Existence and uniqueness of Q-processes with a given finite μ-invariant measure

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Abstract

Let Q be a stable and conservative Q-matrix over a countable state space S consisting of an irreducible class C and a single absorbing state 0 that is accessible from C. Suppose that Q admits a finite μ-subinvariant measure m on C. We derive necessary and sufficient conditions for there to exist a Q-process for which m is μ-invariant on C, as well as a necessary condition for the uniqueness of such a process.

Keywords: Q-matrix; quasi-stationary distributions; construction theory

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1 Introduction

We begin with a totally stable Q-matrix over a countable set S, that is, a collection $Q = (q_{ij},\ i, j \in S)$ of real numbers that satisfies $0 \le q_{ij} < \infty$, $j \ne i$, $q_i := -q_{ii} < \infty$ and

$$\sum_{i \ne j} q_{ij} \le q_i, \qquad i \in S. \tag{1}$$

The Q-matrix is said to be *conservative* if equality holds in (1) for all $i \in S$. For simplicity, we shall assume that Q is conservative. A set of real-valued functions $P(\cdot) = (p_{ij}(\cdot),\ i, j \in S)$ defined on $[0, \infty)$ is called a *standard transition function* or *process* if

$$p_{ij}(t) \ge 0, \qquad i, j \in S,\ t > 0, \tag{2}$$

$$\sum_{j \in S} p_{ij}(t) \le 1, \qquad i \in S,\ t > 0, \tag{3}$$

$$p_{ij}(s + t) = \sum_{k \in S} p_{ik}(s) p_{kj}(t), \qquad i, j \in S,\ s, t > 0,$$

$$\lim_{t \downarrow 0} p_{ij}(t) = \delta_{ij}, \qquad i, j \in S. \tag{4}$$

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P is then *honest* if equality holds in (3) for some (and then all) $t > 0$, and it is called a *Q-transition function* (or *Q-process*) if $p'_{ij}(0+) = q_{ij}$ for each $i, j \in S$.

Under the conditions we have imposed, every Q-process P satisfies the *backward differential equations*,

$$p'_{ij}(t) = \sum_{k \in S} q_{ik} p_{kj}(t), \qquad t > 0, \tag{BE$_{ij}$}$$

for all $i, j \in S$, but might not satisfy the *forward differential equations*,

$$p'_{ij}(t) = \sum_{k \in S} p_{ik}(t) q_{kj}, \qquad t > 0, \tag{FE$_{ij}$}$$

for all $i, j \in S$. The classical construction problem is to find one and then all Q-processes. Feller's recursion [2] provides for the existence of a *minimal* solution $F(\cdot) = (f_{ij}(\cdot),\ i, j \in S)$ to the backward equations (which also satisfies the forward equations); see also Feller [3] and Reuter [14]. This process is the *unique* Q-process if and only if the system of equations

$$\sum_{j \in S} q_{ij} x_j = \nu x_i, \qquad i \in S, \tag{5}$$

has no bounded, non-trivial solution (equivalently, non-negative solution) $x = (x_j,\ j \in S)$ for some (and then all) $\nu > 0$ (Reuter [14]); for the non-conservative case, see Hou [4] and Reuter [18]. When this condition fails, there are infinitely many Q-processes, including infinitely many honest ones (Reuter [14]), and the dimension η of the space of bounded sequences x on S satisfying (5) (a quantity that does not depend on ν) determines the number of "escape routes to infinity" available to the process. A construction of all Q-processes was given by Reuter [15, 16] under the assumption that $\eta = 1$ (the *single-exit case*), and this was later extended to the *finite-exit case* ($\eta < \infty$) by Williams [22].

In the case when (5) has infinitely many bounded non-trivial solutions, the problem of constructing all Q-processes remains unsolved; it seems that there are simply too many solutions of the backward equations to characterize. For this reason, variants of the classical construction have been considered in which various side conditions are imposed. The most recent work centres on an assumption that one is given an *invariant measure* for the Q-matrix, that is, a collection of positive numbers $m = (m_i,\ i \in S)$ that satisfy

$$\sum_{i \in S} m_i q_{ij} = 0, \qquad j \in S. \tag{6}$$

The problem is then to identify Q-processes with m as their invariant measure, that is, $\sum_{i \in S} m_i p_{ij}(t) = m_j$, $j \in S$, $t > 0$. When does there exist such a Q-process, and, when is it the unique Q-process with the given invariant measure? This variant of the classical construction problem has particular significance when m is finite ($\sum_{i \in S} m_i < \infty$), for then one is looking for a Q-process whose *stationary distribution* has been specified. The problem of existence and uniqueness in the single-exit case was solved by Hou and Chen [5] under the assumption that Q is *m-symmetrizable*, that is, $m_i q_{ij} = m_j q_{ji}$, $i, j \in S$. Their results were extended to the non-conservative case by Chen and Zhang [1] and to the general case, where only (6) is assumed, by Pollett [10, 12]. Recently, Zhang, Lin and Hou solved the existence problem in, respectively, the *totally stable case* ($q_i < \infty$ for all $i \in S$) [24], and the *single instantaneous case* ($q_i = \infty$ for a particular $i \in S$ and $q_j < \infty$ for all $j \neq i$) [25]. Since these papers are written in Chinese, and may not be readily accessible, we summarise their results here:

Theorem 1 *Let S be a countable set and let $Q = (q_{ij},\ i, j \in S)$ be a matrix over S that satisfies $q_{ij} \geq 0,\ j \neq i$, and $\sum_{k \neq i} q_{ik} = -q_{ii} \leq +\infty$. Let $m = (m_j,\ j \in S)$ be a strictly positive probability measure that satisfies $\sum_{i \neq j} m_i q_{ij} = -m_j q_{jj}$. If either Q is totally stable or Q is single instantaneous, then there exists a Q-process P for which m is an invariant measure (and hence a stationary distribution) for P.*

This result partly answers an open problem of Williams [23].

In the present paper we look at a slightly different kind of construction problem, where the state space can be decomposed into an irreducible class C and a single absorbing state, and we suppose, rather than an *invariant* measure, a *μ-invariant* measure on C is specified through Q. We seek to determine Q-processes for which m is a μ-invariant measure on C. Since we will not require these processes satisfy the forward equations, we shall relax the μ-invariance for Q to *μ-subinvariance* for Q.

2 Interlude

Many years before the words to "Twinkle Twinkle, Little Star" were written[1], children across France sang the words to "Ah! vous dirai-je, maman" to a tune similar to the one used by the seventeen year old Wolfgang Amadeus Mozart in his piano variations K265/300e. Much later, Daryl Daley produced a third set of lyrics: on the day of the traditional musical concert at the 1998 Oberwolfach Applied Probability Meeting, Daryl penned the following in summary of the day's talks.

Twinkle, twinkle, little dot,
Poisson, Gauss and all that rot,
Placed at random without thought,
Spectral analysed for nought.
Musing while I waited long,
Now I'll tell it all in song.

Twinkle, twinkle, sample path,
Simulated on a graph,
Optimised at Markov time.
When there's just a single line,
Only with a heavy tail, I'll
Prove it with a martingale.

Twinkle, twinkle, abstract queue,
Studied by a chosen few,
Feedback network, bufferless,
Inputs, outputs, what a mess!
Is it stable while I'm there?
If I prove it, who will care?

Asymptotics of the tail
Hardly ever seem to fail.
Minus cust'mers are okay,
They'll be served without delay.
Philip Pollett's Markov chain,
Subinvariant shall remain.

[1] Ann and Jane Taylor, *Rhymes for the Nursery*, 1806.

Several of the contributors to this volume attended the 1998 Oberwolfach Meeting and remember fondly Daryl's rendition of his poem and his expert playing of the Mozart Variations. One of us (PKP) has enjoyed many musical encounters with Daryl, and has benefited for many years from his friendship, his generosity of spirit and his guidance. It is therefore with some considerable pleasure that I dedicate this note to him.

Subinvariant shall remain? Not always. As we shall see, a measure m can be strictly μ-subinvariant for Q, yet it may be possible to identify a process P for which m is μ-invariant.

3 Preliminaries

Suppose that $S = \{0\} \cup C$, where C is an irreducible class (for the minimal Q-process, and hence for any Q-process) and 0 is an absorbing state which is accessible from C, that is, $q_0 = 0$ and $q_{i0} > 0$ for at least one $i \in C$. Then, if μ is some fixed non-negative real number, a collection of strictly positive numbers $m = (m_j,\ j \in C)$ is called a *μ-subinvariant measure (on C) for Q* if

$$\sum_{i \in C} m_i q_{ij} \le -\mu m_j, \qquad j \in C, \tag{7}$$

and *μ-invariant* if equality holds for all $j \in C$. We shall suppose that Q admits a μ-subinvariant measure on C, and then identify Q-processes P such that m is a *μ-invariant (on C) for P*, that is,

$$\sum_{i \in C} m_i p_{ij} = e^{-\mu t} m_j, \qquad j \in C,\ t > 0. \tag{8}$$

The relationship between (7) and (8) has been divined completely for the minimal Q-process F. It was shown by Tweedie [19] that if m is a μ-invariant measure for F, then it is μ-invariant for Q. Conversely (Pollett [8, 9]), if m is a μ-invariant measure for Q, then it is μ-invariant for F if and only if the equations

$$\sum_{i \in C} y_i q_{ij} = -\nu y_j, \qquad 0 \le y_j \le m_j,\ j \in C, \tag{9}$$

have no non-trivial solution for some (and then all) $\nu < \mu$. If $\mu > 0$ and the measure m is assumed to be *finite*, that is $\sum_{i \in C} m_i < \infty$, then much simpler conditions obtain (Pollett and Vere-Jones [13], Nair and Pollett [7]). For example, if F is *honest* (and hence the unique Q-process), then a finite μ-subinvariant measure m for Q is μ-invariant for F if and only if $\sum_{i \in C} m_i q_{i0} = \mu \sum_{i \in C} m_i$. We will see that this condition guarantees, more generally, the *existence* and *uniqueness* a Q-processes P for which the given m is a μ-invariant measure. We note that, in determining such a P, we are effectively identifying a process with a given quasi-stationary distribution (van Doorn [20]): a probability distribution $\pi = (\pi_j,\ j \in C)$ over C is called a *quasi-stationary distribution* if $p_j(t)/\sum_{i \in C} p_i(t) = \pi_j$ for all $t > 0$, where $p_j(t) = \sum_{i \in C} \pi_i p_{ij}(t)$, $t > 0$, so that, conditional on non-absorption, the state probabilities of the underlying continuous-time Markov chain are stationary. It was shown by Nair and Pollett [7] that a distribution $\pi = (\pi_j,\ j \in C)$ is a quasi-stationary distribution if and only if, for some $\mu > 0$, π is a μ-invariant measure for P, in which case if P is honest, then $a_i^P = 1$, for all $i \in C$, where $a_i^P = \lim_{t\to\infty} p_{i0}(t)$ (absorption occurs with probability 1).

4 Existence

It will be convenient to specify transitions functions through their Laplace transforms. If P is a given transition function, then the function $\Psi(\cdot) = (\psi_{ij}(\cdot),\ i,j \in S)$ given by

$$\psi_{ij}(\alpha) = \int_0^\infty e^{-\alpha t} p_{ij}(t)\, dt, \qquad i,j \in S,\ \alpha > 0, \tag{10}$$

is called the *resolvent* of P. If $i,j \in C$, the integral in (10) converges for all $\alpha > -\lambda_P(C)$, where $\lambda_P(C)$ is the *decay parameter* of C (for P); see Kingman [6]. In particular, since C is irreducible, the integral (10) has the same abscissa of convergence for each $i,j \in C$. Notice also that, since 0 is an absorbing state, $\psi_{0j}(\alpha) = \delta_{0j}/\alpha$. Analogous to properties (2)–(4) of P, the resolvent satisfies

$$\psi_{ij}(\alpha) \ge 0, \qquad i,j \in S,\ \alpha > 0, \tag{11}$$

$$\sum_{j\in S} \alpha\psi_{ij}(\alpha) \le 1, \qquad i \in S,\ \alpha > 0, \tag{12}$$

$$\psi_{ij}(\alpha) - \psi_{ij}(\beta) + (\alpha - \beta) \sum_{k\in S} \psi_{ik}(\alpha)\psi_{kj}(\beta) = 0, \qquad i,j \in S,\ \alpha,\beta > 0, \tag{13}$$

$$\lim_{\alpha\to\infty} \alpha\psi_{ij}(\alpha) = \delta_{ij}, \qquad i,j \in S. \tag{14}$$

(Note that (13) is called the *resolvent equation*.) Indeed, any Ψ that satisfies (11)–(14) is the resolvent of a standard transition function P (see Reuter [15, 16]). Furthermore, (12) is satisfied with equality if and only if P is honest, in which case the *resolvent* is said to be honest. Also, the Q-matrix of P can be recovered from Ψ using the following identity:

$$q_{ij} = \lim_{\alpha\to\infty} \alpha(\alpha\psi_{ij}(\alpha) - \delta_{ij}). \tag{15}$$

A resolvent that satisfies (15) is called a Q-resolvent. The resolvent $\Phi(\cdot) = (\phi_{ij}(\cdot),\ i,j \in S)$ of the minimal Q-process has itself a minimal interpretation (see (Reuter [14, 15]); it is the minimal solution to the equations $\alpha\psi_{ij}(\alpha) = \delta_{ij} + \sum_{k\in S} q_{ik}\psi_{kj}(\alpha)$, $i,j \in S$, $\alpha > 0$, which are analogous to (BE_{ij}), and Φ is called the minimal Q-resolvent.

We can identify μ-invariant measures using resolvents. If P is a Q-process with resolvent Ψ and $m = (m_j,\ j \in C)$ is a μ-invariant measure for P, where of necessity

$\mu \leq \lambda_P(C)$ (see Lemma 4.1 of Vere-Jones [21]), then, since the integral in (10) converges for all $\alpha > -\lambda_P(C)$, we have, for all $j \in C$ and $\alpha > 0$, that

$$\sum_{i\in S} m_i \alpha \psi_{ij}(\alpha - \mu) = m_j. \tag{16}$$

We refer to m as being μ-invariant for Ψ if (16) is satisfied. Finally, a simple extension of Lemma 4.1 of Pollett [11] establishes that m is μ-invariant for Ψ if it is μ-invariant for P, and, if $\mu \leq \lambda_P(C)$, then m is μ-invariant for P if it is μ-invariant for Ψ.

We are now ready to state our main result.

Theorem 2 *Let $\mu > 0$ and suppose that Q admits a finite μ-subinvariant measure m on C.*

1. *If the minimal Q-process F is honest, then m is a μ-invariant measure on C for F if and only if*

$$\sum_{i\in C} m_i q_{i0} = \mu \sum_{i\in C} m_i, \tag{17}$$

in which case m is μ-invariant for Q.

2. *If F is dishonest, then there exists a Q-process P for which m is μ-invariant on C if and only if*

$$\sum_{i\in C} m_i q_{i0} \leq \mu \sum_{i\in C} m_i. \tag{18}$$

Proof. Part 1 follows from Theorem 4.1 of Nair and Pollett [7]. The necessity of Part 2 is an immediate consequence of Theorem 3.2 of Nair and Pollett [7]. To complete the proof we shall show that if (18) holds, then there exists a Q-process P for which m is μ-invariant on C.

If m is μ-invariant on C for F, there is nothing to prove; indeed, m is μ-invariant on C for F if and only if $\sum_{i\in C} m_i q_{i0} = \mu \sum_{i\in C} m_i a_i^F$, where recall that $a_i^F = \lim_{t\to\infty} f_{i0}(t)$ (Theorem 4.1 of [7]), this being consistent with (18). Suppose, then, that m is not μ-invariant on C for F. We will specify a (non-minimal) Q-resolvent $\Psi(\cdot) = (\psi_{ij}(\cdot),\ i,j \in S)$ for which m is μ-invariant measure on C. Let

$$\psi_{ij}(\alpha) = \phi_{ij}(\alpha) + \frac{z_i(\alpha) d_j(\alpha)}{(\alpha+\mu)\sum_{k\in C} m_k z_k(\alpha)}, \qquad i,j \in S, \tag{19}$$

where $z(\cdot) = (z_i(\cdot),\ i \in S)$ is given by $z_i(\alpha) = 1 - \sum_{j\in S} \alpha \phi_{ij}(\alpha)$, $i \in S$, and $d(\cdot) = (d_i(\cdot),\ i \in S)$ by

$$d_i(\alpha) = m_i - \sum_{j\in C} m_j (\alpha+\mu) \phi_{ji}(\alpha), \qquad i \in C, \tag{20}$$

and

$$d_0(\alpha) = \frac{e}{\alpha} - \sum_{j \in C} m_j(\alpha + \mu)\phi_{j0}(\alpha), \tag{21}$$

where e satisfies

$$\sum_{i \in C} m_i q_{i0} \le e \le \mu \sum_{i \in C} m_i. \tag{22}$$

Note that $z_0(\alpha) = 0$, and, since F is dishonest, $z_i(\alpha) > 0$ for some (and then all) $i \in C$. Since m is not μ-invariant on C for F, we must have

$$\sum_{i \in C} m_i(\alpha + \mu)\phi_{ij}(\alpha) < m_j, \tag{23}$$

for at least one $j \in C$. Since F satisfies (FE_{ij}) over S, we have, in particular, that

$$\alpha\phi_{i0}(\alpha) = \sum_{j \in C} \phi_{ij}(\alpha) q_{j0}, \qquad i \in C. \tag{24}$$

Hence, from (22), (23), (24) and (18), we have

$$\sum_{i \in C} m_i(\alpha + \mu)\phi_{i0}(\alpha) < \frac{1}{\alpha} \sum_{i \in C} m_i q_{i0} \le \frac{e}{\alpha}.$$

Thus, $d_0(\alpha) > 0$ and $d_j(\alpha) > 0$ for at least one $j \in C$.

Next we shall show that Ψ, given by (19), is a Q-resolvent and that the given m is a μ-invariant measure on C for Ψ. Clearly $\psi_{ij}(\alpha) \ge 0$ for all $i, j \in S$. Since m is finite, we have, from the definition of d, that $\alpha \sum_{j \in S} d_j(\alpha) \le (\alpha + \mu) \sum_{j \in C} m_j z_j(\alpha)$, and so $\sum_{j \in S} \alpha\psi_{ij}(\alpha) \le 1$ for all $i \in S$. In order to prove that Ψ is the resolvent of a standard transition function P, we need only show that Ψ satisfies the resolvent equation (13); see Theorem 1 of Reuter [17]. We shall use the following identities:

$$z_i(\alpha) - z_i(\beta) + (\alpha - \beta) \sum_{k \in C} \phi_{ik}(\alpha) z_k(\beta) = 0, \qquad i \in C, \tag{25}$$

$$d_i(\alpha) - d_i(\beta) + (\alpha - \beta) \sum_{k \in C} d_k(\alpha)\phi_{ki}(\beta) = 0, \qquad i \in C, \tag{26}$$

$$\alpha d_0(\alpha) - \beta d_0(\beta) + (\alpha - \beta) \sum_{k \in C} d_k(\alpha)\beta\phi_{k0}(\beta) = 0 \tag{27}$$

and

$$(\alpha + \mu) \sum_{i \in C} m_i z_i(\alpha) - (\beta + \mu) \sum_{i \in C} m_i z_i(\beta) = (\alpha - \beta) \sum_{i \in C} d_i(\alpha) z_i(\beta). \tag{28}$$

The first three of these can be verified directly using the fact that Φ satisfies the resolvent equation and that $z_0(\alpha) = 0$. The fourth identity follows from the first on multiplying by m_i and summing over i. Using (25)–(28), together with the resolvent equation for Φ, it is easy to verify that Ψ satisfies its own resolvent equation.

Next we need to verify that P is indeed a Q-process, that is $p'_{ij}(0+) = q_{ij}$ for all $i, j \in S$. We shall use a remark of Reuter on Page 83 of [15] (see also Theorem 3.1 of Feller [3]): if one is given a standard transition function P, then it is a Q-process if and only if the backward equations hold, equivalently,

$$\alpha\psi_{ij}(\alpha) = \delta_{ij} + \sum_{k \in S} q_{ik}\psi_{kj}(\alpha), \tag{29}$$

for all $i, j \in S$ and $\alpha > 0$. But, this follows almost immediately from the identity

$$\sum_{k \in C} q_{ik} z_k(\alpha) = \alpha z_i(\alpha), \qquad i \in S, \tag{30}$$

which can be deduced from the backward equations for Φ.

We have shown that Ψ is the resolvent of a Q-process P. To show that m is a μ-invariant measure for P, we again use the definition of d: it is elementary to check that

$$\sum_{i \in S} m_i(\alpha + \mu)\psi_{ij}(\alpha) = m_j, \qquad j \in C, \tag{31}$$

and so the result follows. □

In view of the relationship between quasi-stationary distributions and μ-invariant measures (Proposition 3.1 of Nair and Pollett [7]), we obtain the following corollary of Theorem 2.

Corollary 1 *Let $\mu > 0$, and let $m = (m_j,\ j \in C)$ be a μ-subinvariant probability measure on C for Q.*

1. *If the minimal Q-process F is honest, then m is a quasi-stationary distribution on C for F if and only if (17) holds.*

2. *If the minimal Q-process F is dishonest, then there exists a Q-process P for which m is a quasi-stationary distribution on C if and only if (18) holds.*

5 Uniqueness

Next we shall examine the question of uniqueness. This was considered briefly in the Section 5 of Nair and Pollett [7] under the assumption that Q is a single-exit Q-matrix. Here, in the general case, we give a necessary condition for there to exist uniquely Q-process for which m is μ-invariant on C. Combining the above results with Corollary 5.2 of Nair and Pollett [7], we arrive at the following theorem.

Theorem 3 *Let $\mu > 0$ and suppose that Q admits a finite μ-subinvariant measure m on C.*

1. *If m is μ-invariant for the minimal Q-process F, which is true if and only if*

$$\mu \sum_{i \in C} m_i a_i^F = \sum_{i \in C} m_i q_{i0}, \tag{32}$$

then it is the unique Q-process for which m is μ-invariant on C. When this condition holds, m is μ-invariant on C for Q.

2. *If m is not μ-invariant for the minimal Q-process, there exists uniquely a Q-process for which m is μ-invariant only if (18) holds.*

3. *If Q is single-exit, there exists uniquely Q-process for which m is μ-invariant if and only if (18) holds.*

Proof. Part 1 follows directly from Theorem 2 and the fact that the Q-process is unique in this case. For Part 2, recall that in the proof of Theorem 2 we established that if $\sum_{i \in C} m_i q_{i0} < \mu \sum_{i \in C} m_i$, then there are infinitely many Q-processes for which m is μ-invariant on C. In fact, for each e satisfying (22), we obtain a Q-process, given by (20),(21) and (19), for which m is μ-invariant on C. The final part of the theorem is a direct consequence of Corollary 5.2 of [7]. □

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