Two-link Approximation Schemes for Loss Networks with Linear Structure and Trunk Reservation

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Abstract

Loss networks have long been used to model various types of telecommunication network, including circuit-switched networks. Such networks often use admission controls, such as trunk reservation, to optimize revenue or stabilize the behaviour of the network. Unfortunately, an exact analysis of such networks is not usually possible, and reduced-load approximations such as the Erlang Fixed Point (EFP) approximation have been widely used. The performance of these approximations is typically very good for networks without controls, under several regimes. There is evidence, however, that in networks with controls, these approximations will in general perform less well.

We propose an extension to the EFP approximation that gives marked improvement for a simple ring-shaped network with trunk reservation. It is based on the idea of considering pairs of links together, thus making greater allowance for dependencies between neighbouring links than does the EFP approximation, which only considers links in isolation.

Keywords: Approximation, Blocking probabilities, Erlang Fixed Point, Ring network, Trunk reservation

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1 Introduction

Loss networks have long been used as models of circuit-switched networks. They have also been used to model local area networks, multi-processing architectures, data-base management systems, mobile/cellular radio and broadband packet networks (see for example [6, 13, 18, 23, 27, 32]). When routing is fixed and no controls are employed, closed-form expressions for the blocking probabilities and stationary distribution are easily obtained from these models. However, even for moderately sized networks, these expressions are often difficult to compute, since the number of states rapidly increases as network size

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and capacities increase. To overcome these problems, reduced-load approximations such as the Erlang Fixed Point (EFP) approximation and stochastic knapsack approximation have been widely used [21, 33].

When controls such as trunk reservation are added, even the derivation of closed-form expressions is no longer possible, and the balance equations for the stationary distribution must be solved numerically. The performance of the reduced-load approximations is typically very good for networks without controls, where theoretical justification exists for their use under two different types of limiting regime [19, 35, 39]. However, in networks with controls these approximations will in general perform less well [15], although for some classes of network their performance is still excellent when trunk reservation is applied [16, 26]. In this paper we propose an extension to the EFP approximation for a simple network with trunk reservation controls. It is based on the idea of considering pairs of links together, whereas the traditional EFP approximation considers links in isolation. We choose to consider a ring-shaped network because of its highly linear structure–it is precisely under conditions such as these that the traditional EFP approximation might be expected to perform less well. The simplicity of the network also enables us to obtain a clearer understanding of the performance of our proposed two-link extension to the EFP approximation. We find that it gives considerable improvement for the ring-shaped network with trunk reservation under a range of traffic loadings. We believe that this improvement will carry over to more complicated network structures and routing patterns.

Although the ring network has a simple structure, it has nevertheless been used quite widely in practice, most recently for optical networks, where such networks with wavelength changers at each node are exactly equivalent to circuit-switched networks [4]. For optical ring networks without wavelength changers, the circuit-switched network model gives bounds on the blocking probabilities [5]. As Gerstel *et. al.* [11] point out, ring networks are attractive precisely because of their simplicity, and also because they provide some fault tolerance. The routing patterns that we study here are indeed more simplified than those that would be seen in practice, but we regard this as an essential first step before proceeding to further generalizations.

Consider a circuit-switched network without controls, such as the example depicted in Figure 1. Let K be the number of links (circuit groups) or resources in the network. A route in the network is expressed as a subset of $\{1, 2, \ldots, K\}$, where $\mathcal{R}$ is the set of all routes. Calls are offered to route $r \in \mathcal{R}$ as a Poisson stream of rate ν_r, where we assume that $\mathcal{R}$ indexes independent Poisson processes. Each call on route r requires $a_{jr}(\geq 0)$ circuits from link j, the total number of circuits on link j being C_j. Calls requesting route r are blocked and lost if, on any link $j \in r$, there are fewer than a_{jr} available circuits. Otherwise, the call is connected and simultaneously holds a_{jr} circuits on each link $j \in r$ for the duration of the call. For simplicity, we shall take $a_{jr} \in \{0, 1\}$. Call durations are independent and identically distributed exponential random variables with unit mean, and are independent of the arrival processes.

Let $\boldsymbol{n} = (n_r,\ r \in \mathcal{R})$, where n_r is the number of calls in progress using route r, let $\boldsymbol{C} = (C_j,\ j = 1, \ldots, K)$, and let $\boldsymbol{A} = (a_{jr},\ r \in \mathcal{R},\ j = 1, \ldots, K)$. Then the usual model for a circuit-switched network without controls (see for example [21]) is a continuous-time Markov chain $(\boldsymbol{n}(t),\ t \geq 0)$ taking values in

$$S = S(\boldsymbol{C}) = \left\{ \boldsymbol{n} \in \mathrm{Z}_+^{\mathrm{R}} \ :\ \boldsymbol{A}\boldsymbol{n} \leq \boldsymbol{C} \right\} ,$$

with unique equilibrium distribution $\pi = (\pi(\boldsymbol{n}),\ \boldsymbol{n} \in S)$ given by

$$\pi(\boldsymbol{n}) = \Phi^{-1} \prod_{r \in \mathcal{R}} \frac{\nu_r^{n_r}}{n_r!} , \quad \boldsymbol{n} \in S,$$

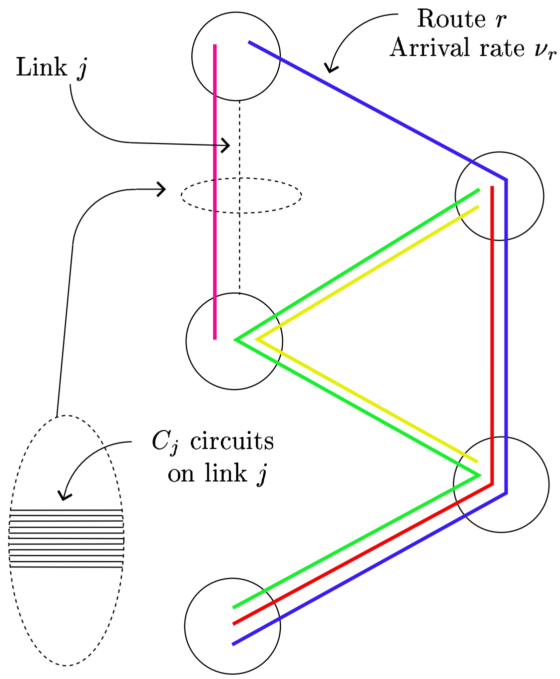

Fig 1. A typical circuit-switched network
(5 nodes, 6 links and 5 routes)

where

$$\Phi = \Phi(\boldsymbol{C}) = \sum_{\boldsymbol{n} \in S(\boldsymbol{C})} \prod_{r \in \mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}\,.$$

The stationary probability that a route r call is blocked is then given by

$$1 - \frac{\Phi(\boldsymbol{C} - \boldsymbol{A}\boldsymbol{e}_r)}{\Phi(\boldsymbol{C})}\,,$$

where $\boldsymbol{e}_r$ is the rth unit vector.

The reduced load approximation for the blocking probabilities is obtained as follows. Let

$$E(\nu, C) = \frac{\nu^C}{C!} \left(\sum_{n=0}^{C} \frac{\nu^n}{n!} \right)^{-1}$$

be Erlang's formula for the loss probability on a single link with C circuits and Poisson traffic offered at rate ν. The Erlang fixed point equations are then given by

$$B_j = E(\rho_j, C_j), \qquad j = 1, 2, \ldots, K,$$

where

$$\rho_j = \sum_r a_{jr} \nu_r \prod_{i \in r - \{j\}} (1 - B_i).$$

Kelly [19] proved that there is a unique vector $(B_1, \ldots, B_K) \in [0,1]^K$ satisfying these equations. The EFP approximation is obtained by using B_j to estimate the probability that link j is full, and

$$L_r = 1 - \prod_{i \in r} (1 - B_i)^{a_{ir}}$$

to estimate the probability that a call on route $r \in \mathcal{R}$ is blocked. Similar equations can be derived for more complex systems (for instance, when routing is not fixed), but they may then have more than one fixed point (see for example [12]). The idea underlying the approximation is that if links blocked independently (they clearly do not) then the arrival process at link j due to route r would be Poisson, thinned by a factor $1 - B_i$ at each of the other links $i \in r$, before being offered to link j. For an excellent overview of loss networks in general, and this approximation in particular, see Kelly [21], and also Ross [33].

The EFP approximation performs particularly well under two limiting regimes. The first is one in which the topology of the network is held fixed, while capacities and arrival rates at the links become large [19]; this has become known as the *Kelly limiting regime*, or (somewhat misleadingly) as the *heavy traffic limit*. Under the second limiting regime, called *diverse routing*, the number of links, and the number of routes which use these links, become large, while the capacities are held fixed and the arrival rates on multi-link routes become small. Examples of this are star networks and fully-connected networks with alternate routing [14, 16, 26, 35, 39].

The above description is for a network without admission or routing controls. In this paper we will be concerned with a form of admission control known as trunk reservation. A *trunk reservation policy* or *threshold policy* is one where a call on route r is accepted on link $j \in r$ provided there will be at least t_{jr} circuits free on that link after that call is accepted. More formally, when the system is in state $\boldsymbol{n}$, a route r call is accepted on link j provided $\sum_s a_{js} n_s \leq C_j - t_{jr} - a_{jr}$. Usually we have $t_{jr} = t_r$, the same for all links on a route. As before, a call is lost if there is any link on its route unable to accept it.

Trunk reservation controls have a number of desirable properties including ease of implementation and, typically, robustness to fluctuations in arrival rates. They were first introduced as a means of admitting only higher-priority traffic when links were relatively full. If route r calls produce a reward w_r when accepted, and the standard assumptions of Poisson arrivals, exponential holding times and independence hold, then such a policy is optimal for a single link [24]. If these assumptions don't hold, or if the system has multiple links, then trunk reservation is no longer optimal, but can still be very close to optimal (see for example [1, 22, 29, 38]). Moreover, it may still be asymptotically optimal for networks with diverse routing [16, 26] or in the Kelly limiting regime [17]. Trunk reservation has also been used as a control in networks with alternative routing to eliminate problems with bistability [12]. More recently, dynamic trunk reservation policies have been proposed as a means of ensuring fair and efficient service to competing streams of traffic [28].

Unfortunately, when trunk reservation is incorporated in the loss network model the equilibrium distribution no longer has product form. Although transition rates are still easily obtained, there is no closed-form solution to the full balance equations, which can then only be solved numerically. For moderate or large networks, solving these equations in real time (which may be desirable for network optimization) will not be feasible. In addition, a desirable feature of network controls is that they only require local knowledge of the system occupancies. In this situation, approximations are especially useful, particularly if they can be applied locally.

The EFP can easily be extended to allow for trunk reservation (see for example [20]). As for standard EFP, the occupancy of a single link is approximated using a birth and death process. Arrival rates are now state-dependent, since arrivals at link j on route r can only be accepted if the occupancy is no more than $C_j - t_{jr} - a_{jr}$. Provided $a_{jr} \in \{0, 1\}$, the stationary distribution for the jth link is approximated by

$$\pi_j(\boldsymbol{\nu}, \boldsymbol{t}, \boldsymbol{C}; n) = \left(\sum_{k=0}^{C_j} \frac{1}{k!} \prod_{i=0}^{k-1} \lambda_{ji} \right)^{-1} \frac{1}{n!} \prod_{i=0}^{n-1} \lambda_{ji}\,,$$

where

$$\lambda_{ji} = \sum_r a_{jr} \nu_r I_{\{i \le C - t_{jr} - a_{jr}\}} \prod_{k \in r - \{j\}} (1 - B_k^{(r)}).$$

The $B_k^{(r)}$ here is an estimate of the probability that a call on route r is blocked on link k, and is given by

$$B_k^{(r)} = \sum_{n=C_k - t_{kr} - a_{kr} + 1}^{C_k} \pi_k(\boldsymbol{\nu}, \boldsymbol{t}, \boldsymbol{C}; n).$$

The probability that a call is blocked on route r is approximated by

$$L_r = 1 - \prod_{j \in r} (1 - B_j^{(r)}).$$

Under the diverse routing limit it seems that the approximation is valid [16, 26]. However, it is known that the extended EFP is not asymptotically correct in the Kelly limiting regime when trunk reservation is in use [15]. Thus, particular care must be taken with its use, and it is correspondingly of greater interest to find refined approximations for networks which do not exhibit diverse routing.

In this paper we examine a simple, highly linear network and develop a refined approximation for the blocking probabilities under trunk reservation. The network we consider consists of a number of links forming a single loop or ring, with single-link and two-link traffic. This paper briefly reviews previous related work, and then develops a new, improved version of the fixed-point equations derived in [2] incorporating trunk reservation, which gives a marked improvement over previous methods.

The remainder of this paper is organized as follows. In section 2 we describe the ring network in greater detail, and discuss approximations for the network without trunk reservation. Section 3 generalizes this to the situation incorporating trunk reservation. In section 4 we give numerical examples that demonstrate that the new approximation does give a marked improvement on the classical extended EFP approximation for networks with trunk reservation. Finally, in section 5 we discuss extensions and directions for further work.

2 The symmetric ring network

Consider a symmetric loss network with K links forming a loop, each link having the same capacity C. Such a network is depicted in Figure 2. There are only two types of route: one-link routes (type 1), and two-link routes (type 2) comprising pairs of adjacent links. Single-link traffic is offered at rate ν_1 on each link, and two-link traffic is offered at rate ν_2 to each two-link route. Associated with each traffic stream is a trunk reservation parameter. Type i routes have trunk reservation parameter t_i, $i = 1, 2$. In this section we assume $t_1 = t_2 = 0$. The EFP approximation then gives

$$L_1 = B \quad \text{and} \quad L_2 = 1 - (1 - B)^2,$$

where B is the unique solution to

$$B = E(\nu_1 + 2\nu_2(1 - B), C).$$

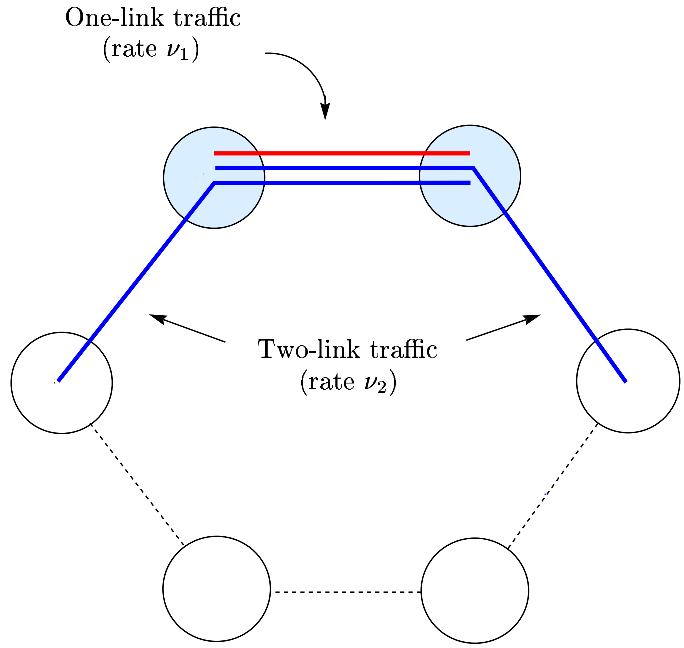

Fig 2. A ring network with 6 nodes showing
one- and two-link traffic using a given link

To illustrate why we might expect the EFP approximation to perform badly in the present context, we shall assess the dependence between two adjacent links. Take links 1 and 2 as reference links and consider the subnetwork depicted in Figure 3. We identify three routes: $\{1\}$, $\{2\}$ and $\{1,2\}$. If m_r denotes the number of calls on route r in the subnetwork, then m_1 is the number of calls occupying capacity on link 1 *but not* on link 2, that is $m_1 = n_1 + n_{K1}$, m_2 is the number occupying capacity on link 2 *but not* on link 1, that is $m_2 = n_2 + n_{23}$, and, $m_{12} (= n_{12})$ is the number of calls occupying capacity on both links. Figure 4 shows the correlation between links 1 and 2 for the network with $C = 12$, $K = 12$ and $\nu_1 = \nu_2 (= \nu)$; to be precise, we have plotted

$$\mathrm{Corr}\left(I_{\{m_1+m_{12}<C\}}, I_{\{m_2+m_{12}<C\}}\right)$$

against the arrival rate ν. We note that the correlation peaks when the total arrival rate $\nu_1 + 2\nu_2$ at a link is approximately equal to the capacity C.

The blocking probabilities can be estimated more accurately by making explicit allowance for the dependencies between adjacent links. The presentation here follows that of Bebbington *et al.* [2], which proposed an improved approximation by considering two-link subnetworks with state-dependent arrival rates. Zachary and Ziedins [37] used a Markov random field approach to obtain improved approximations for general networks without controls. Their approximations give successive multi-link refinements of the EFP approximation for networks without controls. The approximation that they give for the blocking probabilities for the symmetric ring network with one-link and two-link traffic is the same as that given here, although expressed somewhat differently. The expressions are exact for the infinite line network, and they show both convergence and uniqueness for the line and ring networks. Earlier papers by Zachary [36] and Kelly [19] had previously shown that this approximation is exact for an infinite line network carrying only two-link traffic. The idea of using state-dependent arrival rates in two-link subnetworks has also been discussed

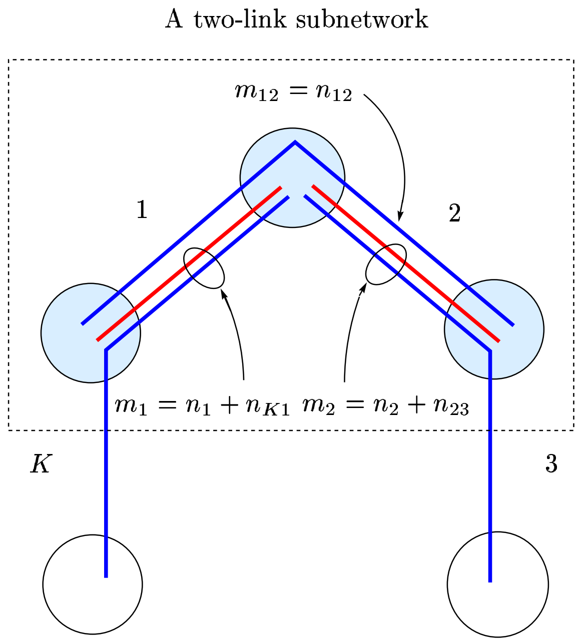

Fig 3. Definition of m_1, m_2 and m_{12}
for the symmetric ring network

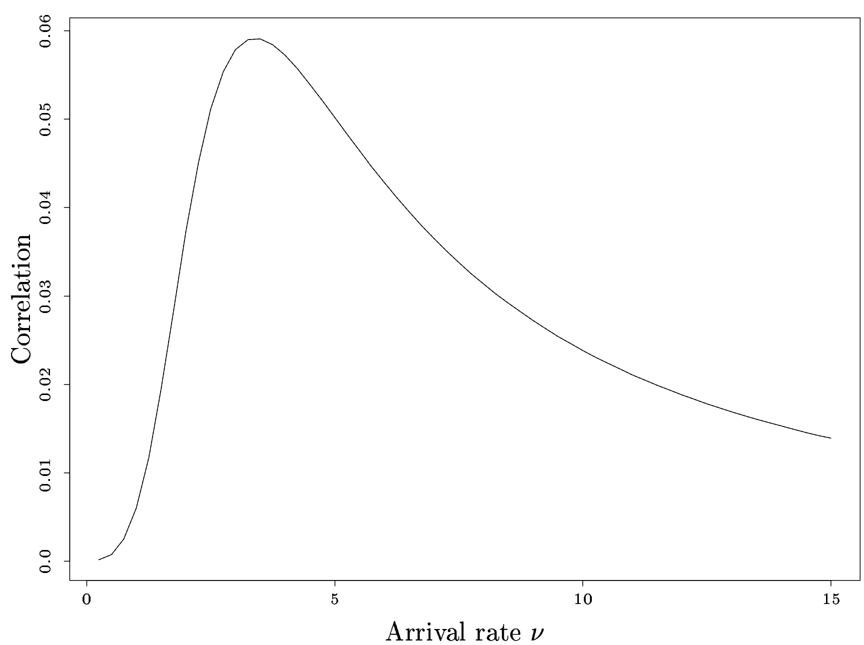

Fig 4. Correlation of spare capacity on adjacent links
($C = 12$, $K = 12$, $\nu_1 = \nu_2 = \nu$)

by Pallant [30] and Pallant and Taylor [31] (see also [7, 8, 9]), although they obtain rather different expressions than the ones given below. For a fuller discussion refer to [3].

The approximation in [2] considers a two-link subnetwork of the ring network as in Figure 2 above and uses knowledge of the state of a given link in estimating the probability that the adjacent link is full. The two-link subnetwork has calls requiring capacity on both links and also calls requiring capacity on just one. Arrivals requesting capacity from both links arrive at rate ν_2. Arrival rates at each of the single links of the subnetwork are state-dependent and given by $\rho_n = \nu_1 + \nu_2(1 - b_n)$, $n \in \{0, 1, \ldots, C-1\}$, where b_n is the probability that link K is fully occupied, conditional on $m_1 = n$ (by symmetry, b_n is also the probability that link 3 is fully occupied, conditional on $m_2 = n$). The stationary distribution for the two-link subnetwork is then given by

$$\pi(m_1, m_2, m_{12}) = \pi(\boldsymbol{m}) = \Phi^{-1} \frac{\nu_2^{m_{12}} \left(\prod_{n=0}^{m_1-1} \rho_n\right) \left(\prod_{n=0}^{m_2-1} \rho_n\right)}{m_1!\, m_2!\, m_{12}!} .$$

An estimate of b_n is found by assuming that b_n does not depend on m_{12}. For $n = 0, \ldots, C-1$, we set

$$b_n = \frac{\sum_{m=0}^{n} p(n-m, C-m, m)}{\sum_{m=0}^{n} \sum_{r=0}^{C-m} p(n-m, r, m)} ,$$

where

$$p(n_1, m_K, n_{K1}) = \frac{\nu_1^{n_1} \nu_2^{n_{K1}} \left(\prod_{s=0}^{m_K-1} \rho_s\right)}{n_1!\, m_K!\, n_{K1}!} .$$

The dependence of b_n on m_{12} in the ring network is due to the cyclic nature of the network, but is expected to be slight for large networks: indeed, Zachary and Ziedins [37] have shown that this dependence decays as the size of the network increases. Once b_n is estimated and π determined, we approximate the probability that a single-link call is blocked by

$$L_1 = \sum_{m_{12}=0}^{C} \sum_{m_2=0}^{C-m_{12}} \pi(C - m_{12}, m_2, m_{12})$$

and the probability that a two-link call is blocked by

$$L_2 = 2L_1 - \sum_{m_{12}=0}^{C} \pi(C - m_{12}, C - m_{12}, m_{12}).$$

Figures 5a and 5b show the relative error in using the EFP approximation and the two-link approximation to estimate the blocking probability of single-link and two-link calls, respectively, in a network with $C = 12$, $K = 12$ and $\nu_1 = \nu_2 (= \nu)$. The improvement over the EFP approximation obtained using this approximation is considerable. Indeed, in this example the maximum error is of order 10^{-8} for both types of traffic. There is an interesting connection between the correlation and relative error of the EFP. Correlation peaks when the total offered load to a link $(\nu_1 + 2\nu_2)$ is equal to the capacity of the link. For single link calls, this is the point at which the EFP for single-link calls is most sensitive, that is, exhibits the greatest variation, as demonstrated in the plot of the relative errors for the two approximations (Figure 5a).

3 Trunk Reservation

In this section we will present an extension of the approach considered in the previous section to allow for trunk reservation.

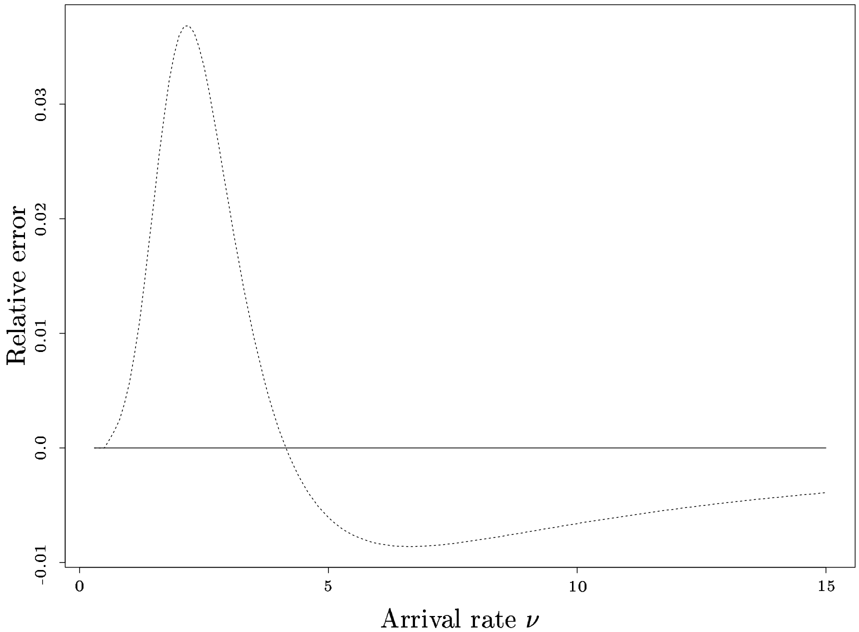

Fig 5a. Relative error in the estimated blocking probability of single-link calls ($C = 12$, $K = 12$, $\nu_1 = \nu_2 = \nu$)
.... EFP ———— Two-link Approx.

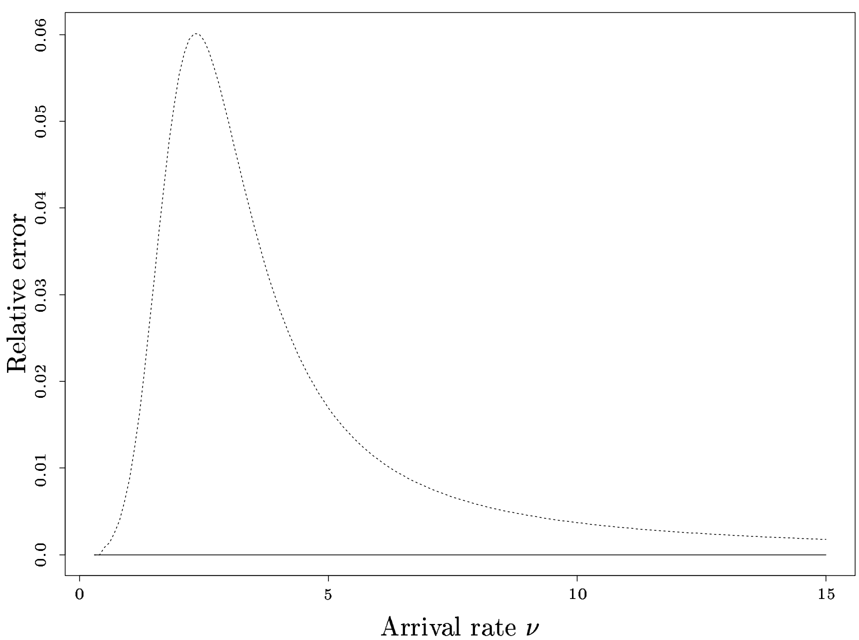

Fig 5b. Relative error in the estimated blocking probability of two-link calls ($C = 12$, $K = 12$, $\nu_1 = \nu_2 = \nu$)
.... EFP ———— Two-link Approx.

We begin by detailing the standard extension of the EFP to the ring network with trunk reservation. As outlined in section 1, the general idea is to approximate the stationary distribution on a single link by a birth and death process with state-dependent arrival rates (see for example [20]). Let $L_1(L_2)$ be the approximate probabilities that a single-link (two-link) call is blocked, and $B^{(1)}$ and $B^{(2)}$ be the approximate probabilities that a link has no circuit available for single-link and two-link calls, respectively. There are two cases to consider, depending on whether single-link or two-link calls have higher priority.

If single-link calls have higher priority, then the trunk reservation parameter against two-link calls is $t_2 > 0$ and the upwards transition rates on a single link are given by

$$\lambda_i = \begin{cases} \nu_1 + 2\nu_2(1 - B^{(2)}) & i < C - t_2 \\ \nu_1 & C - t_2 \leq i < C \\ 0 & i \geq C \end{cases}$$

with downward rates $\mu_i = i$. The stationary distribution for a single link, considered in isolation, is given by

$$\pi(n) = \Phi^{-1} \frac{(\nu_1 + 2\nu_2(1 - B^{(2)}))^{\min(n, C-t_2)} \nu_1^{[n-C+t_2]_+}}{n!},$$

where $[\cdot]_+$ denotes the positive part. We approximate $B^{(1)} = \pi(C)$, $B^{(2)} = \sum_{n=C-t_2}^{C} \pi(n)$ and $L_1 = B^{(1)}$, $L_2 = 1 - (1 - B^{(2)})^2$.

Now suppose that two-link calls have higher priority. Then $t_1 > 0$ and the upwards transition rates are now

$$\lambda_i = \begin{cases} \nu_1 + 2\nu_2(1 - B^{(2)}) & i < C - t_1 \\ 2\nu_2(1 - B^{(2)}) & C - t_1 \leq i < C \\ 0 & i \geq C \end{cases}$$

with downwards transition rates, as previously, $\mu_i = i$. The stationary distribution for a single link in isolation is now given by

$$\pi(n) = \Phi^{-1} \frac{(\nu_1 + 2\nu_2(1 - B^{(2)}))^{\min(n, C-t_1)} (2\nu_2(1 - B^{(2)}))^{[n-C+t_1]_+}}{n!},$$

with $B^{(2)} = \pi(C)$ and $B^{(1)} = \sum_{n=C-t_1}^{C} \pi(n)$. The blocking probabilities are again estimated by $L_1 = B^{(1)}$, $L_2 = 1 - (1 - B^{(2)})^2$. Henceforth EFP will refer, in the presence of trunk reservation, to the extended version outlined above.

Other single-link approximation schemes might also be used. For instance, if capacity requirements and holding times for the two call types are not the same, then a multi-dimensional description of the state of a single link might be used. However, the stationary distribution of the single link no longer has product form, even if no trunk reservation controls are used. Coyle *et al.* [10] suggest two ways of dealing with this difficulty in order to achieve a product-form approximation, citing the source of the approach as Ciardo and Trivedi [7, 8]. We investigated an extension of their approach to incorporate trunk reservation and found that it did not perform as well as the EFP. Thus in this paper, two-link approximation schemes will always be compared with the single-link EFP scheme.

In the previous section we outlined a two-link approximation for the ring network without controls, based on conditional probabilities and state-dependent arrival rates. The two-link approximation that we develop in this section is an extension of that idea. The approximation takes the two-link subnetwork (1,2) (Figure 3) with state space $\{\boldsymbol{m} =$

$(m_1, m_2, m_{12}) : m_1 + m_{12} \le C, m_2 + m_{12} \le C, m_{12} \le C - t_2, m_1 \ge 0, m_2 \ge 0, m_{12} \ge 0\}$ and calculates the equilibrium distribution $\pi(\cdot)$, using the full balance equations,

$$\pi(\boldsymbol{m}) \sum_{\boldsymbol{m}'} q(\boldsymbol{m}, \boldsymbol{m}') = \sum_{\boldsymbol{m}'} \pi(\boldsymbol{m}') q(\boldsymbol{m}', \boldsymbol{m}),$$

where

$$q(\boldsymbol{m}, \boldsymbol{m}') = \begin{cases} \nu_1 I_{\{m_1+m_{12}<C-t_1\}} + \nu_2 a(m_1+m_{12}) I_{\{m_1+m_{12}<C-t_2\}} & \boldsymbol{m}' = (m_1+1, m_2, m_{12}) \\ \nu_1 I_{\{m_2+m_{12}<C-t_1\}} + \nu_2 a(m_2+m_{12}) I_{\{m_2+m_{12}<C-t_2\}} & \boldsymbol{m}' = (m_1, m_2+1, m_{12}) \\ \nu_2 I_{\{m_1+m_{12}<C-t_2\}} I_{\{m_2+m_{12}<C-t_2\}} & \boldsymbol{m}' = (m_1, m_2, m_{12}+1) \\ m_1 I_{\{m_1>0\}} & \boldsymbol{m}' = (m_1-1, m_2, m_{12}) \\ m_2 I_{\{m_2>0\}} & \boldsymbol{m}' = (m_1, m_2-1, m_{12}) \\ m_{12} I_{\{m_{12}>0\}} & \boldsymbol{m}' = (m_1, m_2, m_{12}-1) \\ 0 & \text{otherwise,} \end{cases}$$

are the transition rates from state $\boldsymbol{m}$ to $\boldsymbol{m}'$. Thus $a(m)$ can be thought of as the conditional probability of accepting a two-link call on the link, K, adjacent to link 1, given $m = m_1 + m_{12}$. An analogous expression applies for links 2 and 3 due to the symmetry of the network. In terms of the blocking probabilities, $a(m) = 1 - b(m)$, where

$$b(m) = \frac{\sum_{m_{K1}=0}^{\min(m, C-t_2)} \sum_{m_K=C-t_2-m_{K1}}^{C-m_{K1}} \pi(m_K, m - m_{K1}, m_{K1})}{\sum_{m_{K1}=0}^{\min(m, C-t_2)} \sum_{m_K=0}^{C-m_{K1}} \pi(m_K, m - m_{K1}, m_{K1})}.$$

The $b(m), 0 \le m < C - t_2$, can be interpreted as conditional blocking probabilities, analogous to those obtained for the network without controls. Given that a link has total occupied capacity m, $b(m)$ is the conditional probability that the adjacent link will not accommodate a two-link call.

We have investigated various estimates of the conditional probabilities and found this to be the most satisfactory. Other estimates that we investigated included ones that were considerably more complicated, with the arrival rates depending on (m_1, m_{12}), not just $m_1 + m_{12}$ as they do here.

4 Numerical examples

In this section we present a selection of simulation results illustrating the performance of both the EFP and the two-link approximation that we propose above. All the plots below give the relative errors of the blocking probabilities for the two types of call. Since it is not feasible to calculate the exact stationary distribution for the ring network when trunk reservation is in use, we compared both the EFP approximation and the two-link approximation with the simulated [34] proportion of calls blocked in a system with $K = 24$, after a presample period sufficient to discount transient effects. The simulated blocking probabilities were found to be relatively insensitive to variations in K above 6 or so. The 95% confidence intervals obtained for the blocking probabilities were used to give 95% confidence intervals for the relative errors. Thus if the blocking probability had confidence interval (l_1, l_2) and the approximation is denoted by a, the confidence interval for the relative error was given by $((a - l_2)/l_2, (a - l_1)/l_1)$.

Let us begin with two examples with varying traffic load, where $C = 12$ and $\nu_2 = 2\nu_1$. Figures 6a and 6b show the relative errors for the two types of call under both approximations when $t_1 = 0, t_2 = 3$ (one-link traffic is protected) while Figures 7a and 7b illustrate the case when $t_1 = 3, t_2 = 0$ (two-link traffic is protected). We note that the two-link approximation performs better, in general, than the EFP. The improvement is particularly apparent when two-link traffic is protected. In that case, the relative error of the EFP for two-link calls is quite high (10%) even for relatively low values of the blocking probability (around 3%). When one-link traffic is protected, the result for high traffic load is simply a system containing predominantly one-link calls, and hence there is little inaccuracy in the EFP approximation.

Although it appears that confining our attention to the case where two-link traffic is protected will be more rewarding, we shall briefly further examine the effect of changing the trunk reservation parameter. Again we have $C = 12$, and this time $\nu_1 = 3, \nu_2 = 4.5$, so that the arrival rate is in the critical region. Figures 8a and 8b show the relative errors in the two approximations for one-link and two-link calls respectively. The horizontal axis here gives the trunk reservation against single-link calls, that is, t_1. A negative t_1 should be interpreted here as trunk reservation against *two-link* calls. Thus, for instance, $t_1 = -3$ corresponds to $t_2 = 3, t_1 = 0$. We see that the relative error is most pronounced when there is trunk reservation against single-link calls, with the two-dimensional approximation providing a marked improvement over the EFP.

Our penultimate example examines more closely the behaviour of the approximation at the critical point where $\nu_1 + 2\nu_2 = C$. We take $C = 12$ and $t_1 = 3, t_2 = 0$ as above and allow ν_1 to vary while holding the total load fixed. Figures 9a and 9b show the relative errors. The relative error of the EFP is again highest for the two-link calls, and interestingly we note it is not monotone in the arrival rate for two-link calls.

Our final example investigates the effect of increasing capacities and loads in proportion. We let $\nu_1 = C/6, \nu_2 = 1.75\nu_1$, while varying C. Again we protect two-link traffic, with $t_1 = 3, t_2 = 0$. The relative errors are shown in Figures 10a and 10b. We see that, while the accuracy of the EFP approximation for two-link traffic increases slowly with increasing C, its performance is again markedly worse than the two-link approximation. For one-link calls, while the relative error in the EFP approximation is better than for two-link traffic, the two-link approximation is still superior. Moreover, the EFP approximation exhibits systematic error with increasing C.

As our examples show, the two-link approximation that we have proposed performs better than the EFP approximation in general and, on occasion, markedly better. The convergence of the method is typically fast–for the examples we studied the fixed point had converged to the fourth decimal place within 5 or 6 iterations.

We also performed some experiments to assess the computational complexity of our algorithm. These indicate that the algorithm is NP-hard (in the number of circuits), with the computation time increasing by approximately 50% with each additional circuit, yet insensitive to the loading and the particular trunk reservation regime used. To our knowledge there are no analytical results on the computational complexity for even the standard EFP approximation, although direct methods for evaluating blocking probabilities are known to be #P-complete [25] (and thus at least a hard as NP-complete problems).

5 Conclusions and Future Work

In this paper we have proposed a two-link approximation scheme for the ring network that performs better than the EFP approximation in general and, on occasion, markedly better.

The model studied throughout this paper of a symmetric ring network with only one- and two-link routes was deliberately chosen for ease of exposition and calculation, and to

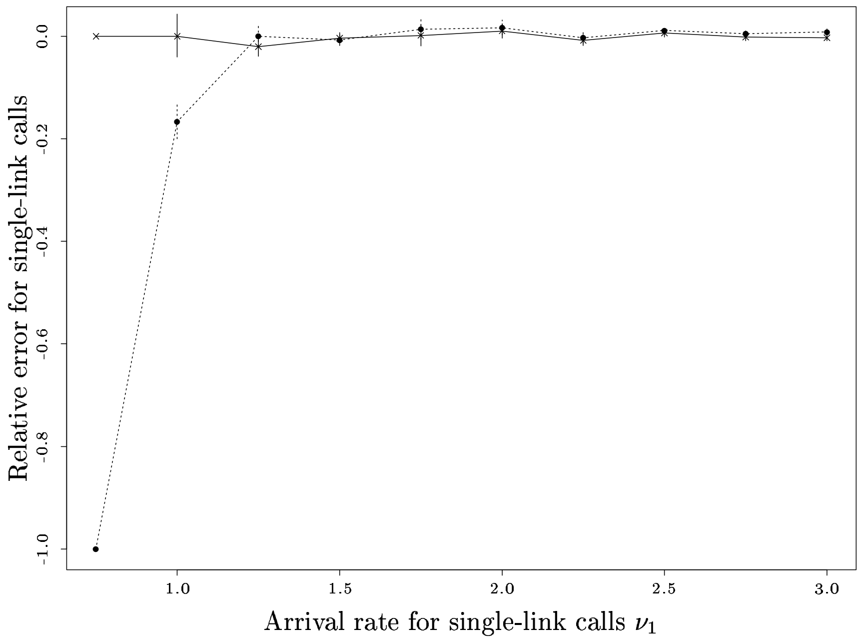

Fig 6a. Relative error in the estimated blocking probability of single-link calls ($C = 12$, $K = 24$, $\nu_2 = 2\nu_1$, $t_1 = 0$, $t_2 = 3$)
············ EFP ———— Two-link Approx.

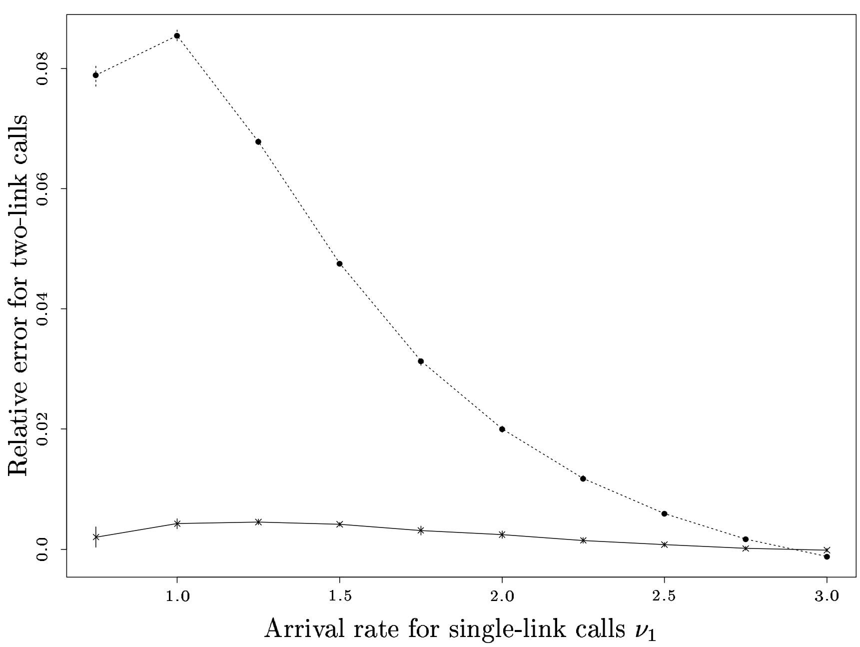

Fig 6b. Relative error in the estimated blocking probability of two-link calls ($C = 12$, $K = 24$, $\nu_2 = 2\nu_1$, $t_1 = 0$, $t_2 = 3$)
············ EFP ———— Two-link Approx.

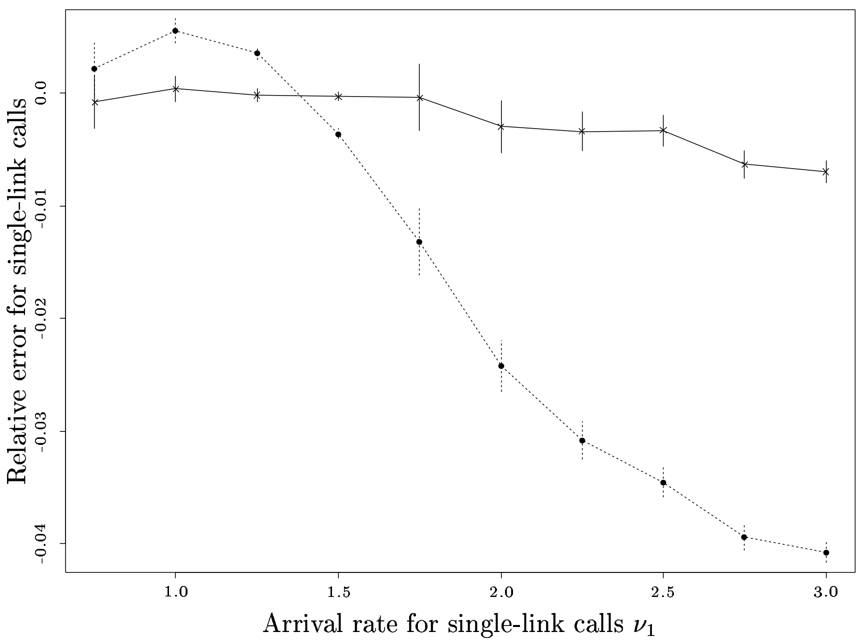

Fig 7a. Relative error in the estimated blocking probability of single-link calls ($C = 12$, $K = 24$, $\nu_2 = 2\nu_1$, $t_1 = 3$, $t_2 = 0$)
············ EFP ———— Two-link Approx.

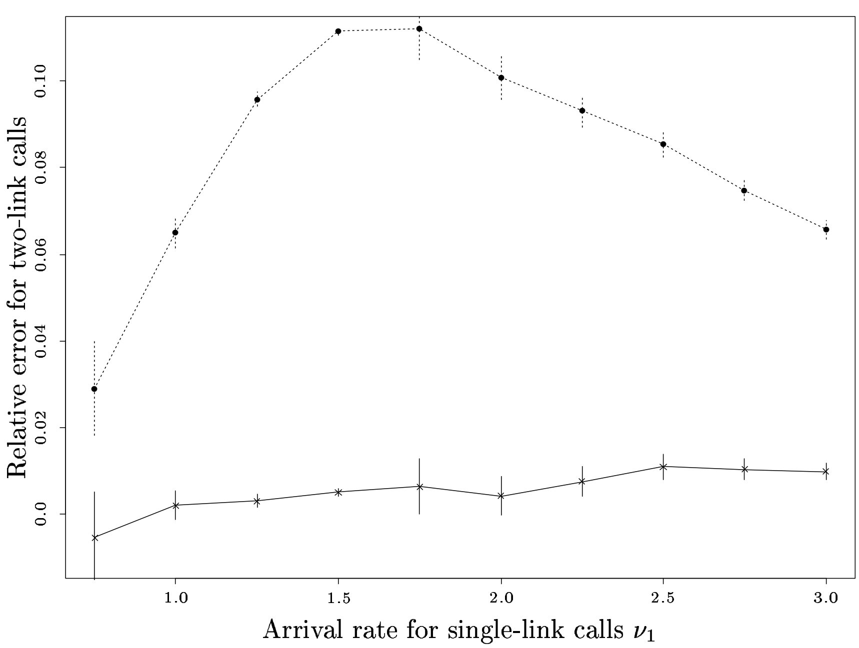

Fig 7b. Relative error in the estimated blocking probability of two-link calls ($C = 12$, $K = 24$, $\nu_2 = 2\nu_1$, $t_1 = 3$, $t_2 = 0$)
············ EFP ———— Two-link Approx.

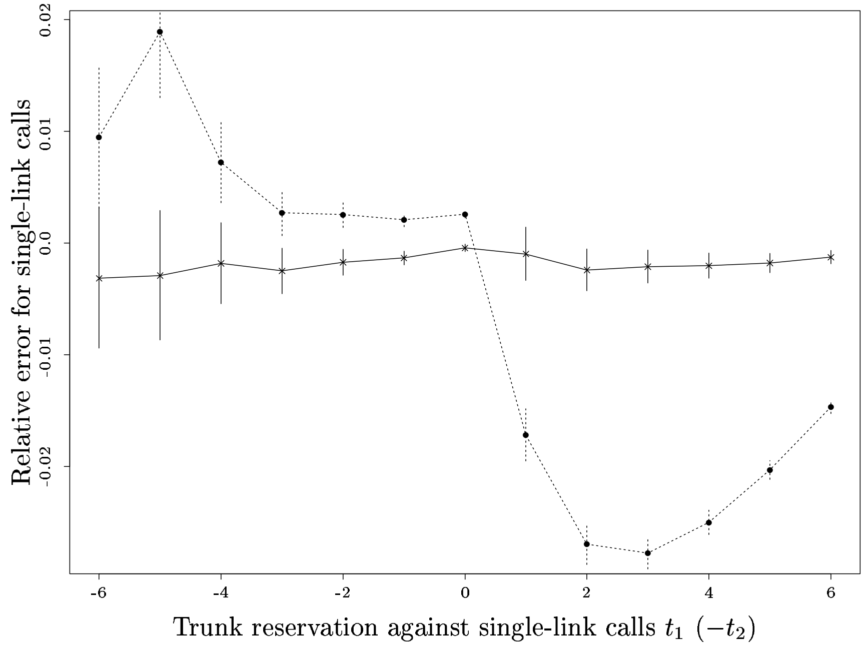

Fig 8a. Relative error in the estimated blocking probability of single-link calls ($C = 12$, $K = 24$, $\nu_1 = 3.0$, $\nu_2 = 4.5$)
············ EFP ———— Two-link Approx.

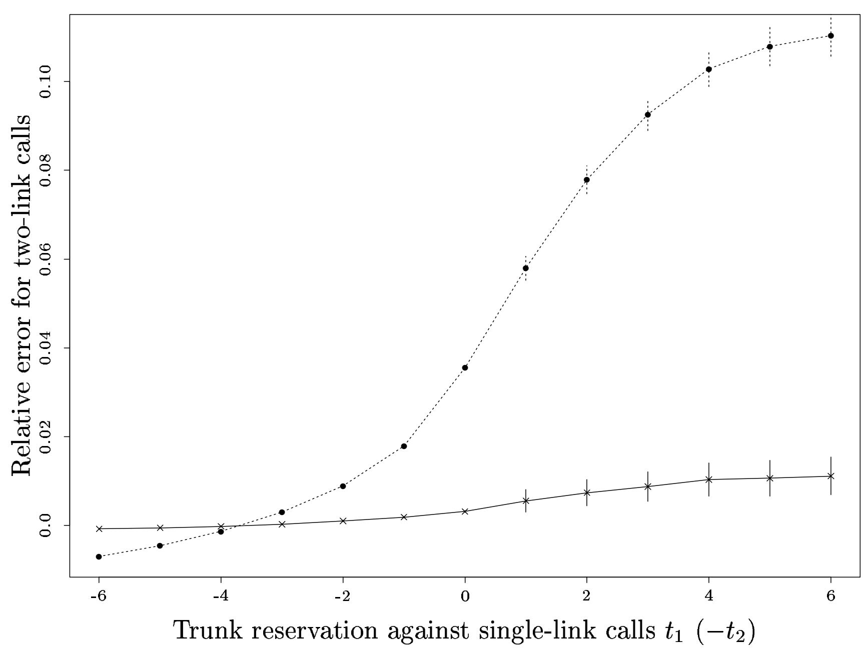

Fig 8b. Relative error in the estimated blocking probability of two-link calls ($C = 12$, $K = 24$, $\nu_1 = 3.0$, $\nu_2 = 4.5$)
············ EFP ———— Two-link Approx.

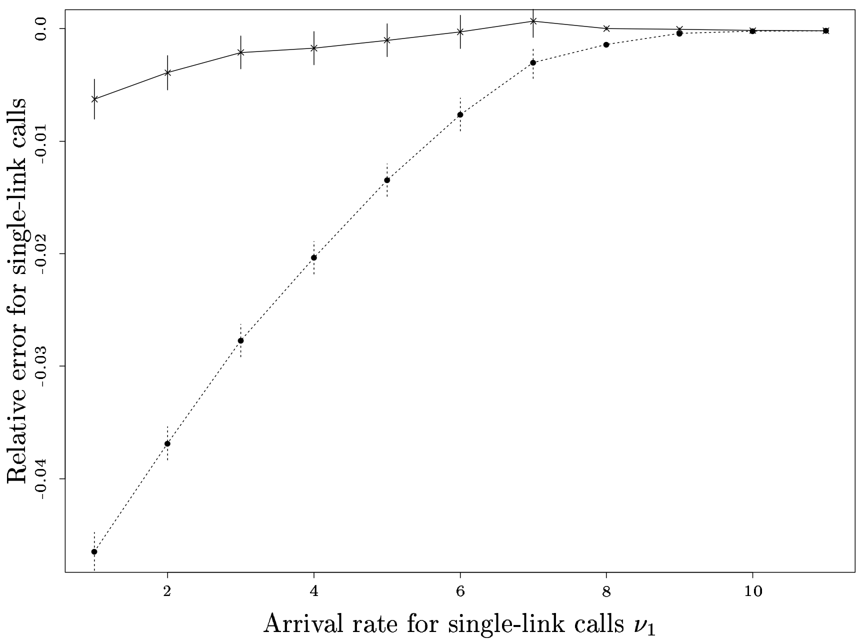

Fig 9a. Relative error in the estimated blocking probability of single-link calls ($C = 12$, $K = 24$, $\nu_1 + 2\nu_2 = C$, $t_1 = 3$, $t_2 = 0$)
············ EFP ———— Two-link Approx.

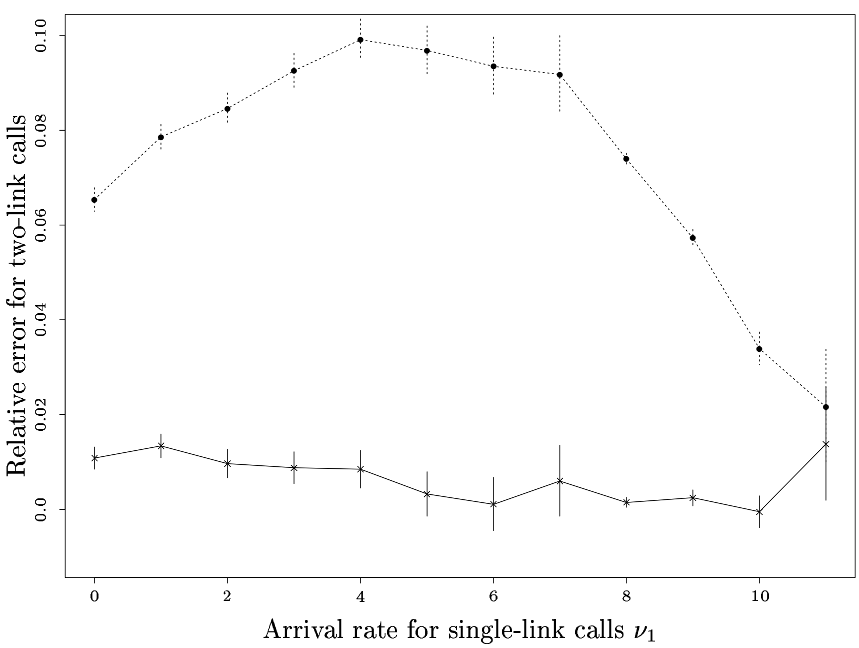

Fig 9b. Relative error in the estimated blocking probability of two-link calls ($C = 12$, $K = 24$, $\nu_1 + 2\nu_2 = C$, $t_1 = 3$, $t_2 = 0$)
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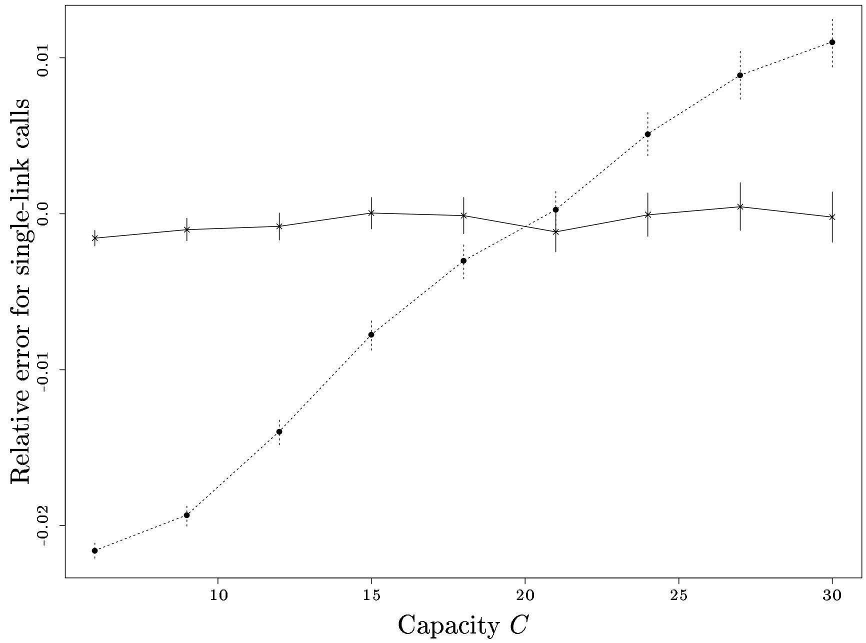

Fig 10a. Relative error in the estimated blocking probability of single-link calls ($K = 24$, $\nu_1 = C/6$, $\nu_2 = 7C/24$, $t_1 = 3$, $t_2 = 0$) ············ EFP ———— Two-link Approx.

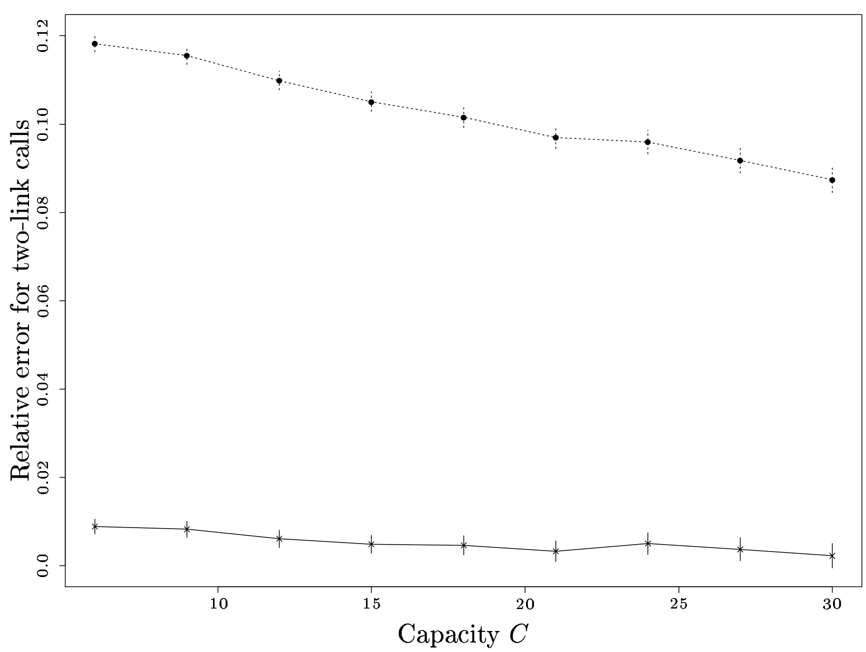

Fig 10b. Relative error in the estimated blocking probability of two-link calls ($K = 24$, $\nu_1 = C/6$, $\nu_2 = 7C/24$, $t_1 = 3$, $t_2 = 0$) ············ EFP ———— Two-link Approx.

provide a case where the EFP was expected to perform inaccurately. There would seem to be several aspects worth considering in a generalization of the method: routes longer than two links, asymmetry in the network, routes requiring multiple circuits from individual links (multi-rate traffic), and finally, intersections of more than two links (that is, networks other than ring networks). We will briefly outline what we see as the difficulties and likely outcome of extending our algorithm in these cases.

Routes longer than two links in the ring network can be dealt with directly by adjusting the expression given for the transition rates $q(\boldsymbol{m}, \boldsymbol{m}')$ in section 3 for the two-link approximation. Acceptance probabilities for such routes are then calculated by multiplying together acceptance probabilities on subnetworks along the route, possibly with the added refinement of using conditional acceptance probabilities for links and subnetworks.

The formulation of the approximation becomes more interesting when the ring network is not symmetric, as then a collection of fixed point equations will need to be solved, giving separate acceptance probabilities for each connected two-link subnetwork. This will lead to a modular, or decomposition, type fixed point solution (see for example [30, 9]), where blocking probabilities for individual subnetworks are calculated using other subnetworks' blocking probabilities (in the exposition above, since all subnetworks were identical, this was considerably simplified). Some form of (weighted) average of the blocking probabilities may then need to be constructed.

We have briefly investigated the possibility of multiple-circuit traffic using a modification of the approximation. The remainder of the formulation (a symmetric ring network with only one- and two-link calls) was unchanged. A fundamental question is how multi-circuit calls are cleared down, in particular how they are represented in the state variable. Since it is not feasible to have a 6-dimensional representation of the network occupancy, multi-circuit calls must have their circuits clear down independently. We tried two ways of weighting these, first assigning weights to find a new rate of cleardown per occupied circuit, and secondly adding a two-circuit cleardown to the possible transitions. Weights were assigned iteratively on the basis of the calculated loss probabilities for the various types of call. The second option was by far the better, and actually performed somewhat better than the extended EFP on the limited range of examples tried. This is in contrast to the decomposition approach of Coyle *et al.*[9].

Let us now consider multiple-link intersections. If we consider only those portions of calls on the links in the intersection, a three-link intersection is equivalent to a three-link ring network. A three-link approximation was examined for the ring network and found to be (very) memory intensive, and hence slow, with no increase in accuracy (although for a different formulation this may not be the case). We note also that Zachary and Ziedins [37] consider a fully-connected four node network without trunk reservation, using the Markov random field approximation, and find that there is no need to consider anything more than a two-link approximation. Hence we would expect the two-link approximation to serve adequately for multiple-link intersections. However, for star-shaped networks the single-link EFP approximation is adequate, and so for networks that have topologies that are primarily star-like the benefits of employing these more complex methods may be slight when compared with the cost of doing so. In practice, the two-link approximation might be used for the linear sections of the network, with the single-link EFP approximation used for those sections of the network that are more star-like in structure and have diverse routing.

Future directions for research are to apply the method to more complicated networks, with the ultimate aim of obtaining a theoretical basis for determining more refined approximations for general networks with trunk reservation and, possibly, other controls.

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