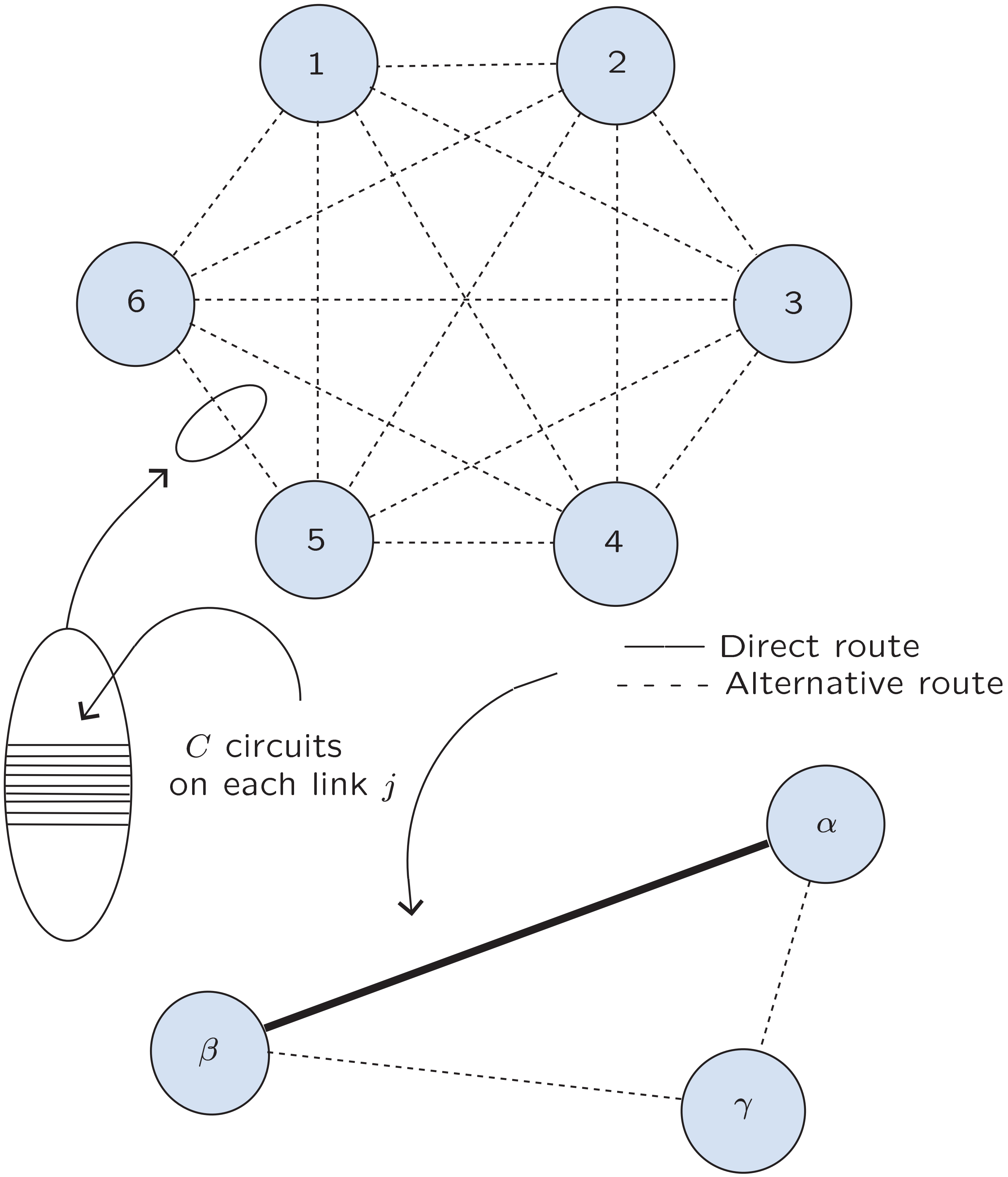


Fig 1. A symmetric fully-connected network

$N(=6)$ nodes and $K = \binom{N}{2} (= 15)$ links

# THE GHK MODEL

Let $n_j^{(K)}(t)$ be the number of links with $j$ circuits in use at time $t$ (for a network with $K$ links).

Let $n^{(K)} = (n_0^{(K)}, n_1^{(K)}, \dots, n_C^{(K)})$. Then, $(n^{(K)}(t),\ t \geq 0)$ is a continuous-time Markov chain which takes values in

$$S^{(K)} = \left\{ n \in \{0, 1, \dots\}^{C+1} \ : \ \sum_{i=0}^{C} n_i = K \right\}.$$

The transition rates, $Q^{(K)} = (q^{(K)}(n,n'), n,n' \in S^{(K)})$, of the process are given by

$$q^{(K)}(n, n+e_{j+1}-e_j) = \nu n_j, \qquad 0 \le j \le C-1,$$

$$q^{(K)}(n, n+e_{j-1}-e_j) = j n_j, \qquad 1 \le j \le C,$$

$$q^{(K)}(n, n+e_{i+1}-e_i+e_{j+1}-e_j) = \nu n_C \frac{n_i n_j}{\binom{K}{2}},$$

$$0 \le i < j \le C-1,$$

$$q^{(K)}(n, n+2(e_{j+1}-e_j)) = \nu n_C \frac{\binom{n_j}{2}}{\binom{K}{2}},$$

$$0 \le j \le C-1,$$

where $e_i$ is the unit vector with 1 as its $i^{\text{th}}$ entry.

## THE BEHAVIOUR OF THE NETWORK

Let $X^{(K)} = (X_0^{(K)}, X_1^{(K)}, \ldots, X_C^{(K)})$, where $X_j^{(K)}(t) = n_j^{(K)}(t)/K$ is the *proportion* of links with $j$ circuits in use at time $t$. The process $(X^{(K)}(t),\ t \geq 0)$ is itself a Markov chain, but one which takes values in a lattice contained in the simplex

$$E = \left\{ x \in [0,1]^{C+1} : \sum_{i=0}^{C} x_i = 1 \right\}.$$

## 2-D "SUMMARY": $(X_C, \bar{X})$

$X_C$ is proportion of links full; this estimates link blocking probability.

And, $\bar{X} = \sum_{i=0}^{C} i X_i$; this estimates the mean number of circuits in use on any given link.

# THE BEHAVIOUR AS THE NETWORK BECOMES LARGE

If, as $K \to \infty$, $X^{(K)}(0) \Rightarrow x_0$, then $X^{(K)}(\cdot) \Rightarrow X(\cdot, x_0)$, where $(X(t, x_0),\ t \geq 0)$ is a deterministic process with initial point $X(0, x_0) = x_0$ and which satisfies

$$\tfrac{d}{dt} X_0(t) = X_1(t) - (\nu + \lambda(t)) X_0(t),$$

$$\tfrac{d}{dt} X_j(t) = (\nu + \lambda(t)) X_{j-1}(t) - (\nu + \lambda(t) + j) X_j(t)$$

$$+ (j+1) X_{j+1}(t), \quad 1 \leq j \leq C-1,$$

$$\tfrac{d}{dt} X_C(t) = (\nu + \lambda(t)) X_{C-1}(t) - C X_C(t),$$

where $\lambda(t) = 2\nu X_C(t)(1 - X_C(t))$.

A CENTER MANIFOLD?

$$\boldsymbol{\Xi} = \left\{ x \,:\, x_i = \frac{\xi^i}{i!} \left( \sum_{j=0}^{C} \frac{\xi^j}{j!} \right)^{-1} \; ; \; \xi \in (0, \infty) \right\}$$

## LAW OF LARGE NUMBERS

**Theorem.** If, as $K \to \infty$, $X^{(K)}(0) \Rightarrow x_0$, then, for all $\epsilon > 0$ and $t \geq 0$,

$$\lim_{K\to\infty} \Pr\left(\sup_{s\leq t}\left|X^{(K)}(s) - X(s, x_0)\right| > \epsilon\right) = 0,$$

where $X(\cdot, x_0)$ is the unique solution to the above system of differential equations with $X(0, x_0) = x_0$.

So, for example, $\{X^{(K)}(s)\}$ converges in probability to $X(s, x_0)$ and, since for each $s$, $X^{(K)}(s)$ is uniformly bounded, dominated convergence implies that

$$\lim_{K\to\infty} \mathsf{E}X^{(K)}(s) = X(s, x_0)$$

on all finite time intervals.

# THE EQUILIBRIUM POINTS

If $x_0 = (p_0, p_1, \dots p_C)$ is an equilibrium point it must be of the form given by

$$p_j = \frac{\xi^j}{j!}\left(\sum_{i=0}^{C}\frac{\xi^i}{i!}\right)^{-1}, \qquad 0 \le j \le C,$$

where $\xi$ solves

$$\xi = \nu + 2\nu E(\xi, C)\,(1 - E(\xi, C))\,.$$

The quantity $E(\xi, C)$, given by

$$E(\xi, C) = \frac{\xi^C}{C!}\left(\sum_{i=0}^{C}\frac{\xi^i}{i!}\right)^{-1},$$

is Erlang's formula for the loss probability of a single link with $C$ circuits and with Poisson traffic offered at rate $\xi$. It is usually more convenient to calculate the equilibrium points by setting $b = E(\xi, C)$ and solving the equation

$$b = E\,(\nu + 2\nu b(1 - b), C)\,.$$

# MODELLING RANDOM FLUCTUATIONS

The central limit law (Theorem 4.2 of [1]) shows that the random fluctuations about any given equilibrium point, $x_0$, are approximately Gaussian. Moreover, it shows that these fluctuations can be approximated by an Ornstein-Uhlenbeck (OU) process.

Let $x_0$ be an equilibrium point. Then, if

$$\lim_{K\to\infty} \sqrt{K}\left(X^{(K)}(0) - x_0\right) = z,$$

the family of processes $\{Z^{(K)}(\cdot)\}$, defined by

$$Z^{(K)}(s) = \sqrt{K}\left(X^{(K)}(s) - x_0\right), \quad 0 \le s \le t,$$

converges weakly to an Ornstein-Uhlenbeck process with initial value $Z(0) = z$ and with a local drift matrix, $B$, and a local covariance matrix, $G$, which can be determined from the parameters of the model.

In particular, $Z(s)$ is normally distributed with mean

$$\mu_s = e^{Bs}z$$

and covariance matrix

$$\Sigma_s = \int_0^s e^{Bu}Ge^{B^Tu}du = \Sigma - e^{Bs}\Sigma e^{B^Ts},$$

where $\Sigma$, the stationary covariance matrix, satisfies

$$B\Sigma + \Sigma B^T + G = 0.$$

We can conclude that, for $K$ large, $X^{(K)}(s)$ has an approximate normal distribution for each $s$, and an approximation for the mean and the covariance matrix of $X^{(K)}(s)$ is given by

$$\mathsf{E}X^{(K)}(s) \simeq x_0 + e^{Bs}(X^{(K)}(0) - x_0)$$

and

$$\mathrm{Cov}\, X^{(K)}(s) \simeq K^{-1}\left(\Sigma - e^{Bs}\Sigma e^{B^Ts}\right).$$

## MEASURING STABILITY

If $C = 1$ (one circuit on each link), set $F(x) = 1 - (\nu + 1)x - 2\nu(1 - x)x^2,\ x \in (0, 1)$, $G(x) = 1 + (\nu - 1)x + 4\nu(1 - x)x^2\ x \in (0, 1)$. Then, $F$ has a unique zero, $x_0$, on $(0, 1)$, for all $\nu > 0$ (stable equilibrium point of $dx/dt = F(x)$). Further, $X^{(K)}(s)$ has an approximate normal distribution with

$$\mathsf{E}X^{(K)}(s) \simeq x_0 + e^{Bs}(X^{(K)}(0) - x_0),$$

where $B = F'(x_0) = 6\nu x_0^2 - 4\nu x_0 - (\nu + 1)(< 0)$, and

$$\mathsf{Var}X^{(K)}(s) \simeq K^{-1}\frac{G(x_0)}{2B}(e^{2Bs} - 1).$$

The magnitude of $B$, and hence the stability of $x_0$, increases as $\nu$ becomes large, but the stationary variance, $G(x_0)/(-2B)$, increases from 0 to a maximum around $\nu = 0.5$ and then decreases to 0.

When $C > 1$, change coordinates:

$$W^{(K)}(s) = AZ^{(K)}(s),$$

where the rows of $A$ are the left-eigenvectors of $B = \nabla F(x_0)$. Since the column sums of $B$ are all equal to 0, $B$ has a zero eigenvalue, and so one of the components of $W^{(K)}$, say $W_0^{(K)}$, is identically zero, since because $\sum_{i=0}^{C} X_i^{(K)} = 1$, we have that $\sum_{i=0}^{C} Z_i^{(K)} = 0$.

The sequence $\{W^{(K)}(\cdot)\}$, where (for convenience) $W^{(K)} = (W_1^{(K)}, W_2^{(K)}, \dots W_C^{(K)})$, converges weakly to an OU process, $W(\cdot)$, whose individual components are, themselves, OU processes. Its local drift matrix is $D = \text{diag}\,(\eta_1, \eta_2, \dots \eta_C)$, where $\eta_1, \eta_2, \dots \eta_C$ are the non-zero eigenvalues of $B$, and its local covariance matrix, $S$, is obtained from the matrix $AGA^T$ by deleting the zeroth row and column.

In particular, $W(s)$ has a properly $C$-dimensional normal distribution with

$$\mathsf{E}W_i(s) = w_i e^{\eta_i s},$$

$$\mathsf{Var}\, W_i(s) = \tfrac{S_{ii}}{2\eta_i}(e^{2\eta_i s} - 1)$$

and

$$\mathsf{Cov}\,(W_i(s),\ W_j(s)) = \frac{S_{ij}}{\eta_i + \eta_j}(e^{(\eta_i+\eta_j)s} - 1),$$

for $i = 1, 2 \ldots, C$, where $w = Az$.

The change of coordinates allows us to use some powerful results of Andrew Barbour, which establish asymptotic results on the time of first exit of $X^{(K)}(\cdot)$ from a region containing $x_0$.

For example, suppose that $x_0$ is a stable and let $\tau(K, c_K)$ be the time when $W^{(K)}(\cdot)$ first crosses the contour

$$\left\{ w \in \mathsf{R}^C : \sum_{i=1}^{C} \sqrt{\frac{2T_{ii}}{w_i^2}} \exp\left(\frac{w_i^2}{2T_{ii}}\right) = c_K^{-1} \exp(c_K^2) \right\},$$

where $T$, the stationary covariance matrix of $W(\cdot)$, has elements $T_{ij} = -S_{ij}/(\eta_i + \eta_j)$ and $\{c_K\}$ is a sequence which tends to $\infty$; to order $c_K^{-1}$, the contour delimits the rectangle

$$\left\{ w \in \mathsf{R}^C \ : \ |w_i| \le c_K \sqrt{(2T_{ii})}, \ i = 1, 2, \ldots, C \right\}.$$

Then, if $c_K = o(K^{\frac{1}{8}})$, the random variable

$$-\tau(K, c_K) \frac{2}{\sqrt{\pi}} \eta c_K \exp(-c_K^2),$$

where $\eta = \sum_{i=1}^{C} \eta_i$, converges weakly to a unit-mean exponential random variable as $K \to \infty$. Thus, provided $c_K = o(K^{\frac{1}{8}})$, the time at which $W^{(K)}(\cdot)$ first crosses the contour is of order $c_K^{-1} \exp(c_K^2)$.

The corresponding result for the $C = 1$ case is more straightforward. One can show that the time that $X^{(K)}(\cdot)$ first leaves the interval

$$\left\{ x \;:\; |x - x_0| \leq K^{-\frac{1}{2}} c_K \right\}$$

is of order

$$\frac{1}{-2Bc_K} \sqrt{\frac{\pi G(x_0)}{-B}} e^{-Bc_K^2/G(x_0)},$$

whenever $c_K = o(K^{\frac{1}{8}})$. Hence, it is asymptotically larger than any power of $K$ if, for example, $c_K = O(K^{\frac{1}{8}}/\log K)$.