Integrals for Continuous-time Markov chains

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Abstract

This paper presents a method of evaluating the expected value of a path integral for a general Markov chain on a countable state space. We illustrate the method with reference to several models, including birth-death processes and the birth, death and catastrophe process.

1 Introduction

Let $(X(t),\ t \geq 0)$ be a continuous-time Markov chain taking values in the non-negative integers $S = \{0, 1, \dots\}$ and consider the path integral

$$\Gamma_0(f) = \int_0^{\tau_0} f(X(t))\, dt\,, \tag{1}$$

where $\tau_0 = \inf\{t > 0 : X(t) = 0\}$ is the first hitting time of state 0, and f is a given function that maps S to $[0, \infty)$. Stefanov and Wang [10] derived an explicit expression for $\mathrm{E}_i(\Gamma_0(f)) := \mathrm{E}(\Gamma_0(f)|X(0) = i)$, $i \geq 1$, in the case of a birth-death process, under the condition that f is eventually non-decreasing. This built on earlier work of Hernández-Suárez and Castillo-Chavez [4], who considered the case $i = 1$ and $f(x) = x$. However, results from potential theory subsume these results and allow one to extend substantially the work of these authors, thus allowing the expectation of $\Gamma_0(f)$ to be evaluated explicitly for a much wider variety of models. We shall provide several illustrations, giving special attention to birth-death processes and the birth, death and catastrophe process. We shall see that, in the case of a birth-death process, the condition on f, imposed by Stefanov and Wang, can be relaxed completely.

We will consider the more general problem of evaluating $\mathrm{E}_i(\Gamma(f))$ for the path integral

$$\Gamma(f) = \int_0^{\infty} f(X(t))\, dt\,. \tag{2}$$

The solution to the original problem can then be obtained by making 0 an absorbing state and setting $f(0) = 0$. As pointed out in [4], it is useful, particularly in biological contexts, to think of $f(x)$ as being the cost (or reward) per unit time of staying in state x. The problem, then, is to evaluate the expected total cost over the life of the process. For convenience, we shall write $f_i = f(i)$ whenever this simplifies notation.

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2 Expectation of the path integral

Let $Q = (q_{ij},\ i,j \in S)$ be the q-matrix of transition rates of the chain (assumed to be stable and conservative), so that q_{ij} represents the rate of transition from state i to state j, for $j \neq i$, and $q_{ii} = -q_i$, where $q_i := \sum_{j \neq i} q_{ij}$ $(< \infty)$ represents the total rate out of state i. For example, in the case of a birth-death process we would have $q_{i,i+1} = \lambda_i$ and $q_{i,i-1} = \mu_i$ with $\mu_0 = 0$, where (λ_i) and (μ_i) are the birth and death rates of the process. It will not be necessary to assume that Q is regular, so that there may actually be many processes with the given set of rates. However, we will take $(X(t),\ t \geq 0)$ to be the *minimal process* associated with Q. Its transition function $P(t) = (p_{ij}(t),\ i,j \in S)$ is the minimal solution to the Kolmogorov backward equations, and has the following interpretation:

$$p_{ij}(t) = \Pr(X(t) = j, N(t) < \infty | X(0) = i)\,,$$

where $N(t)$ is the number of jumps of the process up to time t. For further technical details, see [1]. Whilst it is commonly assumed that Q is regular, which is equivalent to $N(t)$ being almost surely finite for all t, it is often useful in biological applications, and particularly when dealing with birth-death processes, to allow for the possibility that the process might explode by performing infinitely-many transitions in a finite time. The following result can be found in any text on Markov chains that deals with potential theory (for example, Theorem 4.2.4 of [8]).

Proposition 1 If $y_i = \mathrm{E}_i(\Gamma(f))$, where $\Gamma(f)$ is given by (2), then $y = (y_i,\ i \in S)$ is the minimal non-negative solution to the system of equations

$$\sum_{j \in S} q_{ij} z_j + f_i = 0\,, \qquad i \in S, \tag{3}$$

in the sense that y *satisfies* these equations, and, if $z = (z_i,\ i \in S)$ is *any* non-negative solution, then $y_i \leq z_i$ for all $i \in S$.

Returning to the original problem formulated in the introduction, where we sought to evaluate expectations of the path integral $\Gamma_0(f) = \int_0^{\tau_0} f(X(t))\,dt$ with τ_0 being the first hitting time of state 0, we may simply apply Proposition 1 to the chain modified so that 0 is an absorbing state, that is $q_0 = 0$ and $q_{0j} = 0$ for $j \geq 1$, and set $f(0) = 0$ (when $\tau_0 < \infty$, there is no contribution to the path integral (2) for $t > \tau_0$). Of course, this is the context in which we would most likely apply the result: where the expected value of the path integral is evaluated up to absorption.

Corollary 1 If $e_i = \mathrm{E}_i(\Gamma_0(f))$, where $\Gamma_0(f)$ is given by (1), then $e = (e_i,\ i \geq 1)$ is the minimal non-negative solution to the system of equations

$$\sum_{j \geq 1} q_{ij} z_j + f_i = 0\,, \qquad i \geq 1. \tag{4}$$

Remarks. (i) If we set $f(x) = 1$, then $\Gamma_0(f)$ records the time of first leaving the set $E = \{1, 2, \dots\}$, that is $\Gamma_0(f) = \min\{\tau_0, \tau_\infty\}$, where $\tau_\infty = \sup\{t > 0 : N(t) < \infty\}$. If Q is assumed to be regular, so that τ_∞ is almost surely infinite, then Corollary 1 reduces to a well known and widely used result on expected hitting times that can be found in any text on Markov chains (for example, Theorem 3.3.3 of [8]). Despite its obvious use in evaluating expected extinction times, it is apparently not widely known to biologists; in their paper "Four facts every conservation biologist should know about persistence", Mangel and Tier [6] implore their readers to use it: Fact 2 "There is a simple and direct method for the computation of persistence times that virtually all biologists can use".

(ii) If, more generally, τ_0 is replaced by the hitting time of a *set* A, that is $\tau_A = \inf\{t > 0 : X(t) \in A\}$, then the expected value of the resulting path integral $\Gamma_A(f)$ can be evaluated. If $e_i = \mathrm{E}_i(\Gamma_A(f))$ for $i \in A^c$, then $e = (e_i,\ i \in A^c)$ will be the minimal non-negative solution to $\sum_{j \in A^c} q_{ij} z_j + f_i = 0$, $i \in A^c$.

(iii) On dividing equation (4) by f_i (assuming here that $f_j > 0$ for all j), we see that $\mathrm{E}_i(\Gamma_0(f))$ is the same as the expected hitting time of state 0, starting in state i, for the Markov chain with transition rates $Q^* = (q^*_{ij},\ i, j \in S)$ given by $q^*_{ij} = q_{ij}/f_i$, for $i \geq 1$, and $q^*_{0j} = q_{0j}$. This was observed for birth-death processes by McNeil [7]. Indeed, he observed that, conditional on $X(0) = j$, the *distribution* of $\Gamma_0(f)$ is the same as that for τ for the Markov chain with transition rates Q^*. A similar observation can be made in respect of equation (3): $\mathrm{E}_i(\Gamma(f))$ is the same as the expected time to explosion, stating in state i, for the Markov chain with transition rates Q^* given by $q^*_{ij} = q_{ij}/f_i$, for $i \in S$.

3 Birth-death processes

In the case of birth-death processes, explicit formulae can be obtained. This was done for $\mathrm{E}_i(\Gamma_0(f))$ by Stefanov and Wang [10], assuming that there exists a positive number x_0 such that f is non-decreasing for $x \geq x_0$, but, as we shall see, this condition can be relaxed.

Let $q_{i,i+1} = \lambda_i$ for $i \geq 0$ and $q_{i,i-1} = \mu_i$ for $i \geq 1$, where $(\lambda_i,\ i \geq 0)$ and $(\mu_i,\ i \geq 1)$ are sets of strictly positive birth and death rates, with all other transition rates being 0. Under these conditions, S is irreducible. For simplicity, set $\mu_0 = 0$, so that $q_i = \lambda_i + \mu_i$, $i \in S$.

First consider the system of equations (3). For the birth-death process, they can be written $\lambda_i \Delta_i - \mu_i \Delta_{i-1} + f_i = 0$, $i \geq 1$, and $\lambda_0 \Delta_0 + f_0 = 0$, where $\Delta_i = z_{i+1} - z_i$, $i \geq 0$, and can be solved iteratively to obtain

$$\Delta_i = -\frac{1}{\lambda_i \pi_i} \sum_{j=0}^{i} f_j \pi_j\,, \qquad i \geq 0, \tag{5}$$

where the *potential coefficients* $(\pi_j,\ j \in S)$ are given by $\pi_0 = 1$ and

$$\pi_i = \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j}\,, \qquad i \geq 1.$$

Summing (5) from $i = 0$ to $j - 1$ gives $z_j = z_0 - C_j(f)$, where

$$C_j(f) = \sum_{i=0}^{j-1} \frac{1}{\lambda_i \pi_i} \sum_{k=0}^{i} f_k \pi_k\,.$$

Notice that $C_j(f) \uparrow C(f)$ as $j \to \infty$, where

$$C(f) = \sum_{i=0}^{\infty} \frac{1}{\lambda_i \pi_i} \sum_{k=0}^{i} f_k \pi_k \,.$$

Thus, for a finite non-negative solution, we require $C(f) < \infty$ and, then, $z_0 \geq C(f)$. The *minimal* non-negative solution is obtained on setting $z_0 = C(f)$. We have proved the following result.

Proposition 2 For the birth-death process,

$$\mathrm{E}_j(\Gamma(f)) = \sum_{i=j}^{\infty} \frac{1}{\lambda_i \pi_i} \sum_{k=0}^{i} f_k \pi_k \,, \qquad j \in S.$$

Remark. On setting $f(x) = 1$ we obtain the well known result that the expected time to explosion starting in state j is given by

$$\mathrm{E}_j(\tau_\infty) = \sum_{i=j}^{\infty} \frac{1}{\lambda_i \pi_i} \sum_{k=0}^{i} \pi_k \,, \qquad j \in S.$$

Indeed, for the birth-death process, τ_∞ is almost surely infinite if and only if $\mathrm{E}_j(\tau_\infty) = \infty$ for some (and then all) j; see [1].

Next let us consider equations (4). We seek the minimal non-negative solution $(z_i,\ i \geq 1)$ to the system

$$\begin{aligned} \lambda_i z_{i+1} - (\lambda_i + \mu_i) z_i + \mu_i z_{i-1} + f_i = 0 \,, \qquad i \geq 2, \\ \lambda_1 z_2 - (\lambda_1 + \mu_1) z_1 + f_1 = 0 \,. \end{aligned} \tag{6}$$

If we set $z_0 = 0$ and $\Delta_i = z_{i+1} - z_i$, $i \geq 0$, these can be written $\lambda_i \Delta_i - \mu_i \Delta_{i-1} + f_i = 0$, $i \geq 1$, and can be solved iteratively to obtain

$$\Delta_i = z_1 \frac{\mu_1}{\lambda_i \pi_i} - \frac{1}{\lambda_i \pi_i} \sum_{j=1}^{i} f_j \pi_j = \frac{1}{\lambda_i \pi_i} \left(z_1 \mu_1 - \sum_{j=1}^{i} f_j \pi_j \right) , \qquad i \geq 1, \tag{7}$$

where the potential coefficients $(\pi_j,\ j \geq 1)$ are now given by $\pi_1 = 1$ and

$$\pi_i = \prod_{k=2}^{i} \frac{\lambda_{k-1}}{\mu_k} \,, \qquad i \geq 2.$$

Summing (7) from $i = 1$ to $j - 1$ gives, for $j \geq 2$,

$$z_j = z_1 \left(1 + \mu_1 \sum_{i=1}^{j-1} \frac{1}{\lambda_i \pi_i} \right) - \sum_{i=1}^{j-1} \frac{1}{\lambda_i \pi_i} \sum_{k=1}^{i} f_k \pi_k = z_1 + \sum_{i=1}^{j-1} \frac{1}{\lambda_i \pi_i} \left(z_1 \mu_1 - \sum_{k=1}^{i} f_k \pi_k \right) .$$

Using the fact that $\lambda_i \pi_i = \mu_{i+1} \pi_{i+1}$, we arrive at

$$z_j = \sum_{i=1}^{j} \frac{1}{\mu_i \pi_i} \left(z_1 \mu_1 - B_i(f) \right) , \quad \text{where} \quad B_i(f) = \sum_{k=1}^{i-1} f_k \pi_k \,, \tag{8}$$

interpreting an empty sum as being 0. This expression is valid also for $j = 1$. Now let

$$A_j = \sum_{i=1}^{j} \frac{1}{\mu_i \pi_i}$$

and observe that, as $j \to \infty$, $A_j \uparrow A$ and $B_j(f) \uparrow B(f)$, where

$$A = \sum_{i=1}^{\infty} \frac{1}{\mu_i \pi_i} \quad \text{and} \quad B(f) = \sum_{k=1}^{\infty} f_k \pi_k \,.$$

Both of these sums may converge or diverge. We shall consider the cases $A = \infty$ and $A < \infty$ separately.

The case $A = \infty$. Under this condition, the process is non-explosive and τ_0 is almost surely finite (see [1]). Since $A = \infty$ the sequence $\{z_j\}$ will eventually become negative unless $B(f) \le z_1 \mu_1$. So, if $B(f) = \infty$, the solution to (6) will be infinite. Otherwise, the minimal solution is obtained on setting $z_1 \mu_1 = B(f)$. Thus, we have proved the following result, which generalizes Proposition 1 of Stefanov and Wang [10].

Proposition 3 For the birth-death process with $A = \infty$,

$$\mathrm{E}_j(\Gamma_0(f)) = \sum_{i=1}^{j} \frac{1}{\mu_i \pi_i} \sum_{k=i}^{\infty} f_k \pi_k \,, \qquad j \ge 1,$$

and this is finite if and only if $\sum_{k=1}^{\infty} f_k \pi_k < \infty$.

Remark. On setting $f(x) = 1$ we obtain the well known result that if the process hits 0 with probability 1, the expected first passage time to 0 starting in state j is given by

$$\mathrm{E}_j(\tau_0) = \sum_{i=1}^{j} \frac{1}{\mu_i \pi_i} \sum_{k=i}^{\infty} \pi_k \,, \qquad j \ge 1,$$

and this is finite if and only if $\sum_{k=1}^{\infty} \pi_k < \infty$.

The case $A < \infty$. In view of (8), we may write $z_j = z_1 \mu_1 A_j - C_j(f)$, where, now, $C_1(f) = 0$ and, for $j \ge 2$,

$$C_j(f) = \sum_{i=2}^{j} \frac{1}{\mu_i \pi_i} \sum_{k=1}^{i-1} f_k \pi_k = \sum_{i=1}^{j-1} \frac{1}{\lambda_i \pi_i} \sum_{k=1}^{i} f_k \pi_k \,.$$

As $j \to \infty$, $A_j \uparrow A < \infty$ and $C_j(f) \uparrow C(f)$, where

$$C(f) = \sum_{i=2}^{\infty} \frac{1}{\mu_i \pi_i} \sum_{k=1}^{i-1} f_k \pi_k = \sum_{i=1}^{\infty} \frac{1}{\lambda_i \pi_i} \sum_{k=1}^{i} f_k \pi_k \,.$$

If $C(f) = \infty$, then certainly sequence $\{z_j\}$ will eventually become negative. Otherwise, we require $\mu_1 z_1 \ge M(f)$, where $M(f) = \sup_{j \ge 1}(C_j(f)/A_j)$, in order to avoid this happening. The minimal solution is then obtained on setting $z_1 \mu_1 = M(f)$. Thus, we have proved the following result.

Proposition 4 For the birth-death process with $A < \infty$,

$$\mathrm{E}_j(\Gamma_0(f)) = \sum_{i=1}^{j} \frac{1}{\mu_i \pi_i} \left(M(f) - \sum_{k=1}^{i-1} f_k \pi_k \right), \qquad j \geq 1,$$

and this is finite if and only if $C(f) < \infty$.

Remark. This result is not covered by Proposition 1 of Stefanov and Wang [10], for they have effectively assumed that $A = \infty$. Their method of proof relies on a state-space truncation argument, whereby the process is approximated by a finite-state birth-death process on $S_n = \{0, 1, \ldots, n\}$ with a *reflecting barrier* at n. This latter stipulation means that the resulting process may not be the minimal process, though it will when

$$C := \sum_{i=1}^{\infty} \frac{1}{\lambda_i \pi_i} \sum_{k=1}^{i} \pi_k = \infty .$$

However, the limit process will always have $A = \infty$; see [5].

Example 1. In order to illustrate the results of this section, we will consider the birth-death process on $S = \{0, 1, \ldots\}$ with $\lambda_i = \lambda$ and $\mu_i = \mu$ (both strictly positive). This is a simple model for a population that allows immigration at rate λ and removal rate μ. (It is also known as the $M/M/1$ queue, and referred to as such in both [4] and [10].)

Let us first apply Proposition 2. It is easy to see that $\pi_i = \rho^i$, $i \geq 0$, where $\rho = \lambda/\mu$, and so

$$\mathrm{E}_j(\Gamma(f)) = \frac{1}{\lambda} \sum_{i=j}^{\infty} \sum_{k=0}^{i} f_k r^{i-k}, \qquad j \geq 0,$$

where $r = 1/\rho$. To illustrate this further, take $f_k = \alpha^k$, where $\alpha > 0$. If $\alpha = r$, then the expectation is finite if and only if $r < 1$ (that is $\lambda > \mu$), in which case

$$\mathrm{E}_j(\Gamma(f)) = \frac{(1 + (1-r)j) r^j}{\lambda (1-r)^2}, \qquad j \geq 0.$$

If $\alpha \neq r$, then the expectation is finite if and only if $\max\{\alpha, r\} < 1$, which requires $\lambda > \mu$, and in which case

$$\mathrm{E}_j(\Gamma(f)) = \frac{(1-\alpha) r^{j+1} - (1-r) \alpha^{j+1}}{\lambda (r-\alpha)(1-\alpha)(1-r)}, \qquad j \geq 0.$$

Next, we shall evaluate $\mathrm{E}_j(\Gamma_0(f))$ using Propositions 3 and 4. First observe that

$$A_j = \frac{1 - r^j}{\mu(1-r)},$$

and so $A < \infty$ if and only if $r < 1$ (that is $\lambda > \mu$), in which case $A = 1/(\mu(1-r))$.

Consider first the case $\lambda < \mu$. We now have $\pi_i = \rho^{i-1}$, $i \geq 1$. Using Proposition 3, we find that

$$\mathrm{E}_j(\Gamma_0(f)) = \frac{1}{\mu} \sum_{i=0}^{j-1} \sum_{k=i}^{\infty} f_{k+1} \rho^{k-i}, \qquad j \geq 1.$$

To illustrate this further, take $f_k = \alpha^{k-1}$, where $\alpha > 0$. The expectation is finite if and only if $\alpha < r$, in which case

$$\mathrm{E}_j(\Gamma_0(f)) = \frac{r(1-\alpha^j)}{\mu(r-\alpha)(1-\alpha)}, \qquad j \ge 1.$$

Notice that $\{f_j\}$ is a decreasing sequence when $\alpha < 1$, and so Proposition 1 of Stefanov and Wang [10] cannot be used to establish this formula.

Finally, we shall consider the case $\lambda > \mu$, that is, $r < 1$. A simple calculation gives

$$C_j(f) = \frac{r}{\mu(1-r)} \sum_{k=1}^{j-1} f_k(1-r^{j-k}) \quad \text{and} \quad C(f) = \frac{r}{\mu(1-r)} \sum_{k=1}^{\infty} f_k\,.$$

and so

$$\frac{C_j(f)}{A_j} = r \sum_{k=1}^{j-1} f_k \left(\frac{1-r^{j-k}}{1-r^j} \right).$$

The expectation in Proposition 4 will therefore be finite if and only if $\sum_{k=1}^{\infty} f_k < \infty$, in which case

$$\mathrm{E}_j(\Gamma_0(f)) = \frac{1-r^j}{\mu(1-r)} \left(M - r \sum_{k=1}^{j-1} f_k \left(\frac{1-r^{j-k}}{1-r^j} \right) \right), \qquad j \ge 1,$$

where

$$M = \sup_{j \ge 1} \left(r \sum_{k=1}^{j-1} f_k \left(\frac{1-r^{j-k}}{1-r^j} \right) \right).$$

As before, take $f_k = \alpha^{k-1}$, but assume that $\alpha < 1$, so as to ensure that the expectation is finite. We find that

$$\frac{C_j(f)}{A_j} = \begin{cases} \dfrac{r((j-1)r^j - jr^{j-1} + 1)}{(1-r^j)(1-r)}, & \text{if } \alpha = r, \\ \dfrac{r(\alpha^j - \alpha + (1-\alpha^j)r + (\alpha-1)r^j)}{(1-r^j)(1-\alpha)(r-\alpha)}, & \text{if } \alpha \ne r. \end{cases}$$

In both cases this defines an increasing sequence with limit $r/(1-\alpha)$. Therefore, if $\alpha = r$,

$$\mathrm{E}_j(\Gamma_0(f)) = \frac{jr^j}{\mu(1-r)}, \qquad j \ge 1,$$

while if $\alpha \ne r$, then

$$\mathrm{E}_j(\Gamma_0(f)) = \frac{r(r^j - \alpha^j)}{\mu(1-\alpha)(r-\alpha)}, \qquad j \ge 1.$$

Example 2. Next we shall consider an example of a birth-death process that exhibits explosive behaviour. We begin by examining the pure birth process. This has transition rates $q_{i,i+1} = q_i = \lambda_i$, $i \ge 0$, where the birth rates $(\lambda_i,\ i \ge 0)$ are all strictly positive. Equations (3) are $\lambda_i(z_{i+1} - z_i) + f_i = 0$, $i \geq 0$. On summing from $i = 0$ to $j-1$ we get $z_j = z_0 - C_j(f)$, where $C_j(f) = \sum_{i=0}^{j-1} f_i/\lambda_i$. Since $C_j(f) \uparrow C(f) := \sum_{i=0}^{\infty} f_i/\lambda_i$, a finite non-negative solution is obtained whenever $C(f) < \infty$ and $z_0 \geq C(f)$, in which case setting $z_0 = C(f)$ gives the minimal solution. We deduce that $\mathrm{E}_i(\Gamma) = \sum_{j=i}^{\infty} f_j/\lambda_j$. Setting $f_i = 1$ shows that $\mathrm{E}_i(\tau_\infty) = \sum_{j=i}^{\infty} 1/\lambda_j$. So, the expected time to explosion is infinite if and only if $C := \sum_{j=0}^{\infty} 1/\lambda_j = \infty$. Indeed, the process is regular, that is τ_∞ is almost surely infinite for all starting states, if and only if this latter condition holds (see [1]).

Explosive behaviour is easily exhibited. For example, suppose that $X(t)$ describes the number in a population of individuals, *pairs* of whom produce offspring at (per-interaction) rate $\gamma > 0$. Then, $\gamma\binom{i}{2}$ will be the birth rate when the population size is i. Since at least two individuals are needed to get things going, it is convenient to relabel the state space so that $X(t) = i$ indicates that $i+2$ individuals are present at time t. So, we have a pure birth process of the kind described with $\lambda_i = \lambda(i+1)(i+2)$, where $\lambda = \gamma/2$. The process is explosive because

$$C = \sum_{i=0}^{\infty} \frac{1}{\lambda(i+1)(i+2)} = \frac{1}{\lambda} < \infty\,,$$

and the path integral Γ has expectation

$$\mathrm{E}_i(\Gamma) = \frac{1}{\lambda} \sum_{j=i}^{\infty} \frac{f_j}{(j+1)(j+2)}\,.$$

For example, if $f_i = \alpha^i$, where $0 < \alpha \leq 1$, then

$$\mathrm{E}_0(\Gamma) = \frac{1}{\lambda\alpha}\left(1 + \frac{1-\alpha}{\alpha}\log(1-\alpha)\right).$$

The introduction of a linear death component presents no particular difficulties. Let $\lambda_i = \lambda i(i+1)$ and define μ_i by $\mu_i = \mu i$, where $\mu > 0$. In this way, $E = \{1, 2, \dots\}$ is an irreducible class and 0 is the sole absorbing state. Let $\rho = \lambda/\mu$. Then, in the notation established in connection with Propositions 3 and 4, we have $\pi_1 = 1$ and, for $j \geq 2$,

$$\pi_j = \prod_{k=2}^{j} \frac{\lambda_{k-1}}{\mu_k} = \rho^{j-1} \prod_{k=2}^{j} \frac{k(k-1)}{k} = \rho^{j-1}(j-1)!$$

(This formula is valid also for $j = 1$.) The j-th term of A is

$$a_j := \frac{1}{\mu_j \pi_j} = \frac{1}{\mu\rho^{j-1} j!}$$

and so $A < \infty$ because $a_{j+1}/a_j = 1/(\rho(j+1)) \to 0$. Thus, exit from E occurs with probability *less than* 1 (no matter what the ρ). Also, since $\lambda_i/\mu_i = \rho(i+1) \to \infty$, there is very strong positive drift. So, we might expect the process to be explosive. Indeed, this can be established. The j-th term of C is

$$c_j := \frac{1}{\lambda_j \pi_j} \sum_{k=1}^{j} \pi_k = \frac{1}{\lambda\rho^{j-1}(j+1)!} \sum_{k=0}^{j-1} \rho^k k! = \frac{d_{j-1}}{\lambda j(j+1)}\,,$$

where

$$d_j = \frac{1}{\rho^j j!}\sum_{k=0}^{j} \rho^k k! = 1 + \sum_{k=1}^{j} \frac{\rho^{j-k}(j-k)!}{\rho^j j!}\,.$$

Since, for each $k \geq 1$, the summand converges to 0 monotonically as $j \to \infty$, monotone convergence implies that $d_j \to 1$. Hence, because $\sum_{j=1}^{k}\{j(j+1)\}^{-1} = 1$, we may deduce that $C < \infty$.

To illustrate the evaluation of the path integral, suppose that the states in E have Poisson weights: $f_k = e^{-\alpha}\alpha^{k-1}/(k-1)!$, $k \geq 1$, where $\alpha > 0$. Then, if $\alpha\rho \neq 1$,

$$A_j = \frac{\rho}{\mu}\sum_{i=1}^{j}\frac{\rho^{-i}}{i!} \quad \text{and} \quad C_j(f) = \frac{e^{-\alpha}}{1-\alpha\rho}\left(A_j - \frac{1}{\alpha\mu}\sum_{i=1}^{j}\frac{\alpha^i}{i!}\right),$$

while if $\alpha\rho = 1$,

$$A_j = \frac{1}{\alpha\mu}\sum_{i=1}^{j}\frac{\alpha^i}{i!} \quad \text{and} \quad C_j(f) = \frac{e^{-\alpha}}{\mu}\left(1 + \alpha\mu A_{j-1} - \mu A_j\right).$$

It is easy to show that $C_j(f)/A_j \uparrow M$, where

$$M = \frac{e^{-\alpha}}{1-\alpha\rho}\left(1 - \frac{e^{\alpha}-1}{\alpha\rho(e^{1/\rho}-1)}\right), \qquad \text{if } \alpha\rho \neq 1,$$

and

$$M = \frac{e^{-\alpha}+\alpha-1}{e^{\alpha}-1}, \qquad \text{if } \alpha\rho = 1.$$

Using Proposition 4, we get

$$\mathrm{E}_j(\Gamma_0(f)) = \frac{e^{-\alpha}}{\alpha\mu(1-\alpha\rho)}\left(\sum_{i=1}^{j}\frac{\alpha^i}{i!} - \frac{e^{\alpha}-1}{e^{1/\rho}-1}\sum_{i=1}^{j}\frac{\rho^{-i}}{i!}\right), \qquad \text{if } \alpha\rho \neq 1,$$

and

$$\mathrm{E}_j(\Gamma_0(f)) = \frac{e^{-\alpha}}{\mu}\left(\frac{e^{\alpha}}{e^{\alpha}-1}\sum_{i=1}^{j}\frac{\alpha^i}{i!} - \sum_{i=0}^{j-1}\frac{\alpha^i}{i!}\right), \qquad \text{if } \alpha\rho = 1.$$

4 The birth, death and catastrophe process

We shall briefly examine a model [9] that is equivalent to one described by Brockwell [2], which has been used to describe the behaviour of populations that are subject to catastrophic mortality or emigration events. In addition to simple birth and death transitions, it incorporates jumps down of arbitrary size (catastrophes). The process has state space $S = \{0, 1, \dots\}$ and q-matrix given by

$$q_{ij} = \begin{cases} i\rho\sum_{k\geq i} d_k, & j=0, \quad i \geq 1,\\ i\rho d_{i-j}, & j = 1,2,\dots i-1, \quad i \geq 2,\\ -i\rho, & j = i, \quad i \geq 0,\\ i\rho a & j = i+1, \quad i \geq 0,\\ 0, & \text{otherwise}, \end{cases}$$

where $\rho(>0)$ is the rate per capita at which the population size changes, $a(>0)$ is the probability of a birth and d_i, the probability of a catastrophe of size i, is positive for at least one $i \geq 1$ ($a + \sum_{i\geq 1} d_i = 1$). Notice that when $d_i = 0$ for all $i \geq 2$, we recover the simple linear birth-death process, with $\lambda_i = i\rho a$ and $\mu_i = i\rho d_1$. Clearly $E = \{1, 2, \dots\}$ is an irreducible class and 0 is an absorbing state that is accessible from E. It is easily shown that Q is regular. Let d be the probability generating function given by

$$d(s) = a + \sum_{i=1}^{\infty} d_i s^{i+1}, \qquad |s| < 1,$$

and assume that $d'(1-) < \infty$. Brockwell showed that the process is absorbed with probability 1 if and only if the drift D, given by $D = 1 - d'(1-)$, is less than or equal to 0.

We shall evaluate $\mathrm{E}_i(\Gamma_0(f))$ for the path integral (1), where τ_0 is the time to absorption. Considering equations (4), we seek the minimal non-negative solution to the system

$$i\rho a z_{i+1} - i\rho z_i + i\rho \sum_{j=1}^{i-1} d_{i-j} z_j + f_i = 0, \qquad i \geq 1.$$

This can be written

$$\rho a z_{i+1} - \rho z_i + \rho \sum_{j=1}^{i-1} d_{i-j} z_j + g_i = 0, \qquad i \geq 1, \tag{9}$$

where $g_i = f_i/i$. On multiplying by s^i and then summing over i, we find that the generating function $H(s) = \sum_{i=1}^{\infty} z_i s^{i-1}$ of *any* solution $(z_i,\ i \geq 1)$ to (9) must satisfy $b(s)H(s) - a\rho z_1 + g(s) = 0$, where $b(s) = \rho(d(s) - s)$ and $g(s) = \sum_{i=1}^{\infty} g_i s^i$. It is easy to see that b is convex on $(0, 1)$, with $b(0) = a\rho$ and $b(1) = 0$. Furthermore, as Brockwell showed, σ, the smallest zero of b in $(0, 1]$, satisfies $\sigma = 1$ if $D \geq 0$ and $\sigma < 1$ if $D < 0$.

We know, from Lemma V.12.1 of Harris [3], that $1/b(s)$ has a power series expansion with positive coefficients in a neighbourhood of 0. So, let us write $1/b(s) = \sum_{j=0}^{\infty} e_j s^j$, $|s| < \sigma$, where $e_j > 0$, noting that $a\rho = b(0) = 1/e_0$. Letting $\kappa = a\rho z_1$, we obtain

$$H(s) = z_1 + \sum_{i=1}^{\infty} \left(\kappa e_i - \sum_{j=1}^{i} g_j e_{i-j} \right) s^i,$$

and hence, for $i \geq 2$,

$$z_i = \kappa e_{i-1} - \sum_{j=1}^{i-1} g_j e_{i-1-j} = \kappa e_{i-1} - \sum_{j=0}^{i-2} g_{i-1-j} e_j.$$

Now, $\{e_i\}$ is an increasing sequence. This is because $e_0 = 1/(a\rho)$, and, since, $z_i = \kappa e_{i-1}$ when $g \equiv 0$, we have from (9) that

$$\rho a e_i - \rho e_{i-1} + \rho \sum_{j=1}^{i-1} d_{i-j} e_{j-1} = 0, \qquad i \geq 1,$$

and hence that

$$a(e_i - e_{i-1}) = \sum_{j=1}^{i-1} d_{i-j}(e_{i-1} - e_{j-1}) + e_{i-1} \sum_{j=i}^{\infty} d_j\,, \qquad i \geq 1.$$

Therefore, $\{z_i\}$ is the difference of two non-negative increasing sequences, and so, in order to ensure that $\{z_i\}$ itself is non-negative, we require

$$\kappa \geq \sup_{i\geq 1} h_i\,, \quad \text{where} \quad h_i = \frac{1}{e_i} \sum_{j=1}^{i} g_j e_{i-j}\,.$$

The minimal solution is obtained with equality here.

Since $\{e_i\}$ is increasing, we have that $0 < e_{i-j}/e_i \leq 1$ for all $i \geq j \geq 0$. Moreover, because σ is the radius of convergence of $\sum_{j=0}^{\infty} e_j s^j$, we have $\sigma = \lim_{i\to\infty} e_{i-1}/e_i$, whenever this limit exists, implying that $e_{i-j}/e_i \to \sigma^j$ for each j. Hence, formally, $h_i \to g(\sigma)$. If this can be justified, then provided $g(\sigma) < \infty$, we may set $\kappa = g(\sigma)$ to obtain the minimal non-negative solution to (9). By Fatou's lemma, we always have

$$\liminf_{i\to\infty} h_i = \liminf_{i\to\infty} \frac{1}{e_i} \sum_{j=1}^{i} g_j e_{i-j} \geq \sum_{j=1}^{\infty} g_j \sigma^j = g(\sigma)\,,$$

but, in order that $\lim_{i\to\infty} h_i$ exists and equals $g(\sigma)$, further mild conditions must be imposed. For example, if $\sum_{j=1}^{\infty} g_j b_j < \infty$ and $e_{i-j}/e_i \leq b_j$ for all $i \geq j$, or if $\{e_{i-j}/e_i\}$ is monotonic in i, then dominated and monotone convergence, respectively, guarantee that $h_i \to g(\sigma)$. (If, for example, $e_{2i} = 9^i = 3^{2i}$ and $e_{2i+1} = 2.9^i = 2.3^{2i}$ for $i = 0, 1, \ldots$, then it can be show that $\lim_{i\to\infty} h_i$ does not exist.)

Finally, if $g(\sigma) < \infty$ and $h_i \to g(\sigma)$, we get

$$\mathrm{E}_i(\Gamma_0(f)) = g(\sigma) e_{i-1} - \sum_{j=0}^{i-2} g_{i-1-j} e_j\,, \qquad i \geq 1, \tag{10}$$

or, if preferred,

$$\sum_{i=1}^{\infty} \mathrm{E}_i(\Gamma_0(f)) s^{i-1} = (g(\sigma) - g(s))/b(s)\,, \qquad |s| < \sigma.$$

To illustrate this, take $f_i = \alpha^{i-1}$, where $\alpha > 0$, so that $g(s) = -(1/\alpha)\log(1-\alpha s)$, $|s| < 1/\alpha$, and hence $g(\sigma) < \infty$ provided $\alpha < 1/\sigma$. For example, if $\alpha = 1$, so that $f_i = 1$ for all i, then $g(\sigma) < \infty$ if $D < 0$ and $g(\sigma) = g(1) = \infty$ if $D \geq 0$. We may therefore deduce that, when $D < 0$, the expected time to extinction is given by

$$\mathrm{E}_i(\tau_0) = -\log(1-\sigma) e_{i-1} - \sum_{j=0}^{i-2} \frac{e_j}{i-1-j}\,, \qquad i \geq 1,$$

or, if preferred,

$$\sum_{i=1}^{\infty} \mathrm{E}_i(\tau_0) s^{i-1} = \frac{1}{b(s)} \log\left(\frac{1-s}{1-\sigma}\right)\,, \qquad |s| < \sigma.$$

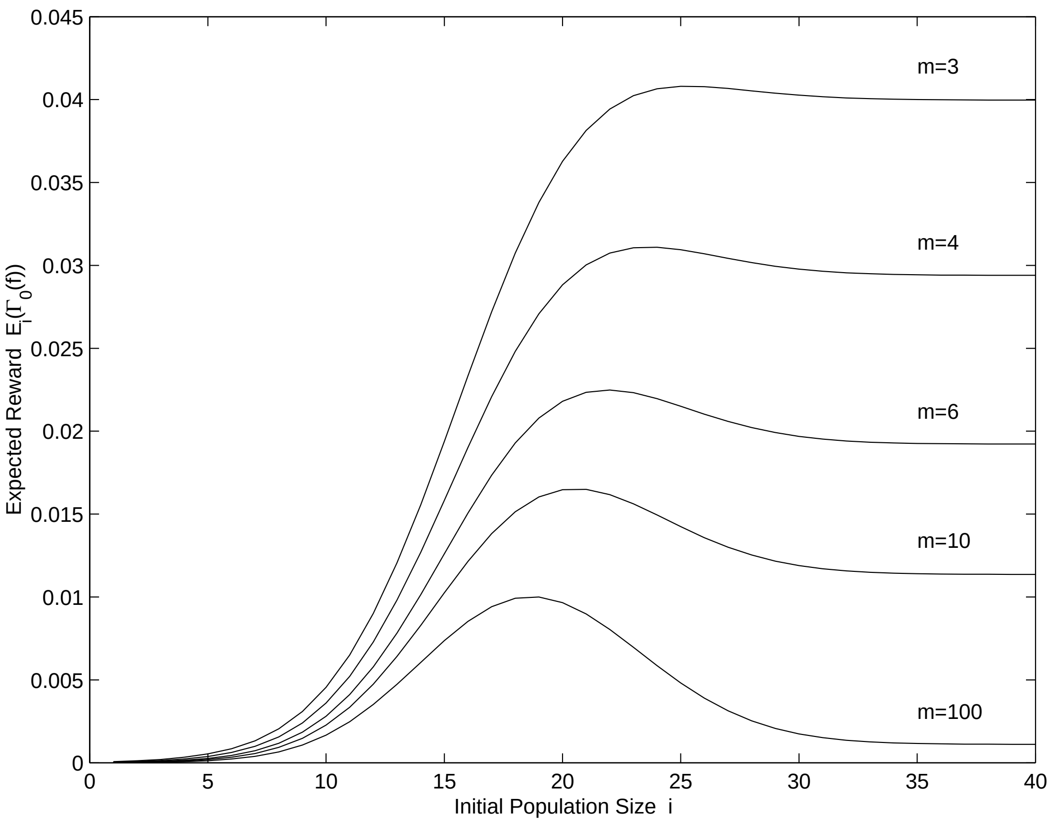

Figure 1: The effect of varying the m, the mean catastrophe size, on $\mathrm{E}_i(\Gamma_0(f))$ with Poisson weights $f_k = e^{-\alpha}\alpha^{k-1}/(k-1)!$ ($a = 0.55$ and $\alpha = 20$).

This is equation (3.1) of [2].

Explicit results can be obtained in the case where the catastrophe size follows a geometric law. Suppose that $d_i = b(1-q)q^{i-1}$, $i \geq 1$, where $b(>0)$ satisfies $a+b=1$, and $0 \leq q < 1$. The simple linear birth-death process is recovered on setting $q = 0$. It is easy to see that $D = a - b/(1-q)$, and so $D < 0$ or $D \geq 0$ according as $c > 1$ or $c \leq 1$, where $c = q + b/a$. We also have

$$b(s) = \frac{(b+qa)s^2 - (1+qa)s + a}{1-qs} = \frac{a(1-s)(1-cs)}{1-qs},$$

and hence if $D < 0$, then $\sigma = 1/c$ (< 1). The coefficients of the power series $1/b(s) = \sum_{j=0}^{\infty} e_j s^j$ are easily evaluated using partial fractions. If $D = 0$, then $e_j = (1 + (1-q)j)/a$, $j \geq 0$. Otherwise, if $D > 0$ or $D < 0$, then

$$e_j = \frac{1 - q - (c-q)c^j}{a(1-c)}, \qquad j \geq 0.$$

(Note that, in all cases, $e_{j-1}/e_j \uparrow \sigma$ and hence $e_{j-k}/e_j \uparrow \sigma^k$.) Thus, $\mathrm{E}_i(\Gamma_0(f))$ may be evaluated explicitly by substituting these expressions into (10), remembering that the expectation will be finite whenever $g(\sigma) < \infty$. For example, when $f_i = \alpha^{i-1}$, where $\alpha > 0$, the expectation is finite if $\alpha < \max\{1, q + b/a\}$. In contrast, $g(\sigma)$ is always finite in the case of Poisson weights: $f_k = e^{-\alpha}\alpha^{k-1}/(k-1)!$, where $\alpha > 0$.

For this latter case, Figure 1 illustrates the effect of varying the mean catastrophe size on the expected value of the path integral.

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