A fixed-point approximation accounting for link interactions in a loss network

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Abstract

This paper is concerned with evaluating the performance of loss networks. We develop a reduced load approximation that improves on the famous Erlang fixed point approximation (EFPA) in a variety of circumstances. We illustrate our results with reference to a line network for which the EFPA may be expected to perform badly.

1 Introduction

We shall use the standard model for a circuit-switched network. The network consists of a finite set of links J and the j-th link comprises a group of C_j circuits. A call on route r seizes a_{jr} circuits from one or more links, and these are released simultaneously once the call is terminated. For simplicity, we will assume that $a_{jr} = 1$ if link j is part of route r; otherwise $a_{jr} = 0$. Denote the set of all routes by R, the routing matrix $(a_{jr}; j \in J, r \in R)$ by A, and write $j \in r$ as an abbreviation for $j \in \{i \in J : a_{ir} > 0\}$. If a call arrives to find insufficient capacity on one or more of the links along its route, then it is blocked and lost. The proportions $(L_r; r \in R)$ of calls that are expected to be lost on the various routes form a natural measure of network efficiency.

The usual state description tracks the number of calls in progress on each of the routes. Let $\boldsymbol{Y} = (Y_r; r \in R)$, where Y_r is the number of route-r calls in progress. Due to the capacity constraints, $\boldsymbol{Y}$ takes values in the subset $S = S(\boldsymbol{C})$ of $\mathbb{N}^R$ given by

$$S(\boldsymbol{C}) = \left\{ \boldsymbol{n} \in \mathbb{N}^R : \sum_{r \in R} a_{jr} n_r \le C_j,\ j \in J \right\}. \tag{1}$$

We will suppose that calls on the various routes arrive in independent Poisson streams, with route-r calls arriving at rate ν_r, and that call durations are exponentially distributed with mean 1. Under these assumptions, $\boldsymbol{Y}$ is a reversible Markov process and its equilibrium distribution has a product form. Define P to be the probability measure under which $(Y_r; r \in R)$ are independent Poisson random variables with means ν_r, $r \in R$. This would be the equilibrium measure for the usage on each of the routes were the system not to have any capacity constraints. The restriction of $\boldsymbol{Y}$ to S is a truncation of a reversible Markov process and its equilibrium probability measure is thus given by

$$\pi(\mathcal{A}) = P(\mathcal{A}|\boldsymbol{Y} \in S), \quad \text{for all } P\text{-measurable } \mathcal{A}. \tag{2}$$

Under π, $\boldsymbol{Y}$ is still reversible (Kelly [1], Corollary 1.10), and thus the form of π can be easily obtained from the detailed balance equations,

$$\nu_r \pi(\boldsymbol{Y} = \boldsymbol{n}) = (n_r + 1)\pi(\boldsymbol{Y} = \boldsymbol{n} + \boldsymbol{e}_r), \qquad \boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}_r \in S. \tag{3}$$

Here $\boldsymbol{e}_r$ represents the unit vector with a 1 in the r-th position.

The form of π can also be derived directly from definition (2). For instance, if $K \subseteq J$ and $R_K = \{r \in R : \sum_{j \in K} a_{jr} > 0\}$ is the set of routes that use at least one link in K, then the marginal distribution of the numbers of calls on routes in R_K is

$$\pi(\boldsymbol{Y}_{R_K} = \boldsymbol{n}_{R_K}) = P(\boldsymbol{Y} \in S | \boldsymbol{Y}_{R_K} = \boldsymbol{n}_{R_K}) P(\boldsymbol{Y}_{R_K} = \boldsymbol{n}_{R_K}) / P(\boldsymbol{Y} \in S),$$

where $\boldsymbol{Y}_{R_K} = \boldsymbol{n}_{R_K}$ is shorthand for $\cap_{r \in R_K} \{Y_r = n_r\}$. Noticing that

$$P(\boldsymbol{Y} \in S | \boldsymbol{Y}_{R_K} = \boldsymbol{n}_{R_K}) = P\left(\sum_{r \notin R_K} a_{jr} Y_r \leq C_j - \sum_{r \in R_K} a_{jr} n_r,\ j \in J\right)$$

is a function, θ_K, only of $\boldsymbol{n}_{\partial R_K} = (n_r : r \in R_K \cap R_{J \setminus K})$, we are lead to

$$\pi(\boldsymbol{Y}_{R_K} = \boldsymbol{n}_{R_K}) = \frac{\theta_K(\boldsymbol{n}_{\partial R_K})}{G(\boldsymbol{C})} \prod_{r \in R_K} \frac{\nu_r^{n_r}}{n_r!}, \tag{4}$$

where $G(\boldsymbol{C})$ is a normalising constant chosen so that the distribution π sums to unity. Expression (4) is due to Zachary and Ziedins [6]; it implies that the equilibrium distribution for the loss network is a Markov random field.

The loss probability on a route can be calculated from π: it can be shown that, for any route $r \in R_K$,

$$L_r = 1 - \frac{\sum_{\boldsymbol{n}_{R_K} \in S_{R_K}} \theta_K(\boldsymbol{n}_{\partial R_K}) P(\boldsymbol{Y}_{R_K} = \boldsymbol{n}_{R_K} - \boldsymbol{e}_r)}{\sum_{\boldsymbol{n}_{R_K} \in S_{R_K}} \theta_K(\boldsymbol{n}_{\partial R_K}) P(\boldsymbol{Y}_{R_K} = \boldsymbol{n}_{R_K})}, \tag{5}$$

where, for any $R_0 \subseteq R$, $S_{R_0} = \left\{ \boldsymbol{n}_{R_0} \in \mathbb{N}^{R_0} : \sum_{r \in R_0} a_{jr} n_r \le C_j,\ j \in J \right\}$ is the projection of S onto $\mathbb{N}^{R_0}$. In the case $K = J$, equation (5) has the concise form $L_r = 1 - G(\boldsymbol{C} - A\boldsymbol{e}_r)/G(\boldsymbol{C})$. Unfortunately, calculating the loss probabilities using $G(\boldsymbol{C})$ is often intractable. In fact, the problem is $\#P$-complete (Louth et al. [4]).

2 The Erlang fixed point approximation

In the EFPA the loss probability for route r is estimated to be

$$L_r = 1 - \prod_{i \in r} (1 - B_i)\,, \tag{6}$$

with $B_1, B_2, \ldots, B_J$ a solution to the system of equations

$$B_j = E(\rho_j, C_j), \quad \rho_j = \sum_{r \in R_j} \nu_r \prod_{i \in r \setminus \{j\}} (1 - B_i)\,, \qquad j \in J, \tag{7}$$

where

$$E(\nu, C) = \frac{\nu^C}{C!} \left(\sum_{n=0}^{C} \frac{\nu^n}{n!} \right)^{-1}$$

is Erlang's formula for the blocking probability on a single isolated link with Poisson traffic offered at rate ν. The rationale of "independent blocking" that leads to the EFPA is well understood, and explained simply in Kelly [3]. Kelly [2] proved that, for the model under consideration, there is unique fixed point $(B_1, \ldots, B_J) \in [0,1]^J$ for the system (7).

The EFPA is known to be effective under a variety of limiting regimes. Kelly [3] proved that the estimates for a network with fixed routing and no controls tend towards the exact probabilities when (i) the link capacities and arrival rates are increased at the same rate, keeping the network topology fixed (Kelly limiting regime), and (ii) the number of links and routes are increased while the link loads are held constant (diverse routing limit [7]). The EFPA performs least well in highly linear networks and in circumstances where the offered traffic loads are roughly equal to the capacities (critically loaded).

3 A two-link approximation

An estimate of the route loss probabilities, which is more accurate than those in (6), can be obtained by taking into account the link interdependencies.

This two-link approximation is achieved by approximating the joint distribution of the usage on *pairs* of links (the EFPA effectively estimates this distribution on single links). The approximation is as follows. For each pair of links i, j, let

$$h_{ij}(u_{i|j}, u_{ij}, u_{j|i}) = \frac{\prod_{m=0}^{u_{i|j}-1} \rho_{i|j}(m)}{u_{i|j}!} \frac{\prod_{m=0}^{u_{ij}-1} \rho_{ij}(m)}{u_{ij}!} \frac{\prod_{m=0}^{u_{j|i}-1} \rho_{j|i}(m)}{u_{j|i}!}\,,$$

for $(u_{i|j}, u_{ij}, u_{j|i}) \in \mathbb{N}^3$ such that $u_{i|j} + u_{ij} \leq C_i, u_{j|i} + u_{ij} \leq C_j$, where

$$\rho_{i|j}(u) = \sum_{r\in R_i \setminus R_j} \nu_r \sum_{u_{ij}=0}^{\min(C_i-u,C_j)} \prod_{k\in r} \big(1 - B_{k|i}(u+u_{ij})\big) \frac{\sum_{v=0}^{C_j-u_{ij}} h_{ij}(u, u_{ij}, v)}{\sum_{w=0}^{C_i-u-1} \sum_{v=0}^{C_j-w} h_{ij}(u, w, v)}, \quad (8)$$

$$\rho_{ij}(u) = \sum_{r\in R_i \cap R_j} \nu_r \sum_{u_{i|j}=0}^{C_i-u-1} \prod_{k\in r} \big(1 - B_{k|i}(u_{i|j}+u)\big) \frac{\sum_{v=0}^{C_j-u-1} h_{ij}(u_{i|j}, u, v)}{\sum_{w=0}^{C_i-u-1} \sum_{v=0}^{C_j-u-1} h_{ij}(w, u, v)}\,, \quad (9)$$

and

$$B_{k|i}(u_i) = \begin{cases} \dfrac{\sum_{u_{ik}=0}^{\min(C_k,u_i)} h_{ki}(C_k-u_{ik}, u_{ik}, u_i-u_{ik})}{\sum_{u_k=0}^{C_k} \sum_{u_{ik}=0}^{\min(u_k,u_i)} h_{ki}(u_k-u_{ik}, u_{ik}, u_i-u_{ik})}\,, & \text{if } k \neq i, \\ 1_{\{u_i=C_i\}}\,, & \text{if } k = i. \end{cases} \quad (10)$$

These equations will be derived in Section 5. They form a set of equations in the unknowns $\boldsymbol{B} = (\boldsymbol{B}_{k|i}; i, k \in J)$, where $\boldsymbol{B}_{k|i} = (B_{k|i}(m); m \leq C_i) \in \mathbb{R}^{C_i}$. Existence of a fixed point is guaranteed by Brouwer's Fixed Point Theorem. To see this, let $\Omega_k = \{\boldsymbol{x}_k \in \prod_{i\in J} \mathbb{R}^{C_i} : \boldsymbol{0} \leq \boldsymbol{x}_k \leq \boldsymbol{1}\}$, and observe that

$$f_{k|i}^{u_i}(\boldsymbol{B}) = \begin{cases} \dfrac{\sum_{u_{ik}=0}^{\min(C_k,u_i)} h_{ki}(C_k-u_{ik}, u_{ik}, u_i-u_{ik})}{\sum_{u_k=0}^{C_k} \sum_{u_{ik}=0}^{\min(u_k,u_i)} h_{ki}(u_k-u_{ik}, u_{ik}, u_i-u_{ik})}\,, & \text{if } k \neq i, \\ 1_{\{u_i=C_i\}}, & \text{if } k = i, \end{cases}$$

defines a continuous mapping from $\Omega = \prod_{k\in J} \Omega_k$ into $[0, 1]$. Thus, with $\boldsymbol{f} = (f_{k|i}^{u_i}; u_i = 0, \ldots, C_i, k, i \in J)$, we have $\boldsymbol{f}(\Omega) \subseteq \Omega$, and therefore $\boldsymbol{f}$ has at least one fixed point in Ω.

The loss probabilities are estimated using $\boldsymbol{h} = (h_{ij}; i, j \in J)$. Losses on two-link routes, for example, have

$$L_r = 1 - \frac{\Phi_{ij}(C_i - 1, C_j - 1)}{\Phi_{ij}(C_i, C_j)}\,, \quad \text{if } r = \{i, j\}, \quad (11)$$

where $\Phi_{ij}(C_i, C_j) = \sum_{u_i=0}^{C_i} \sum_{u_j=0}^{C_j} \sum_{k=0}^{\min(u_i,u_j)} h_{ij}(u_i - k, k, u_j - k)$. Calls that use the single link $r = \{i\}$ are lost with probability

$$B_i = 1 - \frac{\Phi_{ij}(C_i - 1, C_j)}{\Phi_{ij}(C_i, C_j)}\,, \quad (12)$$

where j is any link with a route common to i.

The rationale for the approximation is as follows. The traffic offered to a subsystem consisting of two arbitrary links, i and j, can be classified as either (i) link i only, (ii) link j only, or (iii) common to both links. Correspondingly, let $U_{i|j} = \sum_{r\in R_i\setminus R_j} Y_r$, $U_{j|i} = \sum_{r\in R_j\setminus R_i} Y_r$ and $U_{ij} = \sum_{r\in R_i\cap R_j} Y_r$ be, respectively, the number of calls using link i, the number using link j, and the number on routes using both i and j. This is a natural way to classify the traffic offered to the subsystem. Without capacity constraints, the joint distribution of the link utilisations $U_i = U_{i|j} + U_{ij}$ and $U_j = U_{j|i} + U_{ij}$ is

$$P(U_i = u_i, U_j = u_j) = \sum_{k=0}^{\min(u_i,u_j)} P(U_{i|j} = u_i - k, U_{ij} = k, U_{j|i} = u_j - k)\,,$$

where

$$P(U_{i|j} = u_{i|j}, U_{ij} = u_{ij}, U_{j|i} = u_{j|i}) = \frac{\rho_{i|j}^{u_{i|j}}}{u_{i|j}!}\frac{\rho_{ij}^{u_{ij}}}{u_{ij}!}\frac{\rho_{j|i}^{u_{j|i}}}{u_{j|i}!}e^{-(\rho_{i|j}+\rho_{ij}+\rho_{j|i})}, \qquad (13)$$

with $\rho_{ij} = \sum_{r\in R_i\cap R_j} \nu_r$, $\rho_{i|j} = \sum_{r\in R_i\setminus R_j} \nu_r$ and $\rho_{j|i} = \sum_{r\in R_j\setminus R_i} \nu_r$. To construct the approximation, we replace the aggregate rates ρ_{ij}, $\rho_{i|j}$ and $\rho_{j|i}$ in (13) by "reduced load" rates, and we isolate the subsystem composed of traffic offered to links i and j. Motivated by the form of (13), let us suppose for the moment that $\pi(U_{i|j} = u_{i|j}, U_{ij} = u_{ij}, U_{j|i} = u_{j|i})$ has the form $h_{ij}(u_{i|j}, u_{ij}, u_{j|i})/\Phi_{ij}(C_i, C_j)$. If this were the case, then questions concerning call blocking could be answered easily. For instance, the probability that link i is full would be B_i in expression (12), the probability that either link i or link j are full would be L_r in expression (11), and the conditional probability that link k is full given link i carries u_i calls would be $B_{k|i}(u_i)$ in expression (10). To ensure that the traffic offered to the subsystem is consistent with blocking in other parts of the network, the rates ρ_{ij}, $\rho_{i|j}$ and $\rho_{j|i}$ are replaced by state-dependent reduced rates. For example, expression (8) for $\rho_{i|j}(u_{i|j})$ is just $\rho_{i|j} = \sum_{r\in R_i\setminus R_j} \nu_r$, but reduced by an estimate of the expected blocking on the other links $k \in r$ such that $r \in R_i \setminus R_j$ when link i is carrying $u_{i|j}$ calls that are not also carried by link j.

4 A line network

In order to assess the accuracy of the two-link approximation, we shall consider a network in which l links, labelled $1, \ldots, l$, are joined end-to-end. Suppose that each link is offered a stream of single-link traffic, and that each of the links $i \in \{2, \ldots, l-1\}$ share two-link routes $\{i-1, i\}$ and $\{i, i+1\}$, with each of their neighbouring links. Thus, there are l single-link routes and $l-1$ two-link routes. For simplicity, assume that calls on single-link routes arrive at a common rate ν_1, and that calls on each two-link route arrive at rate ν_2.

The EFPA for the route loss probabilities is $L_i = B_i$, $i = 1, \ldots, l$, and $L_{i,i+1} = 1 - (1 - B_i)(1 - B_{i+1})$, $i = 1, \ldots, l-1$, where $(B_i; i = 1, \ldots, l)$ is the solution to

$$\begin{aligned} B_1 &= \nu_1 + \nu_2 (1 - B_2), \\ B_i &= \nu_1 + \nu_2 (1 - B_{i-1}) + \nu_2 (1 - B_{i+1}), \quad i = 2, \ldots, l-1, \\ B_l &= \nu_1 + \nu_2 (1 - B_{l-1}). \end{aligned}$$

The two-link approximation for this network is as follows. For $u = 0, \ldots, C-1$, set $\rho_{ij}(u) = \nu_2$ for all $i, j = 1, \ldots, l$ such that $j = i-1$ or $j = i+1$, $\rho_{1|2}(u) = \nu_1$, $\rho_{l|l-1}(u) = \nu_1$, and let $(B_{j|i}(u); i, j = 1, \ldots, l, j = i-1 \,\text{or}\, j = i+1)$ be a solution to the system of equations

$$B_{j|i}(u) = \frac{\sum_{u_{ij}=0}^{\min(C,u)} h_{ji}(C - u_{ij}, u_{ij}, u - u_{ij})}{\sum_{u_j=0}^{C} \sum_{u_{ij}=0}^{\min(u_j,u)} h_{ji}(u_j - u_{ij}, u_{ij}, u - u_{ij})}, \qquad i, j = 1, \ldots, l,\ j = i-1 \text{ or } j = i+1,$$

$$\rho_{i|i-1}(u) = \nu_1 + \nu_2 \sum_{k=0}^{C-u-1} \big(1 - B_{i+1|i}(u+k)\big) \frac{\sum_{w=0}^{C-k} h_{i-1,i}(w, k, u)}{\sum_{v=0}^{C-u-1} \sum_{w=0}^{C-v} h_{i-1,i}(w, v, u)}, \qquad i = 2, \ldots, l-1,$$

$$\rho_{i|i+1}(u) = \nu_1 + \nu_2 \sum_{k=0}^{C-u-1} \big(1 - B_{i-1|i}(u+k)\big) \frac{\sum_{w=0}^{C-k} h_{i,i+1}(u, k, w)}{\sum_{v=0}^{C-u-1} \sum_{w=0}^{C-v} h_{i,i+1}(u, v, w)}, \qquad i = 2, \ldots, l-1,$$

where

$$h_{ij}(u_{i|j}, u_{ij}, u_{j|i}) = \frac{\prod_{m=0}^{u_{i|j}-1} \rho_{i|j}(m)}{u_{i|j}!} \frac{\nu_2^{u_{ij}}}{u_{ij}!} \frac{\prod_{m=0}^{u_{j|i}-1} \rho_{j|i}(m)}{u_{j|i}!},$$

for $i, j = 1, \ldots, l$, $j = i-1$ or $j = i+1$. Estimates for the loss probabilities on single-link routes are

$$\begin{aligned} L_i &= 1 - \frac{\Phi_{i,i+1}(C-1, C)}{\Phi_{i,i+1}(C, C)}, \quad \text{for } i = 1, \ldots, l-1, \quad \text{and} \\ L_i &= 1 - \frac{\Phi_{i,i-1}(C-1, C)}{\Phi_{i,i-1}(C, C)}, \quad \text{for } i = 2, \ldots, l, \end{aligned}$$

where

$$\Phi_{ij}(C_i, C_j) = \sum_{u_i=0}^{C_i} \sum_{u_j=0}^{C_j} \sum_{k=0}^{\min(u_i,u_j)} \frac{\prod_{m=0}^{u_i-k-1} \rho_{i|j}(m)}{(u_i-k)!} \frac{\nu_2^k}{k!} \frac{\prod_{m=0}^{u_j-k-1} \rho_{j|i}(m)}{(u_j-k)!},$$

for $i = 1, \ldots, l$ and j a link adjacent to i. For certain links there may be more than one possible estimate for the loss probability. For example, this scheme produces two estimates for the loss probability on link 2:

$$L_2 = 1 - \frac{\Phi_{12}(C, C-1)}{\Phi_{12}(C, C)} \quad \text{and} \quad L_2 = 1 - \frac{\Phi_{23}(C-1, C)}{\Phi_{23}(C, C)}.$$

In practice there is no way of knowing which estimate will be the most accurate. Both estimates achieved greater precision than the EFPA for the network tested here. There is no ambiguity in estimating the two-link loss probabilities; for $i = 1, \ldots, l-1$, $L_{i,i+1} = 1 - \Phi_{i,i+1}(C-1, C-1)/\Phi_{i,i+1}(C, C)$.

In Figures 1, 2 and 3, the relative errors in the loss probability estimates for the EFPA and the two-link approximation are compared. The network tested had 5 links, each with a carrying capacity of 5 calls. The single-link route arrival rate ν_1 was varied over $[0, 10]$ and ν_2 was set at $\nu_1/2$. By symmetry, there are only three single-link routes and two two-link routes to distinguish. In this test case, the two-link approximation provided a significant improvement in accuracy over the EFPA for each of the two-link routes (Figure 3), the single-link route using an end link (top pane of Figure 1) and the single-link route that uses the centre link (bottom pane of Figure 1). The single-link route that uses a link second from the end was the only one with multiple loss estimates. In Figure 2 the relative errors of the estimates of L_2, using the Φ_{12} and Φ_{23}, are compared with the EFPA. Both two-link estimates show a significant improvement over the EFPA.

5 Derivation of the two-link approximation

In this section we derive the fixed-point equations for the two-link approximation of Section 3. Recall the way that we classified traffic offered to links i and j. We had introduced $U_{i|j} = \sum_{r \in R_i \setminus R_j} Y_r$, $U_{j|i} = \sum_{r \in R_j \setminus R_i} Y_r$ and $U_{ij} = \sum_{r \in R_i \cap R_j} Y_r$. When capacity constraints are present, questions concerning $\boldsymbol{U}_{ij} = (U_{i|j}, U_{ij}, U_{j|i})$ are generally not easily answered. Let us now introduce new, independent processes $\tilde{\boldsymbol{U}}_{ij} = (\tilde{U}_{i|j}, \tilde{U}_{ij}, \tilde{U}_{j|i})$, for each pair of links $i, j \in J$. We shall suppose $\tilde{\boldsymbol{U}}_{ij}$ is a continuous-time Markov chain that approximates the π-behaviour of $\boldsymbol{U}_{ij}$ in the space $S_{ij} = S_{ij}(C_i, C_j) =$

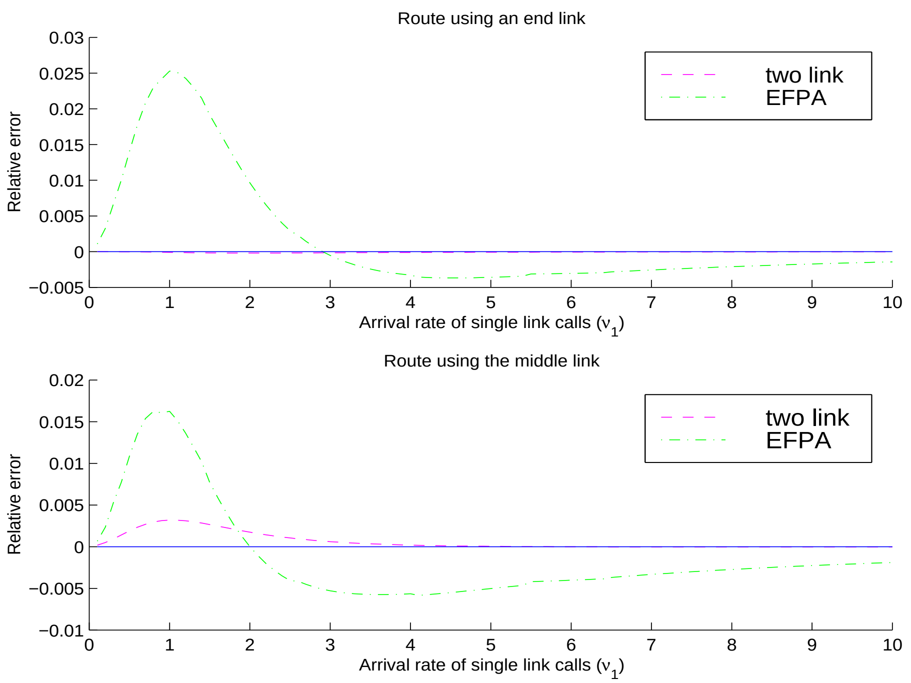

Figure 1: Accuracy for a line network (5 links, $C = 5$, $\nu_2 = \nu_1/2$)

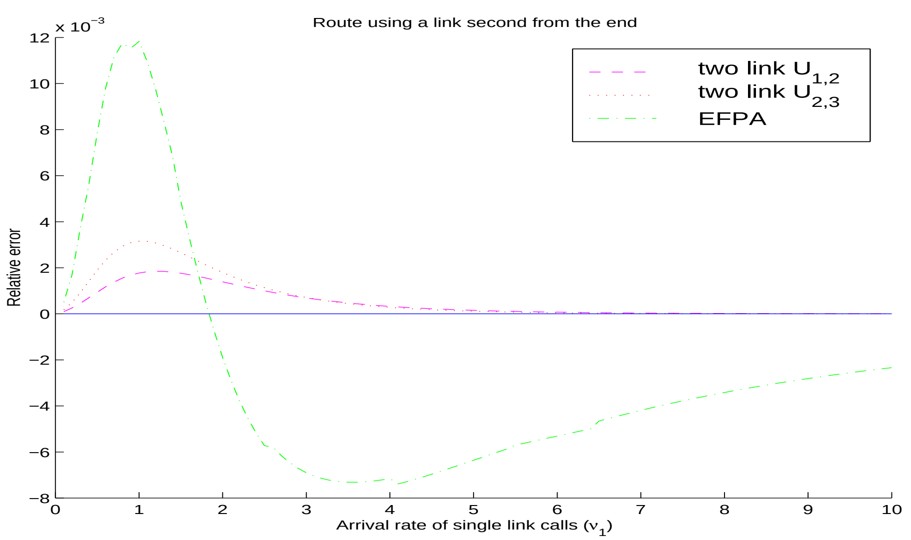

Figure 2: Accuracy for a line network (5 links, $C = 5$, $\nu_2 = \nu_1/2$)

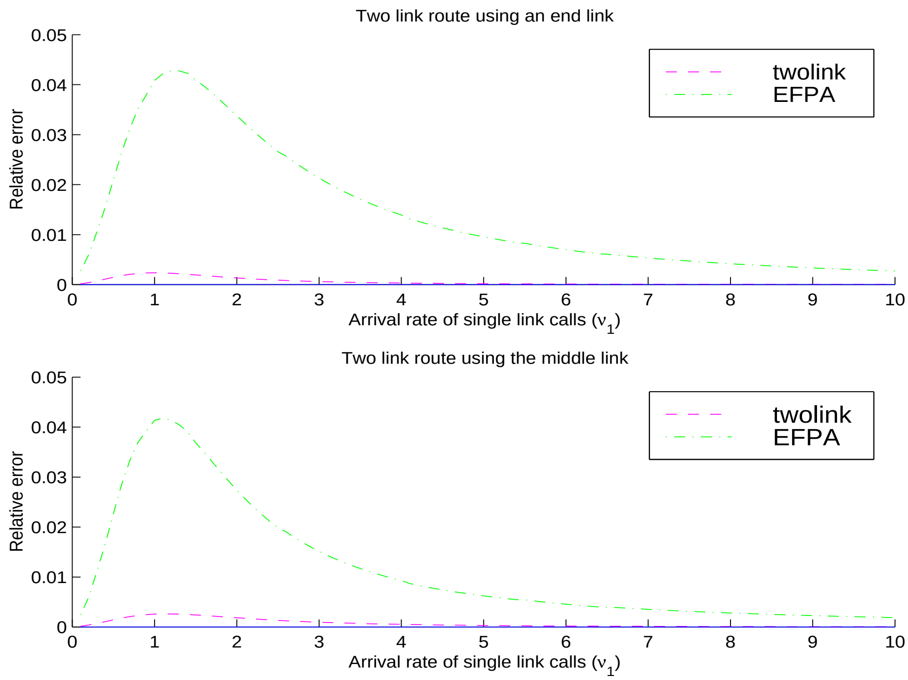

Figure 3: Accuracy for a line network (5 links, $C = 5$, $\nu_2 = \nu_1/2$)

$\{(u_{i|j}, u_{ij}, u_{j|i}) : u_{i|j} + u_{ij} \le C_i, u_{j|i} + u_{ij} \le C_j\}$. Suppose that $\tilde{\boldsymbol{U}}_{ij}$ makes transitions from state $(u_{i|j}, u_{ij}, u_{j|i})$ to the following states at the rates specified:

$$\begin{array}{ll}
(u_{i|j} - 1, u_{ij}, u_{j|i}) & \text{at rate } u_{i|j}, \\
(u_{i|j}, u_{ij} - 1, u_{j|i}) & \text{at rate } u_{ij}, \\
(u_{i|j}, u_{ij}, u_{j|i} - 1) & \text{at rate } u_{j|i}, \\
(u_{i|j} + 1, u_{ij}, u_{j|i}) & \text{at rate } \rho_{i|j}(u_{i|j}) 1_{\{u_{i|j} + u_{ij} \le C_i\}}, \\
(u_{i|j}, u_{ij} + 1, u_{j|i}) & \text{at rate } \rho_{ij}(u_{ij}) 1_{\{u_{i|j} + u_{ij} \le C_i, u_{j|i} + u_{ij} \le C_j\}}, \\
(u_{i|j}, u_{ij}, u_{j|i} + 1) & \text{at rate } \rho_{j|i}(u_{j|i}) 1_{\{u_{j|i} + u_{ij} \le C_j\}},
\end{array}$$

and no other transitions are possible. The stationary distribution for $\tilde{\boldsymbol{U}}_{ij}$ is

$$\mathcal{P}\left(u_{i|j}, u_{ij}, u_{j|i}\right) = \Phi_{ij}^{-1} \frac{\prod_{m=0}^{u_{i|j}-1} \rho_{i|j}(m)}{u_{i|j}!} \frac{\prod_{m=0}^{u_{ij}-1} \rho_{ij}(m)}{u_{ij}!} \frac{\prod_{m=0}^{u_{j|i}-1} \rho_{j|i}(m)}{u_{j|i}!},$$

where $\Phi_{ij} = \Phi_{ij}(C_i, C_j)$ is chosen so that $\mathcal{P}$ sums to 1 over the set S_{ij}:

$$\Phi_{ij}(C_i, C_j) = \sum_{u_i=0}^{C_i} \sum_{u_j=0}^{C_j} \sum_{k=0}^{\min(u_i,u_j)} \frac{\prod_{m=0}^{u_i-k-1} \rho_{i|j}(m)}{(u_i - k)!} \frac{\prod_{m=0}^{k-1} \rho_{ij}(m)}{k!} \frac{\prod_{m=0}^{u_j-k-1} \rho_{j|i}(m)}{(u_j - k)!}.$$

Our aim is to choose $\rho_{i|j}(\cdot)$, $\rho_{ij}(\cdot)$ and $\rho_{j|i}(\cdot)$ such that the behaviour of $\tilde{\boldsymbol{U}}_{ij}$, with its assumed transition structure, best approximates that of $\boldsymbol{U}_{ij}$. We assign to these quantities expected rates, in the following sense (see [5]). Denote the space $\prod_{i,j\in J} S_{ij}$ by $\tilde{S}$. Let $\boldsymbol{\Lambda}_{i|j}(u) = \{(\boldsymbol{u},\boldsymbol{v}) \in \tilde{S}\times\tilde{S} : u_{i|j} = u, v_{i|j} = u+1\}$, for $u = 0,1,\ldots,C_i-1$. Then define $\rho_{i|j}(u)$ to be the expected rate (under $\mathcal{P}$) of transitions in $\boldsymbol{\Lambda}_{i|j}(u)$:

$$\rho_{i|j}(u) = \mathbb{E}_{\mathcal{P}}\left(q(\tilde{\boldsymbol{U}},\boldsymbol{\Lambda}_{i|j}(u,\tilde{\boldsymbol{U}}))\,\middle|\,\tilde{U}_{i|j}=u,\tilde{U}_{i|j}+\tilde{U}_{ij}<C_i\right),$$

where $q(\boldsymbol{u},\boldsymbol{\Lambda}_{i|j}(u,\boldsymbol{u})) = \sum_{r\in R_i\setminus R_j}\nu_r\prod_{k\in r\setminus\{i\}}1_{\{u_{k|i}+u_{ki}<C_k\}}1_{\{u+u_{ij}<C_i\}}$. This expression can be evaluated partially as follows:

$$\begin{aligned}\mathbb{E}&\left(q(\tilde{\boldsymbol{U}},\boldsymbol{\Lambda}_{i|j}(u,\tilde{\boldsymbol{U}}))\middle|\tilde{U}_{i|j}=u,\tilde{U}_{i|j}+\tilde{U}_{ij}<C_i\right)=\\ &\qquad\mathbb{E}\left(\alpha_{i|j}(\tilde{U}_{i|j}+\tilde{U}_{ij},\tilde{U}_{j|i}+\tilde{U}_{ij})\middle|\tilde{U}_{i|j}=u,\tilde{U}_{i|j}+\tilde{U}_{ij}<C_i\right),\end{aligned}$$

where $\alpha_{i|j}(u_i,u_j)=\mathbb{E}(q(\tilde{\boldsymbol{U}},\boldsymbol{\Lambda}_{i|j}(u,\tilde{\boldsymbol{U}}))|E(u_i,u_j))$ and

$$E(u_i,u_j)=\left\{\tilde{U}_{i|k}+\tilde{U}_{ik}=u_i,\ k\in J\setminus\{i\}\right\}\cap\left\{\tilde{U}_{j|k}+\tilde{U}_{jk}=u_j,\ k\in J\setminus\{j\}\right\}$$

is the event that links i and j have utilisations u_i and u_j respectively. The function $\alpha_{i|j}(u_i,u_j)$ is the expected rate of transitions in the set $\{(\boldsymbol{u},\boldsymbol{v})\in\tilde{S}\times\tilde{S} : u_{i|j}+u_{ij}=u_i, u_{j|i}+u_{ij}=u_j, v_{i|j}=u_{i|j}+1\}$. It simplifies to $\alpha_{i|j}(u_i,u_j)=0$ if $u_i=C_i$, and otherwise

$$\alpha_{i|j}(u_i,u_j)=\sum_{r\in R_i\setminus R_j}\nu_r\mathcal{P}\left(\tilde{U}_{k|i}+\tilde{U}_{ik}<C_k,\ k\in r\setminus\{i\}\middle|E(u_i,u_j)\right).$$

Extending the rationale of independent blocking, characteristic of the EFPA, we now assume that pairs of links $\{i,j\}\in J$ index *independent* random processes $\tilde{\boldsymbol{U}}_{ij}$. Under this assumption,

$$\begin{aligned}\alpha_{i|j}(u_i,u_j)&=\sum_{r\in R_i\setminus R_j}\nu_r\prod_{k\in r\setminus\{i\}}\mathcal{P}\big(\tilde{U}_{k|i}+\tilde{U}_{ik}<C_k\big|\tilde{U}_{i|k}+\tilde{U}_{ik}=u_i\big)1_{\{u_i<C_i\}}\\ &=\sum_{r\in R_i\setminus R_j}\nu_r\prod_{k\in r}\big(1-B_{k|i}(u_i)\big),\end{aligned}$$

where $B_{k|i}(u_i)$ is the likelihood that link k is full when link i is known to have u_i circuits in use. This quantity is estimated to be

$$\begin{aligned}B_{k|i}(u_i)\quad&=\frac{\sum_{u_{ik}=0}^{\min(C_k,u_i)}\mathcal{P}\left(\tilde{U}_{k|i}=C_k-u_{ik},\tilde{U}_{ik}=u_{ik},\tilde{U}_{i|k}=u_i-u_{ik}\right)}{\sum_{u_k=0}^{C_k}\sum_{u_{ik}=0}^{\min(u_k,u_i)}\mathcal{P}\left(\tilde{U}_{k|i}=u_k-u_{ik},\tilde{U}_{i,k}=u_{ik},\tilde{U}_{i|k}=u_i-u_{ik}\right)}\\ &=\begin{cases}\dfrac{\sum_{u_{ik}=0}^{\min(C_k,u_i)}h_{ki}(C_k-u_{i,k},u_{ik},u_i-u_{ik})}{\sum_{u_k=0}^{C_k}\sum_{u_{ik}=0}^{\min(u_k,u_i)}h_{ki}(u_k-u_{ik},u_{ik},u_i-u_{ik})}, & \text{if } k\neq i,\\ 1_{\{u_i=C_i\}}, & \text{if } k=i,\end{cases}\end{aligned}$$

with $h_{ki}(u_{k|i}, u_{ki}, u_{k|i})$ proportional to $\mathcal{P}(u_{k|i}, u_{ki}, u_{k|i})$ in S_{ki}. Thus, we have an expression for a reduced load marginal rate of arrivals to link i that do not use link j:

$$\rho_{i|j}(u) = \sum_{r \in R_i \setminus R_j} \nu_r \sum_{v=0}^{\min(C_i-u, C_j)} \prod_{k \in r} \big(1 - B_{k|i}(u+v)\big) \mathcal{P}\big(\tilde{U}_{ij} = v \big| \tilde{U}_{i|j} = u, \tilde{U}_{i|j} + \tilde{U}_{ij} < C_i\big).$$

Equation (8) results on setting

$$\mathcal{P}(\tilde{U}_{ij} = u_{ij} | \tilde{U}_{i|j} = u, \tilde{U}_{i|j} + \tilde{U}_{ij} < C_i) = \frac{\sum_{v=0}^{C_j - u_{ij}} h_{ij}(u, u_{ij}, v)}{\sum_{w=0}^{C_i-u-1} \sum_{v=0}^{C_j - w} h_{ij}(u, w, v)}.$$

Expression (9) for the reduced load rate $\rho_{ij}(u)$ of arrivals corresponding to transitions in $\mathbf{\Lambda}_{ij}(u) = \{(\boldsymbol{u}, \boldsymbol{v}) \in \tilde{S} \times \tilde{S} : u_{ij} = u, v_{ij} = u+1\}$, $u = 0, \ldots, \min(C_i - 1, C_j - 1)$, is derived in a similar way. The quantity $\alpha_{ij}(u_i, u_j)$ representing the expected rate at which calls that cause an increase in the utilisation of both links i and j arrive when $U_i = u_i$ and $U_j = u_j$, is

$$\alpha_{ij}(u_i, u_j) = \mathbb{E}\left(\sum_{r \in R_i \cap R_j} \nu_r \prod_{k \in r \setminus \{i,j\}} 1_{\{\tilde{U}_{k|i} + \tilde{U}_{ki} < C_k\}} \,\middle|\, E(u_i, u_j) \right) 1_{\{u_i < C_i, u_j < C_j\}},$$

which leads to $\alpha_{ij}(u_i, u_j) = \sum_{r \in R_i \cap R_j} \nu_r \prod_{k \in r} (1 - B_{k|i}(u_i))$ if $u_j < C_j$ and $\alpha_{ij}(u_i, u_j) = 0$ if $u_j = C_j$. We set $\rho_{ij}(u)$ to be the expected rate of transitions in $\mathbf{\Lambda}_{ij}(u)$:

$$\begin{aligned}
&\mathbb{E}\left(\alpha_{ij}(\tilde{U}_{i|j} + \tilde{U}_{ij}, \tilde{U}_{j|i} + \tilde{U}_{ij}) \,\middle|\, \tilde{U}_{ij} = u, \tilde{U}_{i|j} + \tilde{U}_{ij} < C_i, \tilde{U}_{j|i} + \tilde{U}_{ij} < C_j\right) \\
&\quad = \sum_{r \in R_i \cap R_j} \nu_r \sum_{u_{i|j}=0}^{C_i - u - 1} \prod_{k \in r \setminus \{j\}} \big(1 - B_{k|i}(u_{i|j} + u)\big) \\
&\qquad\qquad \mathcal{P}\left(\tilde{U}_{i|j} = u_{i|j} \,\middle|\, \tilde{U}_{ij} = u, \tilde{U}_{i|j} + \tilde{U}_{ij} < C_i, \tilde{U}_{j|i} + \tilde{U}_{ij} < C_j\right).
\end{aligned}$$

Expression (9) follows on using

$$\frac{\sum_{v=0}^{C_j - u_{ij} - 1} h_{ij}(u_{i|j}, u, v)}{\sum_{w=0}^{C_i-u-1} \sum_{v=0}^{C_j-u-1} h_{ij}(w, u, v)}$$

to estimate the latter conditional probability. The loss probabilities may then be estimated using Φ_{ij}. Losses on two link routes, $r = \{i, j\}$, have

$$L_r = 1 - \pi(U_i < C_i, U_j < C_j) \approx 1 - \frac{\Phi_{ij}(C_i - 1, C_j - 1)}{\Phi_{ij}(C_i, C_j)}.$$

Calls that use the single link i are lost with probability

$$B_i = 1 - \pi(U_i < C_i) \approx 1 - \frac{\Phi_{ij}(C_i - 1, C_j)}{\Phi_{ij}(C_i, C_j)} .$$

The approximation for B_i depends on j because the distribution of $\tilde{U}_{i|j} + \tilde{U}_{ij}$ is different from that of $\tilde{U}_{i|k} + \tilde{U}_{ik}$. As a result, the loss estimated using Φ_{ij} may be different from the estimate Φ_{ik}.

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