Product Form Approximations for Highly Linear Loss Networks with Trunk Reservation

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Abstract

We are concerned with the performance evaluation of loss networks, in particular under a form of admission control known as trunk reservation. The most important performance measures are the blocking probabilities for the various types of call (effectively the amount of potential revenue lost). For the uncontrolled system, under several regimes, the *Erlang Fixed Point* (EFP) method provides a good approximation for the blocking probabilities, but when these regimes are not operative the method does not perform as well. In many cases this is because the key assumption of independent blocking does not hold. Previously, we have proposed methods for estimating the blocking probabilities which specifically account for the dependencies between neighbouring links in a highly linear network. This paper explores some extensions of these methods to the case where circuits are reserved for certain types of traffic.

keywords: Approximation, Blocking probabilities, Erlang Fixed Point, Ring network

1 Introduction

We shall be concerned with circuit-switched networks of the kind depicted in Figure 1. These consist of a set of links indexed by $j = 1, 2 \dots, K$, with C_j circuits comprising each link j, and a collection of routes $\mathcal{R}$. Each route $r \in \mathcal{R}$ is a set of links. Calls using route r are offered at rate ν_r as a Poisson stream, and use $a_{jr}(\geq 0)$ circuits from link j. $\mathcal{R}$ indexes independent Poisson processes. Calls requesting route r are blocked and lost if, on *any* link j, there are fewer than a_{jr} available circuits. Otherwise, the call is connected and simultaneously holds a_{jr} circuits on each link j for the duration of the call. For simplicity, we shall take $a_{jr} \in \{0, 1\}$. Call durations are independent and identically distributed exponential random variables with unit mean, and are independent of the arrival processes.

Let $\boldsymbol{n} = (n_r,\ r \in \mathcal{R})$, where n_r is the number of calls in progress using route r, let $\boldsymbol{C} = (C_j,\ j = 1, \dots, K)$, and let $\boldsymbol{A} = (a_{jr}, r \in \mathcal{R},\ j = 1, \dots, K)$. Then the usual model for a circuit-switched network (see for example [11]) is a continuous-time Markov chain $(\boldsymbol{n}(t),\ t \geq 0)$ taking values in

$$S = S(\boldsymbol{C}) = \{\boldsymbol{n} \in \mathrm{Z}_+^{\mathrm{R}}\ :\ \boldsymbol{A}\boldsymbol{n} \leq \boldsymbol{C}\}$$

and its unique stationary distribution is given by

$$\pi(\boldsymbol{n}) = \Phi^{-1} \prod_{r \in \mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}, \quad \boldsymbol{n} \in S,$$

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where

$$\Phi = \Phi(\boldsymbol{C}) = \sum_{\boldsymbol{n}\in S(\boldsymbol{C})} \prod_{r\in\mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}.$$

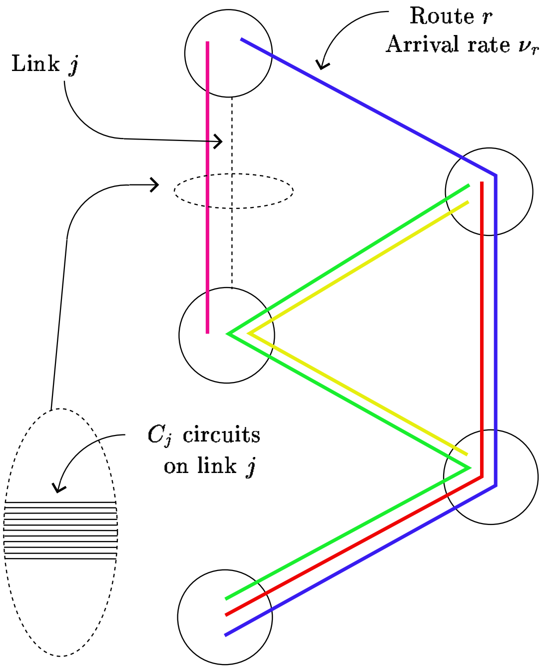

Fig 1. A typical circuit-switched network
(5 nodes, 6 links and 5 routes)

The stationary probability that a route-r call is blocked is then given by

$$1 - \frac{\Phi(\boldsymbol{C} - \boldsymbol{A}\boldsymbol{e}_r)}{\Phi(\boldsymbol{C})},$$

where $\boldsymbol{e}_r$ is the unit vector from $S(\boldsymbol{C})$ describing just one call in progress on route r. However, although we have an explicit expression for the blocking probability in terms of Φ, the latter can't (usually) be computed in polynomial time. Thus, for networks with even moderate capacity, one is forced to use alternative methods, and, arguably the most important of these is the EFP approximation.

Kelly [9] proved that, when the $a_{jr} \in \{0,1\}$, there is a unique vector $(B_1,\ldots,B_K) \in [0,1]^K$ satisfying

$$B_j = E(\rho_j, C_j), \tag{1}$$

$$\rho_j = (1-B_j)^{-1} \sum_r a_{jr}\nu_r(1-L_r), \tag{2}$$

for $j = 1,\ldots,K$, and

$$L_r = 1 - \prod_i (1-B_i)^{a_{ir}},\ r \in \mathcal{R}, \tag{3}$$

where

$$E(\nu, C) = \frac{\nu^C}{C!}\left(\sum_{n=0}^{C} \frac{\nu^n}{n!}\right)^{-1}.$$

$E(\nu, C)$ is *Erlang's Formula* for the loss probability on a single link with C circuits and Poisson traffic offered at rate ν. The EFP approximation is obtained by using B_j to estimate the probability that link j is full, and L_r to estimate the route-r blocking probability.

The rationale for the EFP approximation is one of *independent blocking*. If links along route r were blocked independently (they are clearly not) and if B_j were the link-j blocking probability, then L_r would be the route-r blocking probability:

$$L_r = 1 - \prod_{i \in r}(1 - B_i) = 1 - \prod_{i}(1 - B_i)^{a_{ir}}.$$

Carrying this further, the traffic offered to link j would be Poisson (at rate ρ_j, say) and the *carried traffic* (that which is accepted) on link j would be

$$\sum_r a_{jr}\nu_r(1 - L_r) \ (= (1 - B_j)\rho_j)\ .$$

The EFP approximation therefore stipulates that the link blocking probabilities $(B_1, \dots, B_K)$ should be consistent with this level of carried traffic. On combining (1), (2) and (3) we obtain a set of equations for $(B_1, \dots, B_K)$:

$$B_j = E\Big(\Big(\sum_r a_{jr}\nu_r \prod_{i \in r - \{j\}} (1 - B_i)\Big)\,,\, C_j\,\Big)\,.$$

The existence of the Erlang Fixed Point, namely a fixed point of these equations is easy to prove using the Brouwer fixed point theorem; they define a continuous mapping from a compact convex set $[0, 1]^K$ into itself. The uniqueness is considerably more difficult to prove [9]. We note that for more complex systems, there may be more than one fixed point (see for example [6]).

The EFP approximation performs particularly well under two limiting regimes. The first is one in which the topology of the network is held fixed, while capacities and arrival rates at the links increase together [9]; this has become known as the *Kelly limiting regime*, or (somewhat misleadingly) as the *heavy traffic limit*. Under the second limiting regime, called *diverse routing*, the number of links, and the number of routes which use these links, become large, while the capacities are held fixed and the arrival rates on multi-link routes become small. Examples of this are star networks and fully-connected networks with alternate routing [7, 8, 13, 17, 20]. If neither regime is operative, the EFP method may not perform as well: in particular, in highly linear networks and/or networks with low capacities. Further, adding controls to the network may cause the method to perform badly under otherwise favourable regimes. A particularly useful control is *trunk reservation*. Here, traffic streams are assigned priorities and calls are accepted only if the occupancies of links along their route are below a given threshold, the level of which depends on the type of call. This widely used control mechanism is typically very robust to fluctuations in arrival rate and has the added advantage of eliminating pathological behaviour such as bistability [6]. With such a control operating in a network of reasonable size, the occupancy of neighbouring links may be highly dependent and the equilibrium distribution will no longer have a product form, as it does for the corresponding uncontrolled network. Modelling dependencies in this context is thus critical. For an excellent review of the theory of loss networks, and in particular EFP methods, see Kelly [11].

We focus attention on simple, highly linear networks, since here the EFP approximation is expected to perform relatively poorly.

2 A Symmetric Ring Network

Consider a loss network with K links forming a loop, and each link having the same capacity C. Such a network is depicted in Figures 2 and 3. There are two types of traffic: one-link routes (type-1 traffic) and two-link routes comprising pairs of adjacent links (type-2 traffic). Type-t traffic is offered at rate ν_t on each type-t route. If L_t is the EFP approximation for

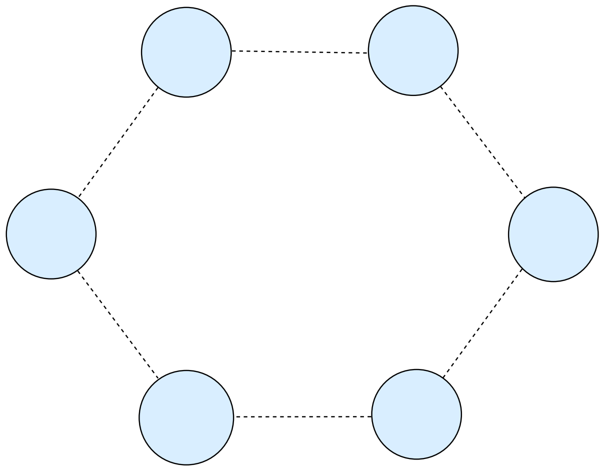

Fig 2. A ring network (6 nodes)

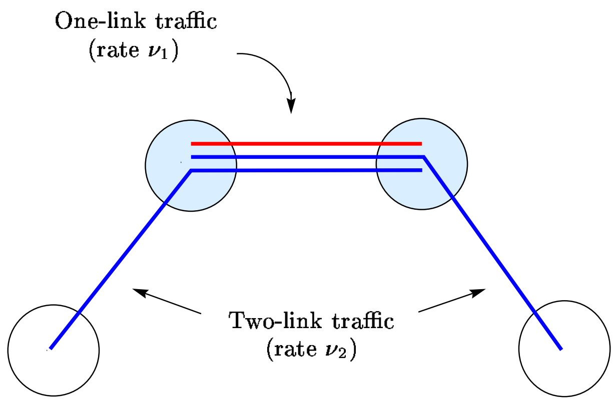

Fig 3. One- and two-link traffic using a given link

the loss probability of type-t calls, then it is easy to show that

$$L_1 = B \quad \text{and} \quad L_2 = 1 - (1 - B)^2,$$

where the Erlang Fixed Point B is the unique solution to

$$B = E(\nu_1 + 2\nu_2(1 - B), C),$$

where recall that $E(\nu, C)$ is Erlang's Formula.

To illustrate what may be a source of problems for the EFP approximation in the present context, we shall assess the dependence between two adjacent links. Figure 4 shows the correlation between links 1 and 2 for the network with $C = 10$, $K = 10$ and $\nu_1 = \nu_2 (= \nu)$; to be precise, we have plotted

$$\mathrm{Corr}\left(I_{\{n_{K1}+n_1+n_{12}<C\}}, I_{\{n_{12}+n_2+n_{23}<C\}}\right)$$

against the arrival rate ν.

The blocking probabilities can be estimated more accurately by specifically accounting for the dependencies between adjacent links. Bebbington *et al.* [1] proposed improvements to

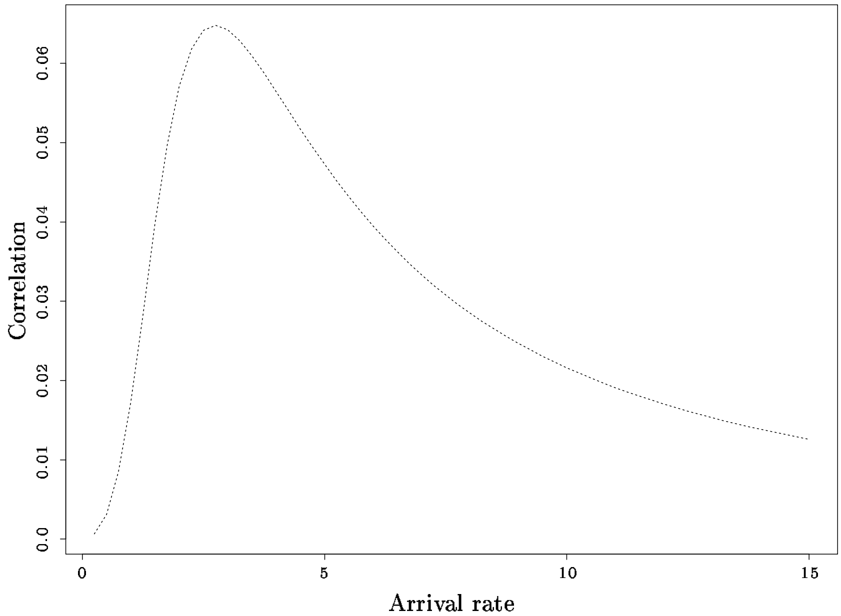

Fig 4. Correlation of spare capacity on adjacent links
($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate)

the fixed point approximations for the ring network without controls, using state-dependent arrival rates in two-link subnetworks. State-dependent arrival rates such as we describe below are also discussed by Pallant and Taylor [15] (see also [4, 5, 2]). We shall now briefly outline the basis of the improved approximations.

Take links 1 and 2 as reference links and consider the subnetwork depicted in Figure 5. We identify three routes: $\{1\}$, $\{2\}$ and $\{1,2\}$. If m_r denotes the number of calls on route r, then m_1 is the number of calls occupying capacity on link 1 *but not* on link 2, that is $m_1 = n_1 + n_{K1}$, m_2 is the number occupying capacity on link 2 *but not* on link 1, that is $m_2 = n_2 + n_{23}$, and, $m_{12} (= n_{12})$ is the number of calls occupying capacity on both links.

The first approximation (*Approximation I*) is obtained by adapting the method of Pallant [14]. In Pallant's method, the network is decomposed into independent subnetworks and the stationary distribution is evaluated for each. For example, if we take our subnetwork to be the one depicted in Figure 5, then its state space will be

$$S = \{(m_1, m_2, m_{12}) : m_i + m_{12} \le C, i = 1, 2\}$$

and its stationary distribution will be

$$\pi(\boldsymbol{m}) = \Phi^{-1} \frac{(\nu_1 + \nu_2(1-B))^{m_1+m_2} \nu_2^{m_{12}}}{m_1!\, m_2!\, m_{12}!},$$

where Φ is a normalizing constant. We then estimate B, the probability that a link adjacent to the two-link subnetwork is fully occupied, using the subnetwork itself; set

$$B = \sum_{m\,:\,m_1+m_{12}=C} \pi(m_1, m_2, m_{12}) = \sum_{m_{12}=0}^{C} \sum_{m_2=0}^{C-m_{12}} \pi(C - m_{12}, m_2, m_{12}). \qquad (4)$$

These expressions are used iteratively to determine a fixed point B, and we then set $L_1 = B$ and

$$L_2 = 2L_1 - \sum_{m_{12}=0}^{C} \pi(C - m_{12}, C - m_{12}, m_{12}).$$

A two-link subnetwork

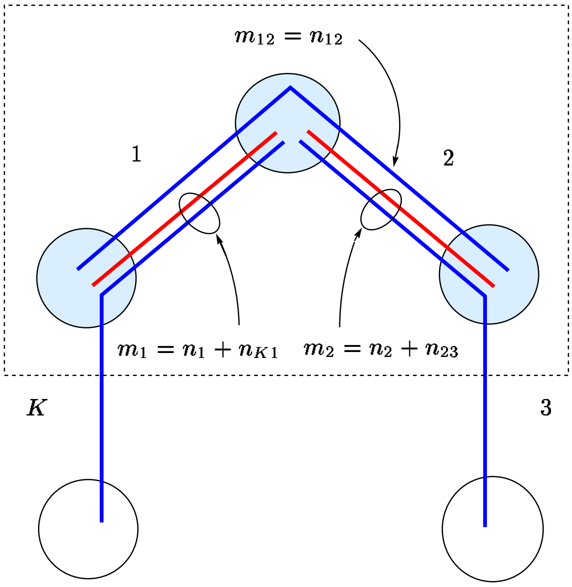

Fig 5. Definition of m_1, m_2 and m_{12}
for the symmetric ring network

A more accurate approximation (*Approximation II*) uses additional knowledge of the state of a given link in estimating the probability that the adjacent link is full. We use *state-dependent* arrival rates, $\rho_n = \nu_1 + \nu_2(1 - b_n)$, $n \in \{0, 1, \ldots, C-1\}$, where b_n is the probability that link K is fully occupied, conditional on $m_1 = n$ (b_n is also the probability that link 3 is fully occupied, conditional on $m_2 = n$), so that

$$\pi(\boldsymbol{m}) = \Phi^{-1} \frac{\nu_2^{m_{12}} \left(\prod_{n=0}^{m_1-1} \rho_n\right) \left(\prod_{n=0}^{m_2-1} \rho_n\right)}{m_1!\, m_2!\, m_{12}!} .$$

Once b_n is estimated and π determined, we set L_1 and L_2 as for Approximation I. An estimate of b_n is found by assuming that b_n does not depend on m_{12}. For $n = 0, \ldots, C-1$, we set

$$b_n = \frac{\sum_{m=0}^{n} p(n-m, C-m, m)}{\sum_{m=0}^{n} \sum_{r=0}^{C-m} p(n-m, r, m)} ,$$

where

$$p(n_1, m_K, n_{K1}) = \frac{\nu_1^{n_1} \nu_2^{n_{K1}} \left(\prod_{s=0}^{m_K-1} \rho_s\right)}{n_1!\, n_{K1}!\, m_K!} .$$

The dependence of b_n on m_{12} is due to the cyclic nature of the network, but is expected to be slight for large networks.

While Approximation I gives some improvement in accuracy over the EFP approximation, the improvement obtained using Approximation II is considerable, with relative error of the order of 10^{-5} of the EFP. See [1] for details.

3 Trunk Reservation

In a *trunk reservation* policy, a call of type-l is accepted on a link with capacity C only if there will be at least r_l circuits free on that link after the call is accepted. Such a policy is usually applied independently for each link involved in a multilink route. If each type

($l = 1, 2$) of call produces a reward w_l when accepted, then such a policy is optimal for a single link, although not necessarily for a multilink system (see, for example, [12]), although it may be asymptotically optimal for some special structures [8, 13].

The EFP can be extended in a straight-forward manner to cover the trunk reservation case, provided all call types have the same capacity requirements and holding time distributions, since in that case a one-dimensional description of the occupancy of each link suffices. If this is not the case, then a higher dimensional description of the occupancy of each link is required and the equilibrium distribution no longer has product form. Coyle *et al.* [3] discuss two ways of dealing with this difficulty, with the object of achieving a product form approximation. This is then combined, in their situation, with a modularization technique developed for Stochastic Petri Net models by Ciardo and Trevidi [4, 5]. We observe that in the two link subnetwork of Section 2, product form is lost immediately on the imposition of a control, such as trunk reservation.

The first method approximates the state of a single link by a birth and death process. This gives an approximate distribution $\{p_n\}$ for the number of occupied circuits as

$$p_n \propto \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}, \tag{5}$$

for $n = 0, 1, \ldots, C$, where λ_i and μ_i are the (circuit) arrival and clear-down rates when i circuits are occupied. Obviously it will be necessary to approximate, as in the EFP, the arrival due to two-link calls by assuming that the neighbouring link blocks independently.

The second method simply ignores the physical trunk reservation, instead *thinning* the unprotected stream by the amount which would be blocked due to trunk reservation. Thus we have the *reduced load* approximation on the two-dimensional state space, which then gives a product-form equilibrium distribution on that two-dimensional state space. (Note that Coyle *et al.* recommend truncating the state-space "appropriately".) In our case, we are interested in a situation with differing traffic types, which take the same number of circuits per link, rather than a different number of circuits on the same number of links, so some adaptation is necessary. We will have the reduced arrival rates

$$\nu_l^* = \nu_l \left(\frac{\sum_{i=0}^{C-r_l-1} p_i}{\sum_{i=0}^{C-1} p_i} \right)^l, \tag{6}$$

$l = 1, 2$, where the exponent l is due to the fact that the two-link traffic can run afoul of trunk reservation on both links. This will be slightly modified for two-link traffic when we consider conditional approximations. The paper [3] does not explain how the state probabilities $\{p_i\}$ are to be calculated. We shall estimate them by means of (5).

The approximate probabilities that a type-1 or type-2 call is blocked will be denoted by L_1, L_2 respectively. We will use $B^{(1)}$ and $B^{(2)}$ for the probabilities that a link has no circuit available for type-1 and type-2 calls, respectively.

Let us suppose first that type-2 traffic is only accepted when the occupancy is below a threshold $C - r_2$, thus reserving the last r_2 circuits on each link for type-1 traffic. Then

$$\lambda_i = \begin{cases} \nu_1 + 2\nu_2(1 - B^{(2)}) & i < C - r_2 \\ \nu_1 & C - r_2 \le i < C \\ 0 & i \ge C \end{cases} \tag{7}$$

and $\mu_i = i$. The first approximation, the standard extension of the EFP (see, for example [10] and hence denoted *Approximation Ea1*, is to let

$$\pi(n) = \Phi^{-1} \frac{(\nu_1 + 2\nu_2(1 - B^{(2)}))^{\min(n, C-r_2)} \nu_1^{[n-C+r_2]_+}}{n!},$$

where $[\cdot]_+$ denotes the positive part, $B^{(1)} = \pi(C)$, and the birth and death approximation (5) is used to approximate

$$B^{(2)} = f(B^{(1)}) = B^{(1)} \sum_{i=0}^{r_2} \frac{C!}{\nu_1^i (C-i)!}, \tag{8}$$

We then have

$$L_1 = B^{(1)}, \quad L_2 = 1 - (1 - B^{(2)})^2. \tag{9}$$

In the case of the reduced load approach, we obtain *Approximation Eb1* with

$$\pi(n) = \Phi^{-1} \frac{\left(\nu_1 + 2\nu_2 \left(\frac{1-B^*}{1-B}\right)^2 (1 - B)\right)^n}{n!},$$

where $B = \pi(C)$ and $B^* = f(B)$ is obtained from (8). The estimates for the one-link and two-link traffic blocking probabilities are

$$L_1 = B, \quad L_2 = 1 - (1 - B^*)^2. \tag{10}$$

Now suppose that type-1 traffic is only accepted when the occupancy is below a threshold $C - r_1$, thus reserving the last r_1 circuits on each link for type-2 traffic. Then

$$\lambda_i = \begin{cases} \nu_1 + 2\nu_2(1 - B^{(2)}) & i < C - r_1 \\ 2\nu_2(1 - B^{(2)}) & C - r_1 \leq i < C \\ 0 & i \geq C \end{cases} \tag{11}$$

and $\mu_i = i$. Again we have the standard extension of the EFP, *Approximation Ea2*, where

$$\pi(n) = \Phi^{-1} \frac{(\nu_1 + 2\nu_2(1 - B^{(2)}))^{\min(n, C-r_1)} (2\nu_2(1 - B^{(2)}))^{[n-C+r_1]_+}}{n!},$$

$B^{(2)} = \pi(C)$, and the birth and death approximation (5) is used to obtain

$$B^{(1)} = g(B^{(2)}) = B^{(2)} \sum_{i=0}^{r_1} \frac{C!}{(2\nu_2(1 - B^{(2)}))^i (C - i)!}. \tag{12}$$

The blocking is again estimated by (9).

In the case of the reduced load approximation, we have *Approximation Eb2* with

$$\pi(n) = \Phi^{-1} \frac{\left(\nu_1 \left(\frac{1-B^*}{1-B}\right) + 2\nu_2(1 - B)\right)^n}{n!},$$

where $B = \pi(C)$ and $B^* = g(B)$ is obtained from (12). The estimates for the type-1 and type-2 traffic blocking probabilities are then

$$L_1 = B^*, \quad L_2 = 1 - (1 - B)^2. \tag{13}$$

Now let us consider how we might attempt to improve on these approximations using the ideas of Section 2.

Consider a two link subnetwork, with state description (m_1, m_2, m_{12}) as in Figure 5, and with trunk reservation parameter $r_2 > 0$ on each link. Then the arrival rate for type-2 calls occupying both links is $\nu_2 I_{\{m_1+m_{12}<C-r_2\}} I_{\{m_2+m_{12}<C-r_2\}}$, and we approximate the arrival rate on link i, $i = 1, 2$, from other sources by $\nu_1 + \nu_2(1 - B^{(2)}) I_{\{m_i+m_{12}<C-r_2\}}$, provided the link is not full. Let $\tilde{B}$ be the probability that a link is at the trunk reservation threshold, i.e., that there are exactly $C - r_2$ circuits in use. Then

$$\tilde{B} = \sum_{m_{12}=0}^{C-r_2} \sum_{m_2=0}^{C-m_{12}} \pi(C - r_2 - m_{12}, m_2, m_{12}). \tag{14}$$

Using the birth and death approximation (5), we can then obtain

$$B^{(1)} = \tilde{B} \frac{\nu_1^{r_2} (C - r_2)!}{C!}, \tag{15}$$

and

$$B^{(2)} = \tilde{B} \sum_{i=0}^{r_2} \frac{\nu_1^i (C - r_2)!}{(C - r_2 + i)!}. \tag{16}$$

We can now produce *Approximation Ia1*, a product-form approximation for the equilibrium distribution (closely related to Approximation I of Section 2) with

$$\pi(m_1, m_2, m_{12}) = \Phi^{-1} \frac{(\nu_1 + \nu_2(1 - B^{(2)}))^{\sum_{l=1,2} \min(m_l, C - r_2)} \nu_1^{\sum_{l=1,2}[m_l - C + r_2]_+} \nu_2^{m_{12}}}{m_1! m_2! m_{12}!},$$

for $m_{12} \leq C - r_2$, where $B^{(1)}$ and $B^{(2)}$ are given by (14), (15), and (16). The estimated blocking probabilities are then

$$L_1 = B^{(1)}, \quad L_2 = 1 - \sum_{m_{12}=0}^{C-r_2-1} \sum_{m_1=0}^{C-r_2-1-m_{12}} \sum_{m_2=0}^{C-r_2-1-m_{12}} \pi(m_1, m_2, m_{12}). \tag{17}$$

Alternatively, using the reduced load approach we obtain *Approximation Ib1*, with

$$\pi(m_1, m_2, m_{12}) = \Phi^{-1} \frac{\left(\nu_1 + \nu_2 \left(\frac{1 - B^*}{1 - B} \right)^2 (1 - B) \right)^{m_1 + m_2} \left(\nu_2 \left(\frac{1 - B^*}{1 - B} \right)^2 \right)^{m_{12}}}{m_1! m_2! m_{12}!},$$

where

$$B = \sum_{m_{12}=0}^{C} \sum_{m_2=0}^{C-m_{12}} \pi(C - m_{12}, m_2, m_{12}) \tag{18}$$

is the blocking probability, and $B^* = f(B)$ is obtained from (8). The estimates for the type-1 and type-2 traffic blocking probabilities are again given by (10).

Now suppose that r_1 circuits on each link are reserved for type-2 traffic. Then the arrival rate for type-2 calls occupying both links is ν_2, and the arrival rate on link i, $i = 1, 2$, from other sources is given by $\nu_1 I_{\{m_i + m_{12} < C - r_1\}} + \nu_2(1 - B^{(2)})$, provided the link is not full. Again, let $\tilde{B}$ be the probability that a link is at the trunk reservation threshold, i.e., that there are exactly $C - r_1$ circuits in use. The two-link subnetwork has

$$\tilde{B} = \sum_{m_{12}=0}^{C-r_1} \sum_{m_2=0}^{C-m_{12}} \pi(C - r_1 - m_{12}, m_2, m_{12}), \tag{19}$$

and the birth and death approximation (5) then gives

$$B^{(1)} = \tilde{B} \sum_{i=0}^{r_1} \frac{(2\nu_2(1 - B^{(2)}))^i (C - r_1)!}{(C - r_1 + i)!}, \tag{20}$$

and $B^{(2)}$ as the (unique) solution of the equation

$$B^{(2)} = \tilde{B} \frac{(2\nu_2(1 - B^{(2)}))^{r_1} (C - r_1)!}{C!}. \tag{21}$$

This results in *Approximation Ia2*, with

$$\pi(m_1, m_2, m_{12}) = \Phi^{-1} \frac{(\nu_1 + \nu_2(1 - B^{(2)}))^{\sum_{l=1,2} \min(m_l, C - r_1)} (\nu_2(1 - B^{(2)}))^{\sum_{l=1,2}[m_l - C + r_1]_+} \nu_2^{m_{12}}}{m_1! m_2! m_{12}!},$$

where $B^{(1)}$ and $B^{(2)}$ are given by (19), (20), and (21). The blocking estimates are

$$L_1 = B^{(1)}, \quad L_2 = 1 - \sum_{m_{12}=0}^{C-1} \sum_{m_1=0}^{C-1-m_{12}} \sum_{m_2=0}^{C-1-m_{12}} \pi(m_1, m_2, m_{12}). \tag{22}$$

With the reduced load approach, we obtain *Approximation Ib2* with

$$\pi(m_1, m_2, m_{12}) = \Phi^{-1} \frac{\left(\nu_1 \left(\frac{1-B^*}{1-B}\right) + \nu_2(1-B)\right)^{m_1+m_2} \nu_2^{m_{12}}}{m_1! m_2! m_{12}!},$$

where B is again given by (18) and $B^* = g(B)$ is found from (12). The blocking estimates are again given by (13).

While Approximation II of Section 2 does not lend itself to exploitation via the birth-death or reduced-load approaches, we can construct a weakened version by replacing the estimate for B in Approximation I with an estimate of the probability that a link is full conditional on the adjacent link having at least one circuit free. We begin by noting that the probability that link 2 has some free capacity in the two-link subnetwork (Figure 5) is given by

$$\sum_{\mathbf{m}: m_2+m_{12}<C} \pi(m_1, m_2, m_{12}) = \sum_{m_{12}=0}^{C-1} \sum_{m_1=0}^{C-m_{12}} \sum_{m_2=0}^{C-1-m_{12}} \pi(m_1, m_2, m_{12}),$$

The probability that link 1 is fully occupied and link 2 has some free capacity is

$$\sum_{\mathbf{m}: m_1+m_{12}=C, m_2+m_{12}<C} \pi(m_1, m_2, m_{12}) = \sum_{m_{12}=0}^{C-1} \sum_{m_2=0}^{C-1-m_{12}} \pi(C - m_{12}, m_2, m_{12}).$$

Hence we replace the estimate of B in (4) with

$$\hat{B} = \frac{\sum_{m_{12}=0}^{C-1} \sum_{m_2=0}^{C-1-m_{12}} \pi(C - m_{12}, m_2, m_{12})}{\sum_{m_{12}=0}^{C-1} \sum_{m_1=0}^{C-m_{12}} \sum_{m_2=0}^{C-1-m_{12}} \pi(m_1, m_2, m_{12})}, \tag{23}$$

which is an estimate of the probability that a link is full given that its adjacent link has at least one unit of free capacity. As a matter of interest, in the uncontrolled system, this is more accurate than Approximation I when the arrival rates are high (relative to the capacities).

In the birth and death approximation, the equations are almost identical to the Approximation I_ cases, but in order to obtain *Approximation IIa1* (type-1 traffic protected), we substitute for (16) with

$$B^{(2)} = \frac{\sum_{m_{12}=0}^{C-r_2-1} \sum_{m_2=0}^{C-r_2-1-m_{12}} \pi(C - r_2 - m_{12}, m_2, m_{12})}{\sum_{m_{12}=0}^{C-r_2-1} \sum_{m_1=0}^{C-m_{12}} \sum_{m_2=0}^{C-r_2-1-m_{12}} \pi(m_1, m_2, m_{12})} \sum_{i=0}^{r_2} \frac{\nu_1^i (C - r_2)!}{(C - r_2 + i)!},$$

where the term before the summation is the probability that a link is at the trunk reservation threshold conditional on the adjacent link being below the threshold. The blocking estimates are again given by (17).

Similarly, in order to obtain *Approximation IIa2* (type-2 traffic protected), we must substitute for (21) with

$$B^{(2)} = \frac{\sum_{m_{12}=0}^{C-r_1} \sum_{m_2=0}^{C-1-m_{12}} \pi(C - r_1 - m_{12}, m_2, m_{12})}{\sum_{m_{12}=0}^{C-1} \sum_{m_1=0}^{C-m_{12}} \sum_{m_2=0}^{C-1-m_{12}} \pi(m_1, m_2, m_{12})} \times \frac{(2\nu_2(1 - B^{(2)}))^{r_1} (C - r_1)!}{C!},$$

where the first term is the probability that a link is at threshold, conditional on the adjacent link not being full. The blocking estimates are again given by (22).

In the reduced load approach we must replace the independent probabilities in (6) by the two link conditional probability that a two-link call is rejected on at least one of the links. If type-1 traffic is protected we thus obtain *Approximation IIb1*, with

$$\pi(m_1, m_2, m_{12}) = \Phi^{-1} \frac{\left(\nu_1 + \nu_2 \frac{(1-B^*)(1-\hat{B}^*)}{(1-B)(1-\hat{B})}(1 - \hat{B})\right)^{m_1+m_2} \left(\nu_2 \frac{(1-B^*)(1-\hat{B}^*)}{(1-B)(1-\hat{B})}\right)^{m_{12}}}{m_1! m_2! m_{12}!},$$

where B is found from (18), $B^* = f(B)$ from (8), $\hat{B}$ from (23), and $\hat{B}^* = f(\hat{B})$ from (8). The estimates for the one-link and two-link traffic blocking probabilities are

$$L_1 = B, \quad L_2 = 1 - (1 - B^*)(1 - \hat{B}^*).$$

While, if we protect type-2 traffic, we obtain *Approximation IIb2* with

$$\pi(m_1, m_2, m_{12}) = \Phi^{-1} \frac{\left(\nu_1 \left(\frac{1-B^*}{1-B}\right) + \nu_2(1-\hat{B})\right)^{m_1+m_2} \nu_2^{m_{12}}}{m_1! m_2! m_{12}!},$$

where B is found from (18), $B^* = g(B)$ from (12), and $\hat{B}$ from (23). The estimates for the type-1 and type-2 traffic blocking probabilities are then

$$L_1 = B^*, \quad L_2 = 1 - (1 - B)(1 - \hat{B}).$$

Coyle *et al.* [3], use the approximations (Ea_), (Eb_) and (Ib_) in analysing a four link ring network, resulting in relative errors of 4%-10% and upwards against simulated values.

4 Numerical Results

In order to investigate the accuracy (or otherwise) of the approximations, they were computed for the system with $C = 10$, and varying traffic rates $\nu_1 = \nu_2$, with $(r_1, r_2) = \{(1,0), (0,1)\}$. The resulting values were compared with the simulated [16] proportion of calls blocked in a system with $K = 24$, over 50 time units, after a presample period of 100 time units, deemed long enough to discount transient effects [18]. 95% confidence intervals for the simulated blocking probabilities are given, calculated from 100 repetitions.

In some cases, notably Eb2, IIb2 and all the _b1 approximations, the algorithms failed to converge in a straightforward manner, with blocking probabilities either exceeding unity, or displaying two-point oscillation. This was remedied by applying a bisection procedure, with the next iterate being the average of the previous two. Iterates exceeding unity were reset to 1 before bisecting.

When type-1 traffic was protected the results are shown in Table 1, while if type-2 traffic is protected we get the results in Table 2.

$\nu_1 (= \nu_2)$		SIM	Ea1	Eb1	Ia1	Ib1	IIa1	IIb1
2.00	L_1	$.0122 \pm .0005$	.0124	.0209	.0126	.0207	.0127	.0209
	L_2	$.1353 \pm .0021$	.1432	.2346	.1780	.2357	.1795	.2301
3.00	L_1	$.0468 \pm .0009$	.0465	.0598	.0464	.0596	.0469	.0604
	L_2	$.3578 \pm .0021$	.3621	.4513	.4543	.4498	.4593	.4459
3.33	L_1	$.0598 \pm .0010$	.0605	.0743	.0596	.0741	.0602	.0752
	L_2	$.4141 \pm .0022$	.4253	.5062	.5299	.5051	.5353	.5011
3.67	L_1	$.0765 \pm .0010$	.0751	.0892	.0731	.0890	.0737	.0902
	L_2	$.4781 \pm .0020$	.4818	.5543	.5949	.5533	.6002	.5493
4.00	L_1	$.0908 \pm .0012$	.0902	.1052	.0865	.1040	.0871	.1054
	L_2	$.5298 \pm .0019$	.5318	.5962	.6501	.5955	.6550	.5916
5.00	L_1	$.1357 \pm .0012$	.1361	.1490	.1253	.1489	.1257	.1507
	L_2	$.6509 \pm .0016$	.6499	.6942	.7699	.6940	.7730	.6905

Table 1: Blocking probabilities for $r_2 = 1$.

How do the approximations do? Firstly we see that, in general, the simplest approximations (Ea1 and Ea2) seem to be the best, and most robust. The other approximations are quite erratic although most of them are occasionally successful. Also, the Ia_ and IIa_

$\nu_1(=\nu_2)$		SIM	Ea2	Eb2	Ia2	Ib2	IIa2	IIb2
2.00	L_1	$.0952 \pm .0021$	.0954	.1219	.0927	.1181	.0928	.1203
	L_2	$.0504 \pm .0015$	.0528	.0668	.0731	.0645	.0732	.0642
3.00	L_1	$.2743 \pm .0024$	.2721	.3080	.2560	.2960	.2567	.3072
	L_2	$.1710 \pm .0019$	.1823	.2033	.2444	.2033	.2450	.1974
3.33	L_1	$.3234 \pm .0022$	.3265	.3615	.3037	.3468	.3045	.3614
	L_2	$.2089 \pm .0020$	.2268	.2474	.3016	.2505	.3023	.2410
3.67	L_1	$.3854 \pm .0025$	.3760	.4093	.3458	.3920	.3467	.4099
	L_2	$.2573 \pm .0019$	.2688	.2884	.3547	.2955	.3555	.2818
4.00	L_1	$.4284 \pm .0023$	.4205	.4519	.3826	.4320	.3835	.4531
	L_2	$.2923 \pm .0020$	.3078	.3262	.4033	.3377	.4040	.3195
5.00	L_1	$.5434 \pm .0023$	.5282	.5537	.4666	.5268	.4674	.5563
	L_2	$.3889 \pm .0019$	.4068	.4212	.5225	.4471	.5230	.4150

Table 2: Blocking probabilities for $r_1 = 1$.

approximations seem to systematically overestimate L_2 and (in the $r_1 = 1$ case) underestimate L_1.

This would seem to indicate that adapting the approximations of [1] in this way is unsatisfactory, since it fails to outperform the simple EFP/birth-death approximation. The relative error in the latter, for the few cases examined, is given in Table 3. From this it becomes

$\nu_1(=\nu_2)$		Ea1 ($r_2 = 1$)	Ea2 ($r_1 = 1$)
2.00	L_1	.016	.002
	L_2	.059	.047
3.00	L_1	.006	.007
	L_2	.011	.064
3.33	L_1	.024	.009
	L_2	.027	.086
3.67	L_1	.018	.024
	L_2	.006	.047
4.00	L_1	.007	.019
	L_2	.004	.055
5.00	L_1	.003	.027
	L_2	.002	.046

Table 3: Relative errors in the robust approximations.

obvious that the greatest systematic deficiencies are when two-link traffic is protected, and in predicting the blocking probability for two-link traffic. The later is understandable, due to the additional level of approximation in calculating L_2 from L_1. The level of the relative error in Approximation Ea_ is comparable to that found when using the EFP approximation in the uncontrolled system [1].

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