

PATH INTEGRALS FOR CONTINUOUS-TIME MARKOV CHAINS

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Abstract

This note presents a method of evaluating the distribution of a path integral for Markov chains on a countable state space.

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1. Introduction

Let $(X(t), t \geq 0)$ be a continuous-time Markov chain taking values in the nonnegative integers $S = \{0, 1, \dots\}$ and let A be a fixed subset of S . Consider the path integral

$$\Gamma = \int_0^\tau f_{X(t)} dt, \quad (1)$$

where $f : A \rightarrow [0, \infty)$ and $\tau = \inf\{t > 0 : X(t) \notin A\}$ is the first exit time of A . Here f_j may be regarded as the reward per unit time of staying in state j , so that Γ is the total reward over the period that the chain spends in A . We shall describe a method of obtaining the Laplace transform of the distribution of Γ , which is a simple extension of the corresponding result (f identically 1) on the distribution of first passage times [10]. Explicit results are available for special Markov chains. For example, when the state space is finite, the distribution of the time spent in A can be exhibited explicitly (see [8]). In the case of birth–death processes, Flajolet and Guillemin [4] and Ball and Stefanov [2] obtained results on transforms of the distributions of first passage times, and other fundamental characteristics, in terms of continued fractions.

2. Distribution of the path integral

Let $Q = (q_{ij}, i, j \in S)$ be the q -matrix of transition rates of the chain (assumed to be stable and conservative), so that, for $j \neq i$, q_{ij} represents the rate of transition from state i to state j , and $q_{ii} = -q_i$, where $q_i := \sum_{j \neq i} q_{ij} (< \infty)$ represents the total rate out of state i . It will not be necessary to assume that Q is regular, so that there may actually be many processes with the given set of rates. However, we will take $(X(t), t \geq 0)$ to be the *minimal chain* associated with Q .

We shall evaluate the Laplace transform of the distribution of path integral (1), conditional on the chain starting in state $i \in A$, making the harmless assumption that $q_j > 0$ for all $j \in A$, so that A contains no absorbing states. With the notation $E_i(\cdot) = E(\cdot \mid X(0) = i)$,

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let $y_i(\theta) = E_i(e^{-\theta\Gamma})$, with the understanding that $y_i(\theta) = 1$ when $i \notin A$. The following result is a simple extension of the standard characterization of hitting times (see, for example, Theorem 9 of [10, p. 86]).

Proposition 1. *For each $\theta > 0$, $\mathbf{y}(\theta) = (y_i(\theta), i \in S)$ is the maximal solution to the system*

$$\sum_{j \in S} q_{ij} z_j = \theta f_i z_i, \quad i \in A, \tag{2}$$

with $0 \leq z_j \leq 1$ for $j \in A$ and $z_j = 1$ for $j \notin A$, in the sense that $\mathbf{y}(\theta)$ satisfies these equations, and, if $\mathbf{z} = (z_i, i \in S)$ is any such solution, then $y_i(\theta) \geq z_i$ for all $i \in S$.

Proof. Fix $i \in A$ and condition on the time of the first jump and the state visited at that time, to get

$$E_i(e^{-\theta\Gamma}) = \int_0^\infty \sum_{k \neq i} e^{-\theta f_i u} E_k(e^{-\theta\Gamma}) \frac{q_{ik}}{q_i} q_i e^{-q_i u} du.$$

It follows that

$$\sum_{k \neq i} q_{ik} y_k(\theta) = (\theta f_i + q_i) y_i(\theta), \tag{3}$$

and we deduce immediately that $\mathbf{y}(\theta)$ satisfies (2).

To show that $\mathbf{y}(\theta)$ is the maximal solution to (2), let $T_j(n)$, $n \geq 1$, be the total time that the process spends in state j during the period up to the time of the n th jump, and let $y_i(n, \theta) = E_i(e^{-\theta\Gamma(n)})$, where $\Gamma(n) = \sum_{j \in S} f_j T_j(n)$, with the understanding that $y_i(n, \theta) = 1$ for $i \notin A$ (so $\Gamma(n)$ is the contribution to the path integral over the period). Clearly, $\Gamma(n) \uparrow \Gamma$, so that monotone convergence gives $y_i(n, \theta) \downarrow y_i(\theta)$. Again fix $i \in A$. Using a similar conditioning argument, we find that, for any $n \geq 1$,

$$E_i(e^{-\theta\Gamma(n+1)}) = \int_0^\infty \sum_{k \neq i} e^{-\theta f_i u} E_k(e^{-\theta\Gamma(n)}) \frac{q_{ik}}{q_i} q_i e^{-q_i u} du,$$

and we deduce that

$$\sum_{k \neq i} q_{ik} y_k(n, \theta) = (\theta f_i + q_i) y_i(n + 1, \theta), \quad n \geq 1. \tag{4}$$

But, we also have $y_i(1, \theta) = q_i / (\theta f_i + q_i)$, and so (4) is valid for $n = 0$ provided that we put $y_k(0, \theta) = 1$ for all $k \in S$. Now suppose that $\mathbf{z} = (z_j, j \in S)$ is any solution to (2) with $0 \leq z_j \leq 1$ for $j \in A$ and $z_j = 1$ for $j \notin A$, so that, in particular,

$$\sum_{k \neq i} q_{ik} z_k = (\theta f_i + q_i) z_i, \quad i \in A. \tag{5}$$

For $i \notin A$, we have $y_i(\theta) = z_i = 1$. On comparing (4) and (5), we see that, if $y_i(n, \theta) \geq z_i$ for $i \in A$ and any fixed $n \geq 0$, we will then also have $y_i(n + 1, \theta) \geq z_i$ for $i \in A$. But, $z_i \leq 1 = y_i(0, \theta)$, and so, by mathematical induction, $y_i(n, \theta) \geq z_i$ for $i \in A$ and every $n \geq 0$. Letting $n \rightarrow \infty$ shows that $y_i(\theta) \geq z_i$ for $i \in A$, and this completes the proof.

Formal differentiation of (3) suggests a corresponding result on the expected value of the path integral, conditional on the chain starting in state $i \in A$. In fact, using similar arguments, we can arrive at the following result, which is a simple extension of Theorem 10 of [10, p. 86].

Proposition 2. *The minimal nonnegative solution to the system*

$$\sum_{j \in A} q_{ij} z_j + f_i = 0, \quad i \in A, \tag{6}$$

is $\mathbf{e} = (e_i, i \in A)$, where $e_i = E_i(\Gamma)$.

Remark 1. If we set $f_i = 1$ for all $i \in A$, then $\Gamma = \tau$, and so the above results can be used to determine the distribution and the expectation of τ . In the case when \mathbf{Q} is regular, these reduce to well-known and widely used results on hitting times; see Section 9.2 of [1].

Of particular interest are the cases (i) $A = S$ with S irreducible, and (ii) $S = A \cup \{0\}$, with A irreducible and 0 being an absorbing state that is accessible from A . In both cases, Γ counts the reward over the lifetime of the chain. Note that, if \mathbf{Q} is not regular, then, in case (i), τ is the explosion time of the chain (which is almost surely finite for all starting states). The above results might therefore be useful in biological applications, where we may wish to account for explosive behaviour by allowing the chain to perform infinitely-many transitions in a finite time. Case (ii) was considered by Stefanov and Wang [9] for birth–death processes. They derived an explicit expression for the expectation $E_i(\Gamma)$, building on earlier work of Hernández-Suárez and Castillo-Chavez [5], who studied the case $i = 1$ and $f_j = j$.

On dividing (2) by f_i , we see that, conditional on $X(0) = i$, Γ has the same distribution as τ for the Markov chain with transition rates $\mathbf{Q}^* = (q_{ij}^*, i, j \in S)$ given by $q_{ij}^* = q_{ij}/f_i$ for all $i \in A$ such that $f_i > 0$, and $q_{ij}^* = q_{ij}$ otherwise. This was observed for birth–death processes by McNeil [6]. It is intuitively reasonable. If T_j is the total time that the process spends in state j during the period up to time τ , and N_j is the number of visits to j during that period, then

$$\Gamma = \sum_{j \in A} f_j T_j \quad \text{and} \quad T_j = \sum_{n=1}^{N_j} X_{jn},$$

where $\{X_{jn}, n = 1, 2, \dots\}$ are independent and identically distributed (i.i.d.) exponential random variables with parameter q_j . Since the distribution of N_j does not depend on the holding times, but rather on the transition probabilities $p_{ij} = q_{ij}/q_i$ of the jump chain, then, for states j with $f_j > 0$, $f_j T_j$ has the same distribution as the sum of N_j i.i.d. exponential random variables with parameter q_j/f_j . Therefore, since (in an obvious notation) $p_{ij}^* = p_{ij}$, and $q_i^* = q_i/f_i$ for all $i \in A$ such that $f_i > 0$, we would expect Γ to have the same distribution as τ for the modified chain. This observation will be useful in studying specific models for which the distribution and the expectation of τ are known in sufficient generality to accommodate state-dependent transition rates. For example, in the case of birth–death processes, there are explicit expressions for the expected value of various hitting times, and, expressions for transforms of their distributions are available in terms of continued fractions (see [4], [2]), while in several special cases the hitting time densities are known explicitly (see, for example, [3]). For several illustrations of the method, we refer the reader to [7].

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References

- [1] ANDERSON, W. (1991). *Continuous-Time Markov Chains: An Applications-Oriented Approach*. Springer, New York.
- [2] BALL, F. AND STEFANOV, V. (2001). Further approaches to computing fundamental characteristics of birth–death processes. *J. Appl. Prob.* **38**, 995–1005.
- [3] DI CRESCENZO, A. (1998). First-passage-time densities and avoiding probabilities for birth-and-death processes with symmetric sample paths. *J. Appl. Prob.* **35**, 383–394.
- [4] FLAJOLET, P. AND GUILLEMIN, F. (2000). The formal theory of birth-and-death processes, lattice path combinatorics and continued fractions. *Adv. Appl. Prob.* **32**, 750–778.
- [5] HERNÁNDEZ-SUÁREZ, C. AND CASTILLO-CHAVEZ, C. (1999). A basic result on the integral for birth–death Markov processes. *Math. Biosci.* **161**, 95–104.
- [6] MCNEIL, D. (1970). Integral functionals of birth and death processes and related limiting distributions. *Ann. Math. Statist.* **41**, 480–485.
- [7] POLLETT, P. AND STEFANOV, V. (2002). Path integrals for continuous-time Markov chains. Tech. Rep., Department of Mathematics, University of Queensland. Available at <http://www.maths.uq.edu.au/~pkp/>.
- [8] RUBINO, G. AND SERICOLA, B. (1989). Sojourn times in finite Markov processes. *J. Appl. Prob.* **27**, 744–756.
- [9] STEFANOV, V. AND WANG, S. (2000). A note on integrals for birth–death processes. *Math. Biosci.* **168**, 161–165.
- [10] SYSKI, R. (1992). *Passage Times for Markov Chains*. IOS Press, Amsterdam.