Complex Networks at UQ

Phil Pollett

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The University of Queensland
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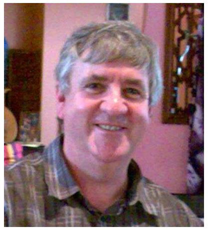
AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics
and Statistics of Complex Systems

MASCOS Qld 2009

Research staff



Phil
Pollett (CI)

(Networks
and Risk)

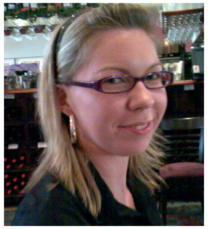
Ross
McVinish (RF)

(Networks
and Risk)

Iadine
Chadès (RF)

(Risk)

PhD students



Fionnuala
Buckley
April 2007

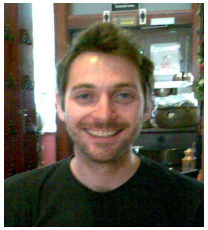
(Networks)

Dejan
Jovanović
March 2009

(Networks)

Andrew
Smith
July 2009

(Networks)



Daniel
Pagendam
March 2007

(Risk)

Nimmy
Thaliath
February 2009

(Risk)

Robert
Cope
July 2009

(Risk)

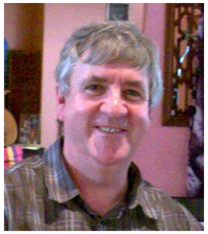
Honours/Masters students

Alex Ridley (Networks), Chung Kai Chan (Risk)

Expertise

Phil Pollett

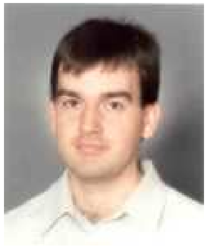
Mathematical modelling, stochastic process theory and applications: ecology, epidemiology, parasitology, telecommunications and chemical kinetics.



A current project: *Stochastic models for metapopulation networks*.

Ross McVinish

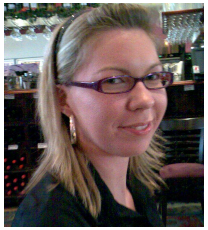
Lévy processes and stochastic processes displaying long memory, Bayesian nonparametrics, computation for Bayesian statistics and time series analysis.



A current project: *Limits of large metapopulations with site-dependent extinction*.

PhD Projects

Fionnuala Buckley (April 2007 –)



Discrete-time Stochastic Metapopulation Models

A metapopulation is a population that occupies several geographically separated habitat patches. Although the individual patches may become empty through local extinction, they may be recolonized through migration from other patches. Empirical evidence suggests that a balance between migration and extinction is reached that enables metapopulations to persist for long periods, and there has been considerable interest in developing methods that account for their persistence populations and which provide an effective means of studying their long-term behaviour before extinction occurs.

For many populations, extinction and colonization happen in distinct phases, corresponding to stages in the organism's life cycle, and the natural stochastic model is a (time-inhomogeneous) Markov chain in discrete time. We have developed a device that accounts for colonization potential of occupied patches, and we are developing deterministic and distributional approximations to analyse these models that build on our direct methods for "mainland-island" models.

PhD Projects

Dejan Jovanović (March 2009 –)

Fault Detection in Complex and Distributed Systems

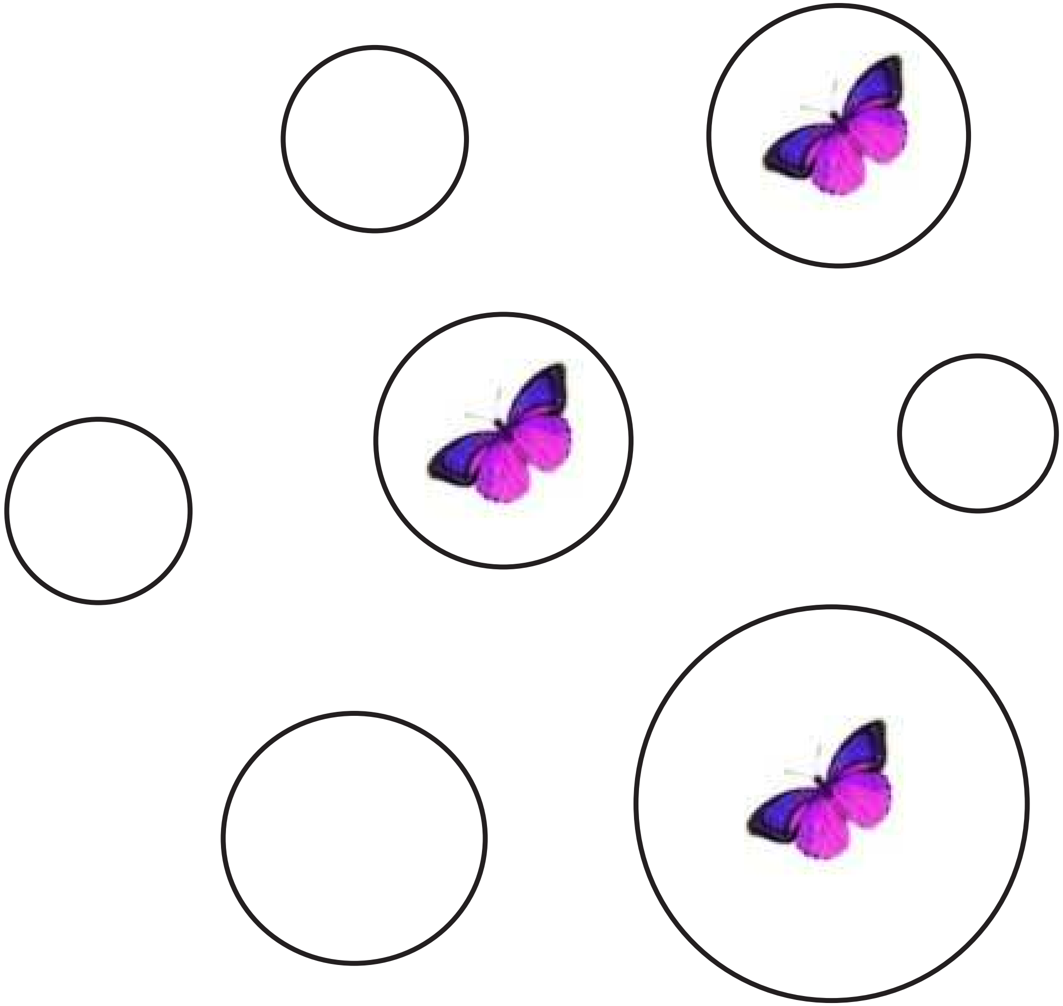
The primary goal is to develop a theoretical framework based on Markov processes in order to detect, identify and isolate faults in complex and distributed systems. The aim is to improve overall safety and reduce any negative impact on the environment due to a fault. There are three main tasks. The first is development of local stochastic models, which need to be capable of interpreting the local environment's state. At the core is estimation of transition probabilities. The second is extracting the features of local models in the case of non-faulty and faulty operating conditions. In order to assist local models to achieve satisfactory results, design and implementation of a multi-agent system is proposed. The next task is planning an optimal action to protect the environment. Finally, the framework has to allow for the possibility of incorporating local expert knowledge about the system.

Andrew Smith (July 2009 –)

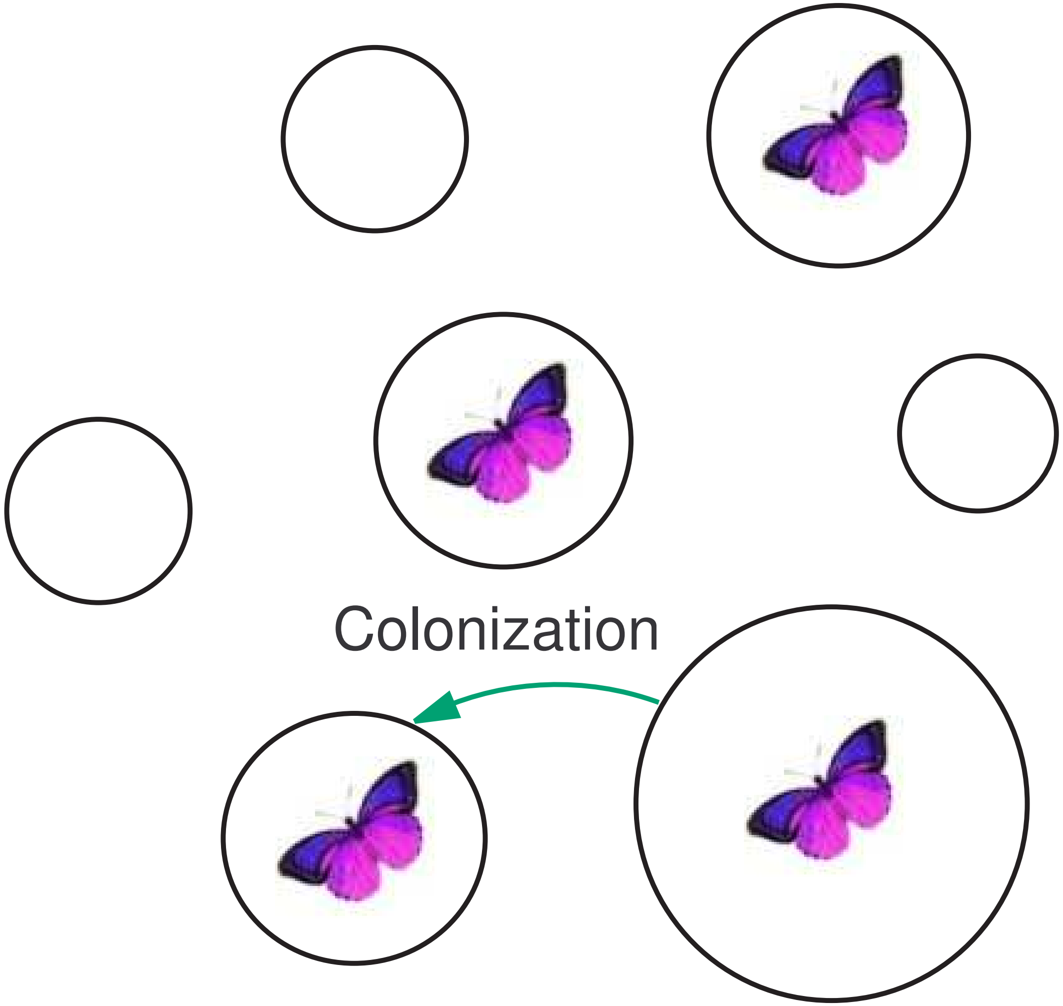
Models for Spatially Structured Metapopulations

We aim to extend continuous-time stochastic models for metapopulations to account for the spatial arrangement of patches. We begin by looking at basic patch-occupancy models for population networks, ones that merely record which patches are occupied. The main aim is to exploit recent developments in stochastic network theory by adapting models that were developed originally for the study of telecommunications systems. By recording the numbers of individuals in the various patches we can incorporate local patch dynamics, spatial structure and migration patterns. We will adopt the powerful diffusion approximation technique that has been used so effectively elsewhere in the analysis of patch-occupancy models.

Metapopulations



Metapopulations



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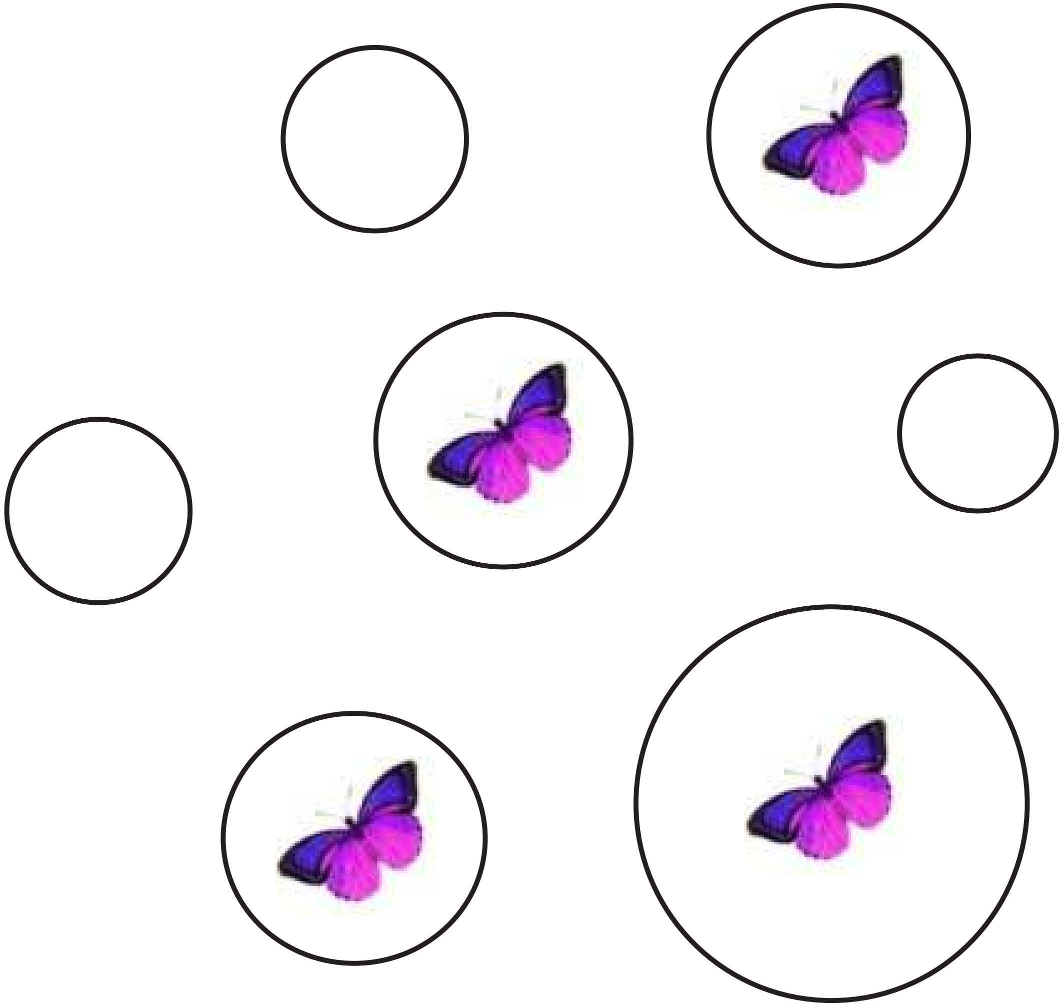

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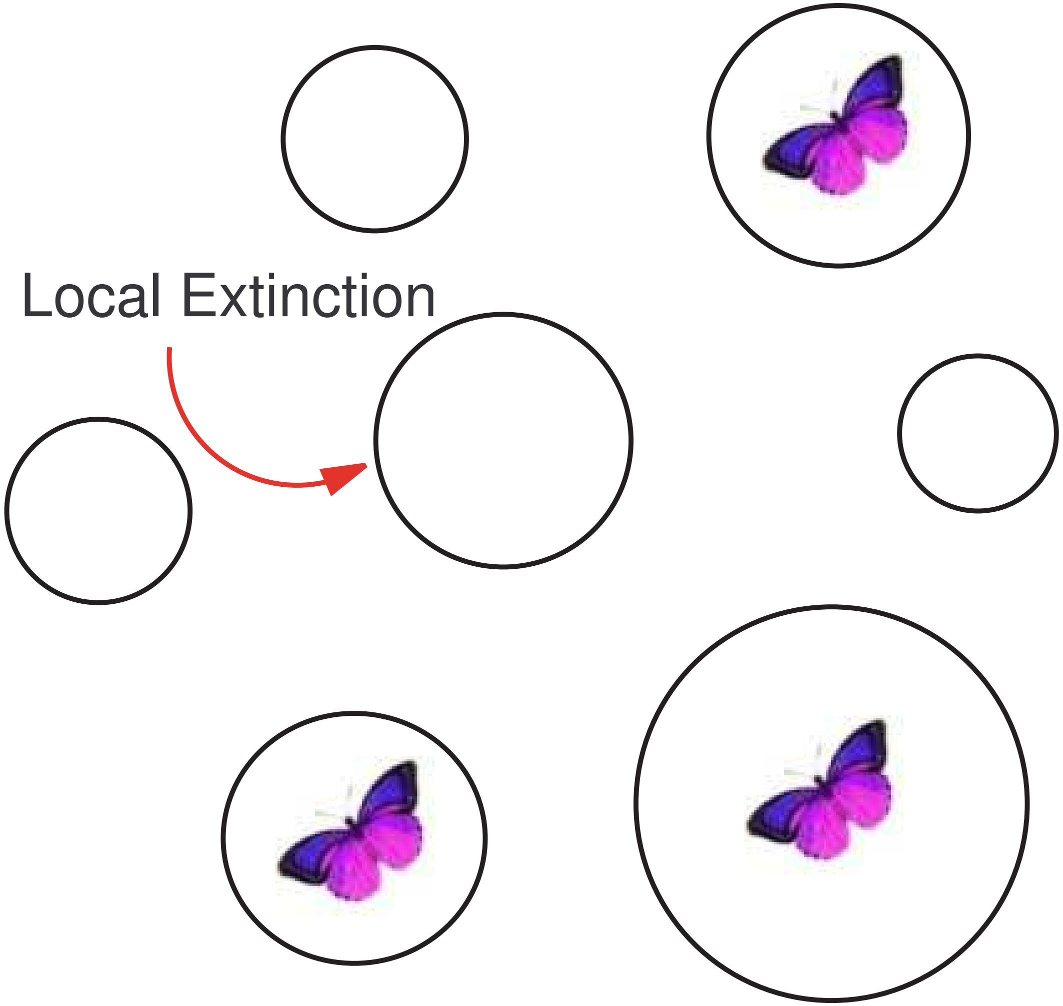

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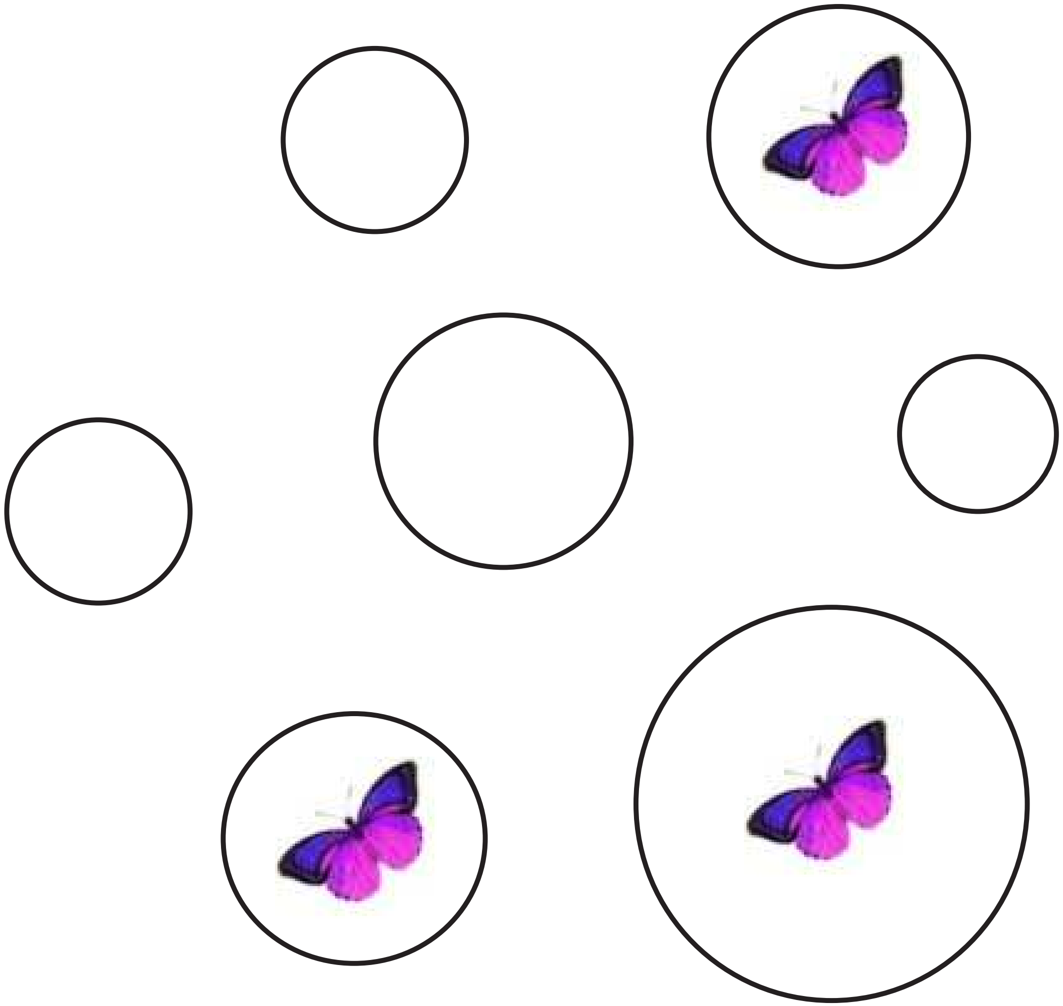
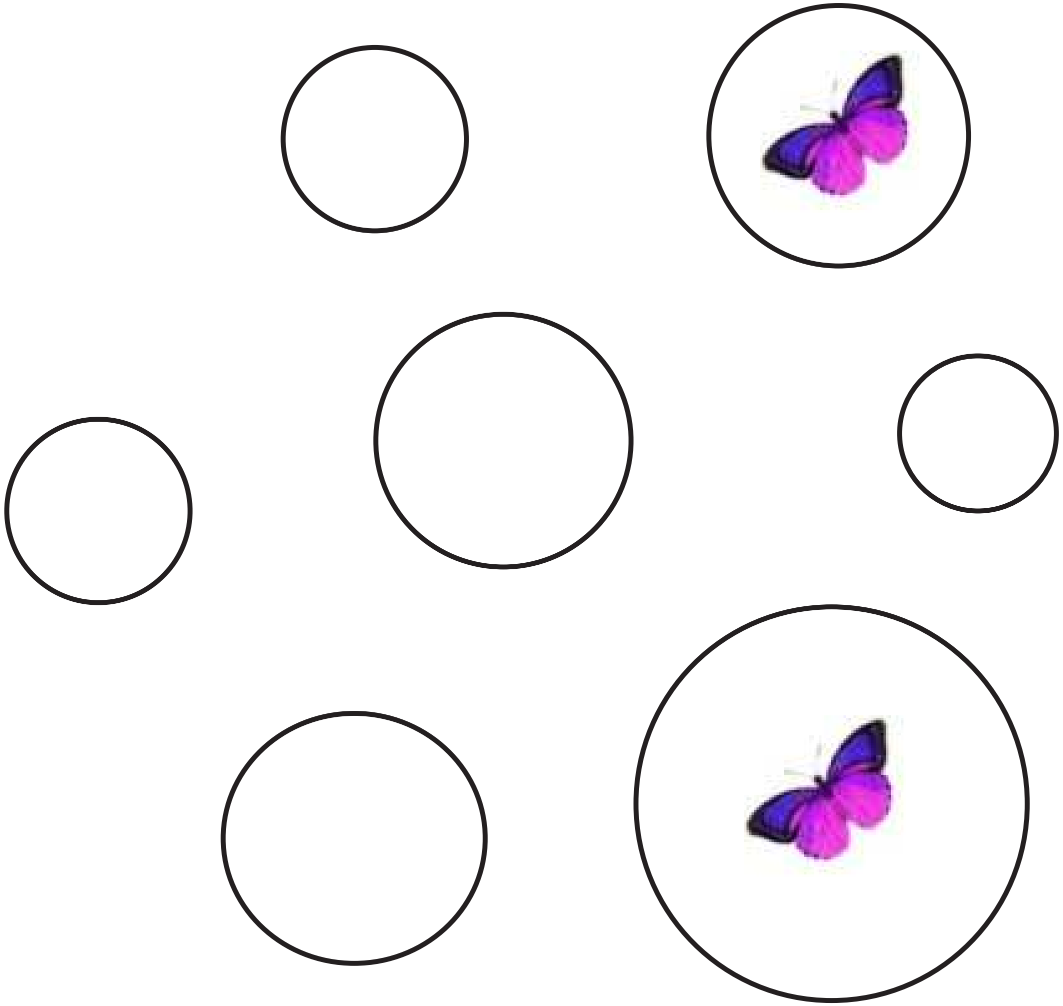


Metapopulations



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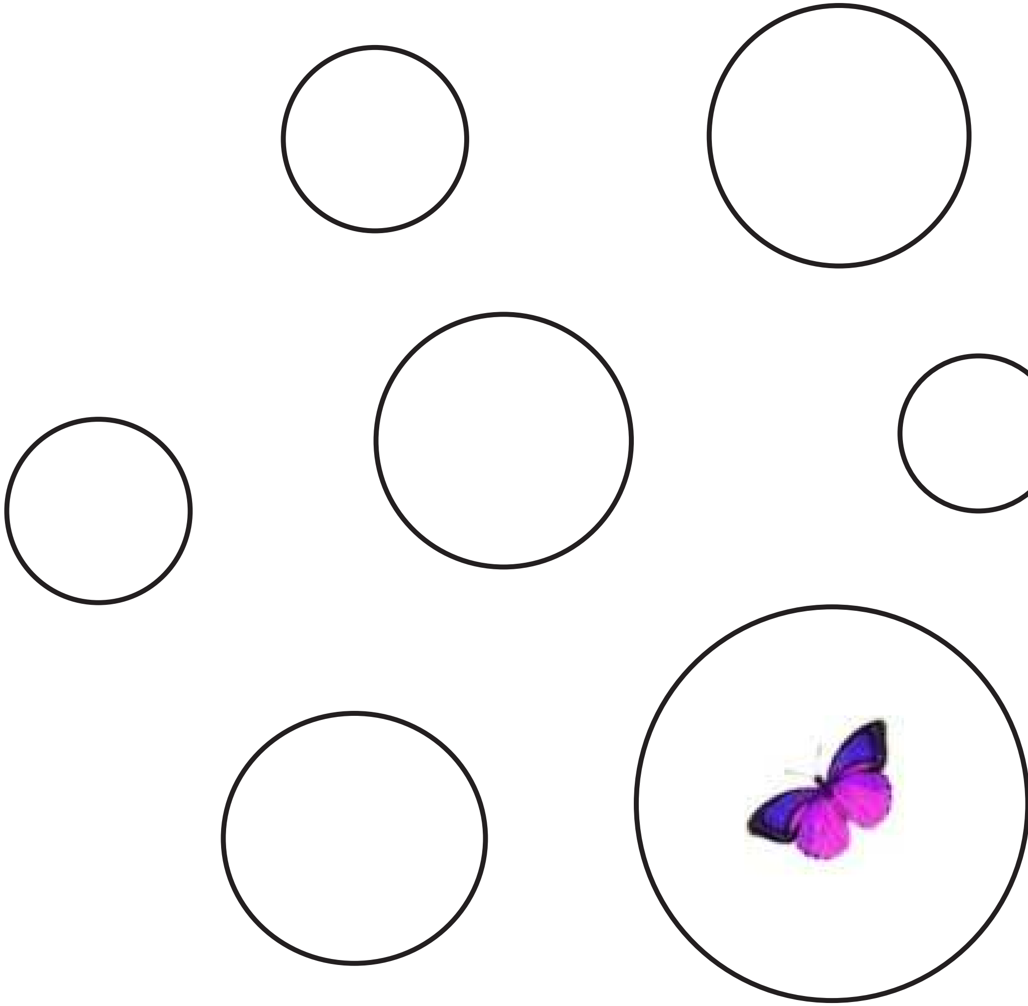

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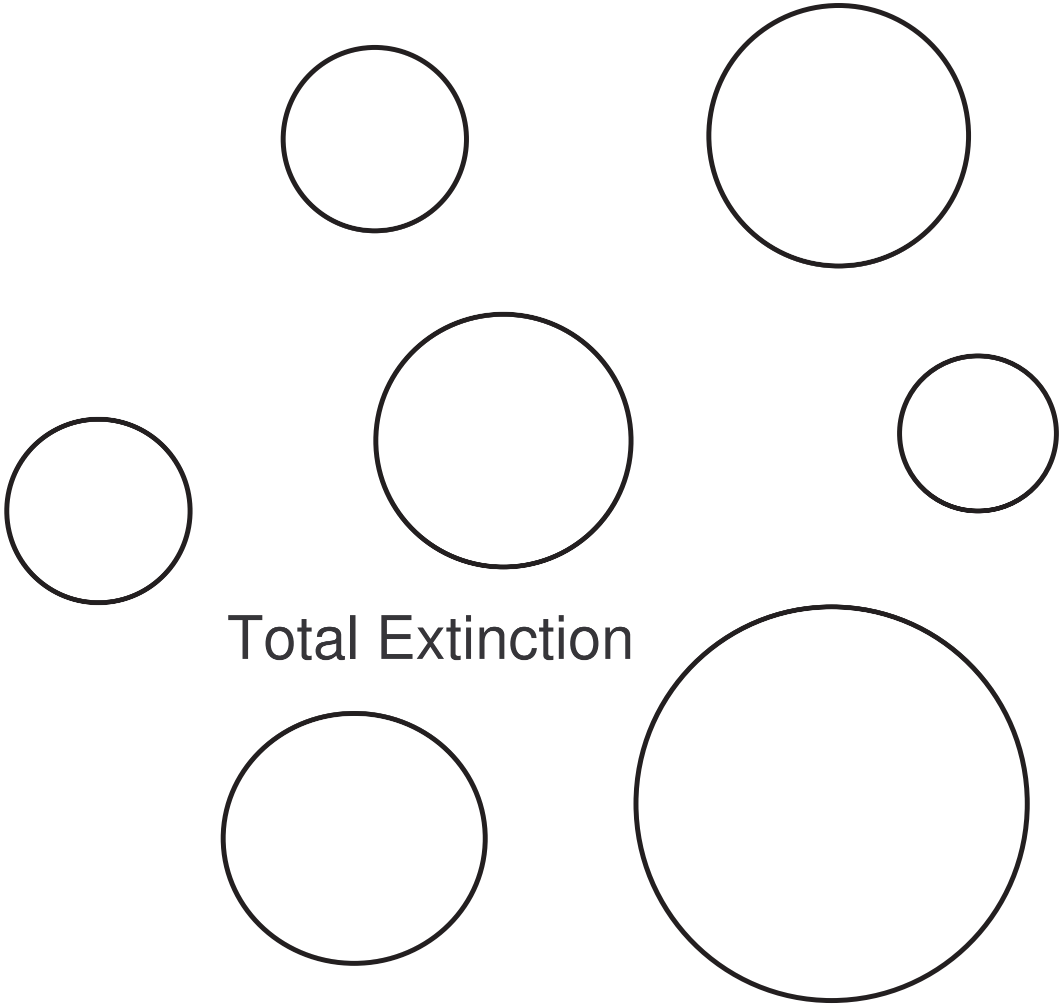

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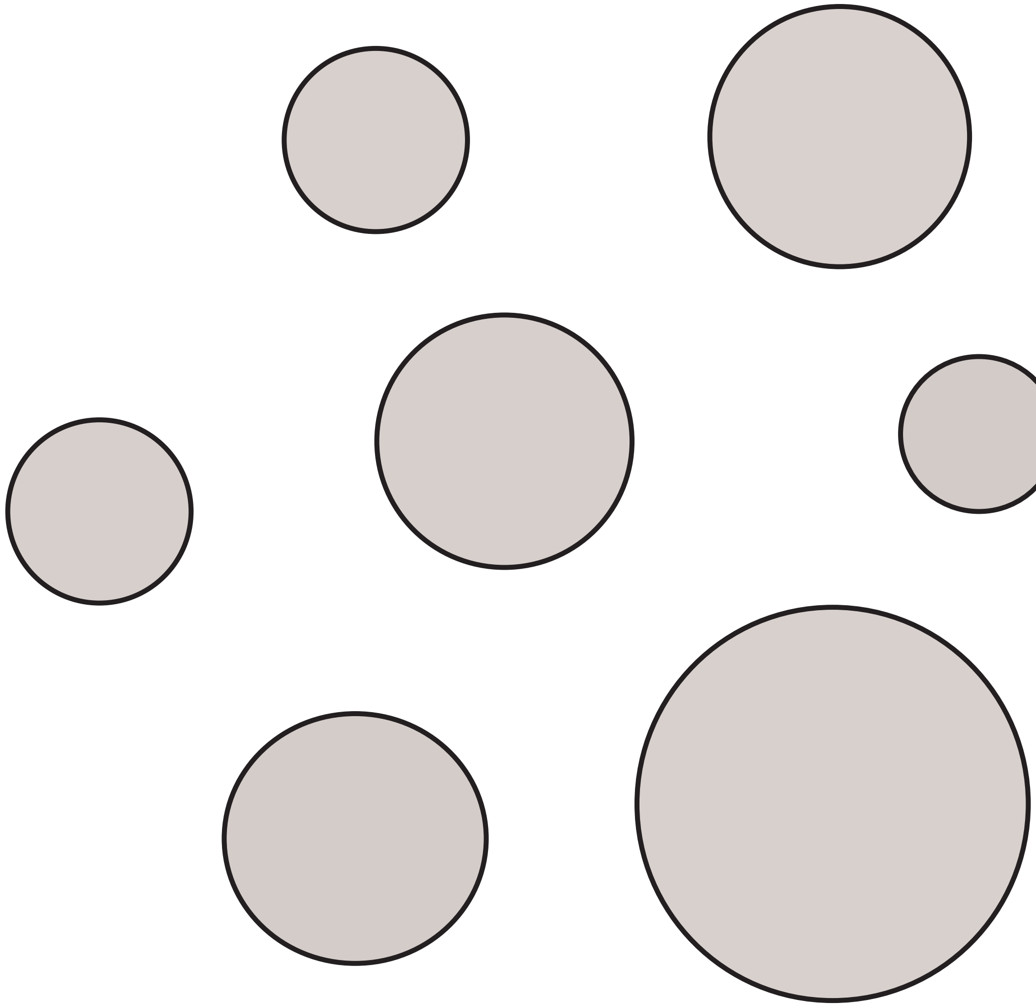

Recent highlights - Ross McVinish

Let $X_t^{(n)} = (X_{1,t}^{(n)}, \ldots, X_{n,t}^{(n)})$, where $X_{i,t}^{(n)}$ is a binary variable indicating whether or not patch i is occupied. (For each n, $(X_t^{(n)}, t = 0, 1, \ldots, T)$ is a Markov chain.)

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Extinction: occupied patch i remains occupied independently with probability S_i (random).

Recent highlights - Ross McVinish

Theorem Suppose that

$$n^{-1}\sum_{i=1}^{n} S_i^k \xrightarrow{P} s(k) \quad \text{and} \quad n^{-1}\sum_{i=1}^{n} S_i^k X_{i,0}^{(n)} \xrightarrow{P} d_0(k),$$

for all $k = 0, 1, \ldots, T$. Then, there is a deterministic triangular array $(d_t(k))$ $(t = 0, \ldots, T$, $k = 0, \ldots, T - t)$ defined by

$$d_{t+1}(k) = d_t(k+1) + f(d_t(0))(s(k+1) - d_t(k+1)),$$

such that

$$n^{-1}\sum_{i=1}^{n} S_i^k X_{i,t}^{(n)} \xrightarrow{P} d_t(k).$$

Recent highlights

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Definition (Kurtz*) The model is *density dependent* if there is a subset E of $\mathbb{R}^k$ and a continuous function $f : \mathbb{Z}^k \times E \to \mathbb{R}$, such that

$$q_{n,n+l} = N f_l\left(\frac{n}{N}\right), \quad l \neq 0 \quad (l \in \mathbb{Z}^k).$$

*Kurtz, T. (1970) Solutions of ordinary differential equations as limits of pure jump Markov processes, *J. of Appl. Probab.* 7, 49–58.

Recent highlights

Let $X_{i,t}^{(N)} = n_{i,t}/N$ be the *proportion* of individuals at site i.

Call $(X_t^{(N)})$ the *population density process*.

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Idea: $(X_t^{(N)})$ looks deterministic when N is large.

Recent highlights

Theorem Let $F(x) := \sum_{l \neq 0} l f_l(x)$ $(x \in E)$ and suppose that F is Lipschitz.

If $\lim_{N\to\infty} X_0^{(N)} = x_0$, then $(X_t^{(N)})$ converges (uniformly in probability over $[0,t]$) to (x_t), the unique (deterministic) trajectory satisfying

$$x_s' = F(x_s) \quad (x_s \in E,\ s \in [0,t]).$$

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Recent highlights

Define $(Z_t^{(N)})$ (scaled fluctuations about the deterministic trajectory) by

$$Z_s^{(N)} = \sqrt{N}\left(X_s^{(N)} - x_s\right) \qquad (0 \leq s \leq t).$$

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This next result follows almost immediately from $\ldots$

*Kurtz, T. (1971) Limit theorems for sequences of jump Markov processes approximating ordinary differential processes. *J. Appl. Probab.* 8, 344–356.

Recent highlights

Theorem Suppose that F is Lipschitz and has uniformly continuous first derivative on E, and that the $k \times k$ matrix $G(x)$, defined for $x \in E$ by $G_{ij}(x) = \sum_{l \neq 0} l_i l_j f_l(x)$, is uniformly continuous on E.

Let (x_t) be the unique deterministic trajectory starting at x_0 and suppose that $\lim_{N\to\infty} \sqrt{N}\left(X_0^{(N)} - x_0\right) = z$.

Then, $(Z_t^{(N)})$ converges weakly in $D[0,t]$ (the space of right-continuous, left-hand limits functions on $[0,t]$) to a Gaussian diffusion (Z_t) with initial value $Z_0 = z$ and with mean and covariance given by $\mu_s := \mathbb{E}(Z_s) = M_s z$, where $M_s = \exp(\int_0^s B_u\,du)$ and $B_s = \partial F(x_s)$, and

$$V_s := \mathrm{Cov}(Z_s) = M_s \left(\int_0^s M_u^{-1} G(x_u)(M_u^{-1})^\top \, du\right) M_s^\top .$$

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Jun Zhao (Workshop 29/06/2009) "Sync theory for complex networks with non-identical nodes"?

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Recent highlights

Corollary If x_{eq} satisfies $F(x_{\mathrm{eq}}) = 0$, then, under the conditions of the previous theorem, the family $(Z_t^{(N)})$ defined by

$$Z_s^{(N)} = \sqrt{N}(X_s^{(N)} - x_{\mathrm{eq}}) \qquad (0 \le s \le t),$$

converges weakly in $D[0,t]$ to an *OU process* (Z_t) with initial value $Z_0 = z$, local drift matrix $B = \partial F(x_{\mathrm{eq}})$ and local covariance matrix $G(x_{\mathrm{eq}})$. In particular, Z_s is normally distributed with mean and covariance given by $\mu_s := \mathbb{E}(Z_s) = e^{Bs}z$ and

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Recent highlights

Note that

$$V_t = \int_0^t e^{Bu} G(x_{\text{eq}}) e^{B^\top u}\, du = V_\infty - e^{Bt} V_\infty e^{B^\top t},$$

where V_∞, the stationary covariance matrix, satisfies

$$BV_\infty + V_\infty B^\top + G(x_{\text{eq}}) = 0.$$

We conclude that, for N large, $X_t^{(N)}$ has an approximate Gaussian distribution with $\text{Cov}(X_t^{(N)}) \simeq V_t/N$.